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FUZZY ALGEBRAIC VARIETIES

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ABSTRACT. The concept of a fuzzy algebraic variety is introduced in order to bring the current knowledge of fuzzy commutative ring theory to bear on the solution of nonlinear systems of equations of fuzzy singletons. It is shown for every finite-valued fuzzy ideal A of a polynomial ring in several indeterminates over a field with $A(0) = 1$ that the fuzzy algebraic variety of A can be expressed as a union of fuzzy irreducible algebraic varieties, no one of which is contained in the union of the others.

Introduction. Rosenfeld's application [**12**] of the pioneering work of Zadeh [**15**] inspired the fuzzification of various algebraic structures. Liu [**2**] and Mukherjee and Sen [**10**] presented some of the earliest work on the fuzzification of an ideal of a ring. Since then the notions of fuzzy prime ideal, fuzzy primary ideal, the radical of a fuzzy ideal, and the fuzzy primary representation of a fuzzy ideal have been introduced and examined [**1,3,4,5,6,7,10,11,13,14,16,17,18,19**]. There are several natural ways to define these concepts, many of which have appeared in the literature. In the interesting work of Zadehi [**16**] and Kumbhojkar and Bapat [**1**], the various types of fuzzy prime ideals, fuzzy primary ideals, and the radical of a fuzzy ideal have been compared. In [**7**] various types of radicals of fuzzy ideals and fuzzy primary representations of fuzzy ideals have been compared in order to prepare the way for the study of nonlinear systems of equations of fuzzy singletons.

Up to this point, the fuzzification of concepts and results of commutative ring theory have had no apparent application. The purpose of this paper is to give some meaning to fuzzy commutative ring theory developed to this point and to put some direction to its further study. We bring fuzzy commutative ring theory to bear on a natural application area, namely, the solution of nonlinear systems of equations of fuzzy singletons. Let R denote the polynomial ring $F[x_1, \dots, x_n]$ where F is a field and $x_1, \dots, x_n$ are algebraically independent indeterminants over

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F. Let L be a field containing F. L may be taken to be the algebraic closure of F or an algebraically closed field with infinite transcendence degree over F. Let L^k denote the set of all ordered k-tuples with entries from L, k a positive integer. Our approach is to consider those fuzzy ideals A of R which are finite-valued and are such that $A(0) = 1$ since these are precisely the fuzzy ideals of R which have fuzzy primary representations, [**6**]. We define the fuzzy algebraic variety $\mathcal{M}(A)$ of A and show that, from an irredundant fuzzy primary representation of A, $\mathcal{M}(A)$ is a finite union of irreducible fuzzy algebraic varieties, no one of which is contained in the union of the others, Theorem 2.7. We then apply this result to the solution of a nonlinear system of equations of fuzzy singletons, Example 2.9. We show that there exists a fuzzy ideal A of R which represents this system and the irredundant primary representation of $\sqrt{A}$ displays the solution of the system in a manner similar to that of the crisp situation. Hence, we have thus shown that in this sense the current definitions of fuzzy prime ideal, fuzzy primary ideal, and radical of a fuzzy ideal have been appropriately defined.

A fuzzy subset of a set Z is a function of Z into the closed interval $[0,1]$. If X and Y are fuzzy subsets of Z, then we write $X \subseteq Y$ if $X(z) \leq Y(z)$ for all $z \in Z$. If $\{X_i \mid i \in I\}$ is a collection of fuzzy subsets of Z, we define the fuzzy subsets $\cap_{i\in I} X_i$ and $\cup_{i\in I} X_i$ of Z by for all $z \in Z$, $(\cap_{i\in I} X_i)(z) = \inf\{X_i(z) \mid i \in I\}$ and $(\cup_{i\in I} X_i)(z) = \sup\{X_i(z) \mid i \in I\}$. Let X be a fuzzy subset of Z. We let $\mathrm{Im}\,(X)$ denote the image of X and $|\mathrm{Im}\,(X)|$ the cardinality of $\mathrm{Im}\,(X)$. We say that X is finite-valued if $|\mathrm{Im}\,(X)| < \infty$. We let $X^* = \{z \in Z \mid X(z) > 0\}$, the *support* of X, and $X_t = \{z \in Z \mid X(t) \geq t\}$ for all $t \in [0,1]$. If X is a fuzzy ideal of R [**4**, Lemma 1.7], then X^* is an ideal of R. Also X is a fuzzy ideal of R if and only if X_t is an ideal of R for all $t \in \mathrm{Im}\,(X)$ [**14**, Theorem 1.2]. For $z \in Z$ and $t \in [0,1]$, we let z_t denote the fuzzy subset of Z defined by $z_t(z) = t$ and $z_t(x) = 0$ if $x \neq z$. z_t is called a *fuzzy singleton* of Z. If z_t and y_s are fuzzy singletons of R, we define $z_t + y_s = (z+y)_r$ and $z_t y_s = (zy)_r$ where $r = \min\{t,s\}$. If S is a subset (fuzzy subset) of R, we let $\langle S\rangle$ denote the ideal (fuzzy ideal) of R generated by S. We recall that a fuzzy subset A of R is a fuzzy ideal of R if and only if for all $x, y \in R$, $A(x-y) \geq \min\{A(x), A(y)\}$ and $A(xy) \geq \max\{A(x), A(y)\}$. If A is a fuzzy ideal of R, then the radical of A, $\sqrt{A}$, is the intersection of all fuzzy prime ideals of R which contain A, [**5**, Definition 4.3, **16**, Theorem 3.8].

1. Fuzzy algebraic varieties. If I is an ideal of R, we let $\mathcal{M}(I)$ denote the algebraic variety of I, [**8**, page 203]. If Z is a subset of L^k, we let $\mathcal{J}(Z)$ denote the set of all $f \in R$ which vanish at all points of Z. Then $\mathcal{J}(Z)$ is an ideal of R, [**8**, page 203]. We now give definitions for the fuzzy counterparts of $\mathcal{M}$ and $\mathcal{J}$. Let c be a strictly decreasing function of $[0,1]$ into itself such that $c(0) = 1$, $c(1) = 0$, and for all $t \in [0,1]$, $c(c(t)) = t$. The following approach has the advantage that c may be changed to fit the application. For example, c may belong to the Sugeno class of fuzzy complements for one application and the Yeager class for another.

Definition 1.1. Let X be a finite-valued fuzzy subset of L^k, say $\mathrm{Im}\,(X) = \{t_0, t_1, \dots, t_n\}$ where $t_0 < t_1 < \cdots < t_n$. Define the fuzzy subset $\mathcal{J}(X)$ of R as follows:

$$\mathcal{J}(X)(f) = \begin{cases} c(t_n) & \text{if } f \in R - \mathcal{J}(X_{t_n}); \\ c(t_i) & \text{if } f \in \mathcal{J}(X_{t_{i+1}} - \mathcal{J}(X_{t_i}), i = 1, \dots, n-1; \\ c(t_0) & \text{if } f \in \mathcal{J}(X_{t_1}). \end{cases}$$

If $n = 0$, then we define $\mathcal{J}(X)(0) = 1$.

Definition 1.2. Let A be a finite-valued fuzzy ideal of R, say $\mathrm{Im}\,(A) = \{s_0, s_1, \dots, s_m\}$ where $s_0 < s_1 < \cdots < s_m$. Define the fuzzy subset $\mathcal{M}(A)$ of L^k as follows:

$$\mathcal{M}(A)(b) = \begin{cases} c(s_m) & \text{if } b \in L^k - \mathcal{M}(A_{s_m}); \\ c(s_i) & \text{if } b \in \mathcal{M}(A_{s_{i+1}}) - \mathcal{M}(A_{s_i}),\ i = 1, \dots, m-1; \\ c(s_0) & \text{if } b \in \mathcal{M}(A_{s_1}). \end{cases}$$

$\mathcal{M}(A)$ is called a *fuzzy algebraic variety* (of A).

In Definition 1.1, it is possible for $\mathcal{J}(X_{t_{i+1}}) = \mathcal{J}(X_{t_i})$ or $R = \mathcal{J}(X_{t_n})$. In this case $c(t_i) \notin \mathrm{Im}\,(\mathcal{J}(X))$, $i = 1, \dots, n$. Similarly, it is possible for $c(s_i) \notin \mathrm{Im}\,(\mathcal{M}(A))$ for some $i = 1, \dots, m$.

Proposition 1.3. *Let X be defined as in Definition* 1.1. *Then*

(1) $\mathcal{J}(X)_{c(t_i)} = \mathcal{J}(X_{t_{i+1}})$ *for* $i = 0, 1, \dots, n-1$;

(2) *if* $0 \leq s \leq c(t_n)$, *then* $\mathcal{J}(X)_s = R$;

(3) *if* $c(t_{i+1}) < s < c(t_i)$, *then* $\mathcal{J}(X)_s = \mathcal{J}(X_{c(s)})$ *for* $i = 0, 1, \ldots, n-1$;

(4) *if* $c(t_0) < s \leq 1$, *then* $\mathcal{J}(X)_s = \varnothing$.

Proof. (1) $f \in \mathcal{J}(X)_{c(t_i)}$ if and only if $\mathcal{J}(X)(f) \geq c(t_i)$ if and only if $f \in \mathcal{J}(X_{t_{i+1}})$ by Definition 1.1.

(2) $\mathcal{J}(X)_s = R$ since $c(t_n) \leq$ the smallest element in $\mathrm{Im}\,(\mathcal{J}(X))$.

(3) $f \in \mathcal{J}(X)_s$ if and only if $\mathcal{J}(X)(f) \geq s$ if and only if $\mathcal{J}(X)(f) \geq c(t_i)$ if and only if $f \in \mathcal{J}(X)_{c(t_i)}$ if and only if $f \in \mathcal{J}(X_{t_{i+1}})$ by (1) if and only if $f \in \mathcal{J}(X_{c(s)})$ since $X_{t_{i+1}} = X_{c(s)}$.

(4) $\mathcal{J}(X)_s = \varnothing$ since $c(t_0)$ is the largest element in $\mathrm{Im}\,(\mathcal{J}(X))$. $\square$

For X as defined in Definition 1.1, $\mathcal{J}(X)_{c(t_i)} = \mathcal{J}(X_{t_{i+1}})$ is an ideal of R for $i = 0, 1, \ldots, n-1$. Thus, since $\mathrm{Im}\,(X) \subseteq \{c(t_i) \mid i = 0, 1, \ldots, n\}$, $\mathcal{J}(X)$ is a fuzzy ideal of R [**14**, Theorem 1.2].

Proposition 1.4. *Let A be defined as in Definition* 1.2. *Then*

(1) $\mathcal{M}(A)_{c(s_i)} = \mathcal{M}(A_{s_{i+1}})$ *for* $i = 0, 1, \ldots, m-1$;

(2) *if* $0 \leq t \leq c(s_m)$, *then* $\mathcal{M}(A)_t = L^k$;

(3) *if* $c(s_{i+1}) < t < c(s_i)$, *then* $\mathcal{M}(A)_t = \mathcal{M}(A_{c(t)})$ *for* $i = 0, 1, \ldots, m-1$;

(4) *if* $c(s_0) < t \leq 1$, *then* $\mathcal{M}(A) = \varnothing$.

Proof. (1) $b \in \mathcal{M}(A)_{c(s_i)}$ if and only if $\mathcal{M}(A)(b) \geq c(s_i)$ if and only if $b \in \mathcal{M}(A_{s_{i+1}})$ by Definition 1.2.

(2) $\mathcal{M}(A)_t = L^k$ since $c(s_m) \leq$ the smallest element in $\mathrm{Im}\,(\mathcal{M}(A))$.

(3) $b \in \mathcal{M}(A)_t$ if and only if $\mathcal{M}(A)(b) \geq t$ if and only if $\mathcal{M}(A)(b) \geq c(s_i)$ if and only if $b \in \mathcal{M}(A)_{c(s_i)}$ if and only if $b \in \mathcal{M}(A_{s_{i+1}})$ by (1) if and only if $b \in \mathcal{M}(A_{c(t)})$ since $A_{s_{i+1}} = A_{c(t)}$.

(4) $\mathcal{M}(A)_t = \varnothing$ since $c(s_0)$ is the largest element in $\mathrm{Im}\,(\mathcal{M}(A))$. $\square$

Proposition 1.5. *Let X and A be as defined in Definitions* 1.1 *and* 1.2*, respectively. Then*

(1) $|\mathrm{Im}\,(\mathcal{M}(\mathcal{J}(X))| = |\mathrm{Im}\,(\mathcal{J}(X))|$;

(2) $|\mathrm{Im}\,(\mathcal{J}(\mathcal{M}(A))| = |\mathrm{Im}\,(\mathcal{M}(A))|$.

Proof. (1) Suppose that $\mathrm{Im}\,(\mathcal{J}(X)) = \{r_0, r_1, \ldots, r_h\}$ with $r_0 < r_1 < \cdots < r_h$. Then $\mathcal{J}(X)_{r_{i+1}} \subset \mathcal{J}(X)_{r_i}$ and so $\mathcal{M}(\mathcal{J}(X)_{r_{i+1}}) \supset \mathcal{M}(\mathcal{J}(X)_{r_i})$ from the crisp case and since for $r \in \mathrm{Im}\,(\mathcal{J}(X))$, $\mathcal{J}(X)_r = \mathcal{J}(X_t)$ for some $t \in \mathrm{Im}\,(X)$. Hence, $\mathrm{Im}\,(\mathcal{M}(\mathcal{J}(X)) = \{c(r_0), c(r_1), \ldots, c(r_h)\}$.

(2) Suppose that $\mathrm{Im}\,(\mathcal{M}(A)) = \{q_0, q_1, \ldots, q_j\}$ with $q_0 < q_1 < \cdots < q_j$. Then $\mathcal{M}(A)_{q_{i+1}} \subset \mathcal{M}(A)_{q_i}$ and so $\mathcal{J}(\mathcal{M}(A)_{q_{i+1}}) \supset \mathcal{J}(\mathcal{M}(A)_{q_i})$. Hence, $\mathrm{Im}\,(\mathcal{J}(\mathcal{M}(A)) = \{c(q_0), c(q_1), \ldots, c(q_j)\}$. $\square$

Proposition 1.6. *Let X and A be defined as in Definitions* 1.1 *and* 1.2*, respectively. Then*

(1) *for all $t \in [0,1]$, $\mathcal{M}(\mathcal{J}(X))_t = \mathcal{M}(J(X_t))$;*

(2) *for all $s \in [0,1]$, $\mathcal{J}(\mathcal{M}(A))_s = \mathcal{J}(\mathcal{M}(A_s))$.*

Proof. (1) For $i = 0, 1, \ldots, n-1$, $\mathcal{M}(\mathcal{J}(X_{t_{i+1}})) = \mathcal{M}(\mathcal{J}(X)_{c(t_i)})$ by Proposition 1.3 (1). Suppose that $c(t_i) \in \mathrm{Im}\,(\mathcal{J}(X))$. Let $s_{n-i} = c(t_i)$ for $i = 0, 1, \ldots, n$. Then $\mathcal{M}(\mathcal{J}(X)_{c(t_i)}) = \mathcal{M}(\mathcal{J}(X)_{s_{n-i}}) = \mathcal{M}(\mathcal{J}(X))_{c(s_{n-i-1})} = \mathcal{M}(\mathcal{J}(X))_{t_{i+1}}$. Suppose that $c(t_i) \notin \mathrm{Im}\,(\mathcal{J}(X))$. Then by Proposition 1.4 (3), $\mathcal{M}(\mathcal{J}(X)_{c(t_i)}) = \mathcal{M}(\mathcal{J}(X))_{c(c(t_i))} = \mathcal{M}(\mathcal{J}(X))_{t_i} = \mathcal{M}(\mathcal{J}(X))_{t_{i+1}}$ where the latter equality holds since $c(t_i) \notin \mathrm{Im}\,(\mathcal{J}(X))$ implies $t_i = c(c(t_i)) \notin \mathrm{Im}\,(\mathcal{M}(\mathcal{J}(X)))$ and so $\mathcal{M}(\mathcal{J}(X))(b) \geq t_i$ if and only if $\mathcal{M}(\mathcal{J}(X))(b) \geq t_{i+1}$. For $0 \leq t \leq t_0$, $\mathcal{M}(\mathcal{J}(X_t)) = \mathcal{M}(\mathcal{J}(L^k)) = \mathcal{M}(\langle 0 \rangle) = L^k = \mathcal{M}(\mathcal{J}(X))_t$ since t_0 is the smallest value in $\mathrm{Im}\,(X)$. Suppose that $t_n < t \leq 1$. Then $\mathcal{M}(\mathcal{J}(X_t)) = \mathcal{M}(\mathcal{J}(\varnothing)) = \mathcal{M}(R) = \varnothing = \mathcal{M}(\mathcal{J}(X))_t$ since t_n is the largest valued in $\mathrm{Im}\,(X)$. For any other t, $\mathcal{M}(\mathcal{J}(X_t)) = \mathcal{M}(\mathcal{J}(X)_{c(t)}) = \mathcal{M}(\mathcal{J}(X))_t$.

(2) For $i = 0, 1, \ldots, m-1$, $\mathcal{J}(\mathcal{M}(A_{s_{i+1}})) = \mathcal{J}(\mathcal{M}(A)_{c(s_i)})$ by Proposition 1.4 (1). Suppose that $c(s_i) \in \mathrm{Im}\,(\mathcal{M}(A))$. Let $t_{m-i} = c(s_i)$ for $i = 0, 1, \ldots, m$. Then $\mathcal{J}(\mathcal{M}(A)_{c(s_i)}) = \mathcal{J}(\mathcal{M}(A)_{t_{m-i}}) = \mathcal{J}(\mathcal{M}(A))_{c(t_{m-i-1})} = \mathcal{J}(\mathcal{M}(A))_{s_{i+1}}$. Suppose that $c(t_i) \notin \mathrm{Im}\,(\mathcal{M}(A))$.

Then, by Proposition 1.3 (3), $\mathcal{J}(\mathcal{M}(A)_{c(s_i)}) = \mathcal{J}(\mathcal{M}(A))_{c(c(s_i))} = \mathcal{J}(\mathcal{M}(A))_{s_i} = \mathcal{J}(\mathcal{M}(A))_{s_{i+1}}$ where the latter equality holds since $c(s_i) \notin \operatorname{Im}(\mathcal{M}(A))$ implies $s_i = c(c(s_i)) \notin \operatorname{Im}(\mathcal{J}(\mathcal{M}(A)))$ and so $\mathcal{J}(\mathcal{M}(A))(f) \geq s_i$ if and only if $\mathcal{J}(\mathcal{M}(A))(f) \geq s_{i+1}$. For $0 \leq s \leq s_0$, $\mathcal{J}(\mathcal{M}(A_s)) = \mathcal{J}(\mathcal{M}_0(R)) = \mathcal{J}(\varnothing) = R = \mathcal{J}(\mathcal{M}(A))_s$ since s_0 is the smallest value in $\operatorname{Im}(A)$. Suppose that $s_m < s \leq 1$. Then $\mathcal{J}(\mathcal{M}(A_s)) = \mathcal{J}(\mathcal{M}(\varnothing)) = \mathcal{J}(L^k) = \varnothing = \mathcal{J}(\mathcal{M}(A))_s$ since s_m is the largest value in $\operatorname{Im}(A)$. For any other s, $\mathcal{J}(\mathcal{M}(A_s)) = \mathcal{J}(\mathcal{M}(A)_{c(s)}) = \mathcal{J}(\mathcal{M}(A))_s$. $\square$

Proposition 1.7. *Let X and A be defined as in Definitions* 1.1 *and* 1.2, *respectively. Then*

(1) $\mathcal{J}(\mathcal{M}(\mathcal{J}(X))) = \mathcal{J}(X)$;

(2) $\mathcal{M}(\mathcal{J}(\mathcal{M}(A))) = \mathcal{M}(A)$.

Proof. By Propositions 1.5 and 1.6 and the crisp case, we have the following arguments.

(1) $\mathcal{J}(\mathcal{M}(\mathcal{J}(X)))_{c(t_i)} = \mathcal{J}(\mathcal{M}(\mathcal{J}(X)_{c(t_i)})) = \mathcal{J}(\mathcal{M}(\mathcal{J}(X_{t_{i+1}}))) = \mathcal{J}(X_{t_{i+1}}) = \mathcal{J}(X)_{c(t_i)}$ where $i = 0, 1, \dots, n-1$. If $c(t_{i+1}) < s < c(t_i)$, then $\mathcal{J}(\mathcal{M}(\mathcal{J}(X)))_s = \mathcal{J}(\mathcal{M}(\mathcal{J}(X)_s)) = \mathcal{J}(\mathcal{M}(\mathcal{J}(X_{c(s)}))) = \mathcal{J}(X_{c(s)}) = \mathcal{J}(X)_s$ where $i = 0, 1, \dots, n-1$. For $0 \leq s \leq c(t_n)$, $\mathcal{J}(\mathcal{M}(\mathcal{J}(X)))_s = R = \mathcal{J}(X)_s$. For $c(t_0) < s \leq 1$, $\mathcal{J}(\mathcal{M}(\mathcal{J}(X)))_s = \varnothing = \mathcal{J}(X)_s$. Thus, $\mathcal{J}(\mathcal{M}(\mathcal{J}(X)))_s = \mathcal{J}(X)_s$ for all $s \in [0,1]$.

(2) $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_{c(s_i)} = \mathcal{M}(\mathcal{J}(\mathcal{M}(A)_{c(s_i)})) = \mathcal{M}(\mathcal{J}(\mathcal{M}(A_{s_{i+1}}))) = \mathcal{M}(A_{s_{i+1}}) = \mathcal{M}(A)_{c(s_i)}$ where $i = 0, 1, \dots, m-1$. If $c(s_{i+1}) < t < c(s_i)$, then $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_t = \mathcal{M}(\mathcal{J}(\mathcal{M}(A)_t)) = \mathcal{M}(\mathcal{J}(\mathcal{M}(A_{c(t)}))) = \mathcal{M}(A_{c(t)}) = \mathcal{M}(A)_t$. For $0 \leq t \leq c(s_m)$, $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_t = L^k = \mathcal{M}(A)_t$. For $c(s_0) < t \leq 1$, $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_t = \varnothing = \mathcal{M}(A)_t$. Thus, $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_t = \mathcal{M}(A)_t$ for all $t \in [0,1]$. $\square$

Theorem 1.8. *Let M be a fuzzy subset of L^k. Then M is a fuzzy algebraic variety if and only if M is finite-valued and for all $t \in \operatorname{Im}(M)$, M_t is an algebraic variety.*

Proof. Suppose that M is a fuzzy algebraic variety. Then $M = \mathcal{M}(A)$ for some finite-valued fuzzy ideal A of R. Hence, for all $t \in \operatorname{Im}(M)$, there exists $s \in \mathrm{Im}\,(A)$ such that $t = c(s)$. Thus, either $M_t = \mathcal{M}(A)_{c(s)} = \mathcal{M}(A_{s'})$, where s' is the successor of s in $\mathrm{Im}\,(A)$ or $\mathcal{M}(A)_{c(s)} = L^k = \mathcal{M}(\langle 0\rangle)$. Since $\mathcal{M}(A_{s'})$ and $\mathcal{M}(\langle 0\rangle)$ are algebraic varieties, M_t is an algebraic variety. Conversely, suppose that M is finite-valued and M_t is an algebraic variety for all $t \in \mathrm{Im}\,(M)$. Then $M_t = \mathcal{M}(I^{(t)})$ for some ideal $I^{(t)}$ of R, $t \in \mathrm{Im}\,(M)$. Now $\mathcal{M}(I^{(t)}) = \mathcal{M}(\mathcal{J}(\mathcal{M}(I^{(t)})))$. Thus, $M_t = \mathcal{M}(J^{(t)})$ for some ideal $J^{(t)}$ of R such that if $t,\ t' \in \mathrm{Im}\,(M)$ with $t < t'$, then $J^{(t)} \subset J^{(t')}$, namely, $J^{(t)} = \mathcal{J}(\mathcal{M}(I^{(t)}))$. Let $\mathrm{Im}\,(M) = \{t_0, t_1, \dots, t_n\}$ where $t_0 < t_1 < \cdots < t_n$. Define the fuzzy subset A of R by $A(f) = c(t_n)$ if $f \in R - J^{(t_n)}$, $A(f) = c(t_i)$ if $f \in J^{(t_{i+1})} - J^{(t_i)}$, and $A(f) = c(t_0)$ if $f \in J^{(t_1)}$. Then A is a fuzzy ideal of R. Now $\mathcal{M}(A)_{t_i} = \mathcal{M}(A)_{c(c(t_i))} = \mathcal{M}(A_{c(t_{i-1})}) = \mathcal{M}(J^{(t_i)}) = M_{t_i}$ and so $M = \mathcal{M}(A)$. Thus, M is a fuzzy algebraic variety. $\square$

Proposition 1.9. *If A is a nonconstant fuzzy prime ideal of R, then $A = \mathcal{J}(\mathcal{M}(A))$.*

Proof. Now $\mathrm{Im}\,(A) = \{s, 1\}$ where $s < 1$ and A_1 is a prime ideal of R, [**4, 14**]. Suppose that $A_1 \neq \langle 0\rangle$. Then $\mathrm{Im}\,(\mathcal{J}(\mathcal{M}(A))) = \mathrm{Im}\,(A)$. By Proposition 1.6, $\mathcal{J}(\mathcal{M}(A))_1 = \mathcal{J}(\mathcal{M}(A_1)) = A_1$ since A_1 is a prime ideal of R. Now $\mathcal{J}(\mathcal{M}(\mathcal{J}(A)))_s = R = A_s$. Hence, $\mathcal{J}(\mathcal{M}(A)) = A$. Suppose that $A_1 = \langle 0\rangle$. Then $\mathcal{M}(A)(b) = c(1) = 0$ if $b \in L^k - \mathcal{M}(A_{s_1}) = L^k - \mathcal{M}(\langle 0\rangle) = \varnothing$ and $\mathcal{M}(A)(b) = c(s)$ if $b \in \mathcal{M}(A_{s_1}) = L^k$. Now $\mathcal{J}(\mathcal{M}(A))(f) = c(c(s)) = s$ if $f \in R - \mathcal{J}(\mathcal{M}(A)_{c(s)}) = R - \langle 0\rangle$ and $\mathcal{J}(\mathcal{M}(0)) = 1$. $\square$

Theorem 1.10. *Suppose that A is defined as in Definition* 1.2 *and $A(0) = 1$. Then $\mathcal{M}(A) = \mathcal{M}(\sqrt{A})$.*

Proof. $\mathrm{Im}\,(\sqrt{A}) \subseteq \mathrm{Im}\,(A)$ since A is finite-valued [**7**, Definition 1.1, Theorems 3.5, 3.10]. Now $\mathcal{M}(A)_t = \mathcal{M}(A_{c(t)}) = \mathcal{M}(\sqrt{A_{c(t)}}) = \mathcal{M}((\sqrt{A})_{c(t)})$ [**7**, Lemma 3.9, Theorem 3.10] $= \mathcal{M}(\sqrt{A})_t$ if $c(s_{i+1}) < t < c(s_i)$ for $i = 1, \dots, m-1$ since $\mathrm{Im}\,(\mathcal{M}(\sqrt{A})) \subseteq \mathrm{Im}\,(\mathcal{M}(A))$ where the latter inclusion follows since $\mathcal{M}(A_{s_{i+1}}) = \mathcal{M}(A_{s_i})$ if and only if $\mathcal{M}((\sqrt{A})_{s_{i+1}}) = \mathcal{M}((\sqrt{A})_{s_i})$ and so $c(s_i) \notin \mathrm{Im}\,(\mathcal{M}(A))$ implies

$c(s_i) \notin \mathrm{Im}\,(\mathcal{M}(\sqrt{A}))$. By Proposition 1.4, $\mathcal{M}(A)_{c(s_i)} = \mathcal{M}(A_{s_{i+1}}) = \mathcal{M}(\sqrt{A_{s_{i+1}}}) = \mathcal{M}((\sqrt{A})_{s_{i+1}})$. Let j be the largest nonnegative integer such that $j \leq i$, $s_j \in \mathrm{Im}\,(\sqrt{A})$ and let i^* be the smallest nonnegative integer such that $i^* \geq i$ and $s_{i^*+1} \in \mathrm{Im}\,(\sqrt{A})$. Then $\mathcal{M}(\sqrt{A_{s_{i+1}}}) = \mathcal{M}(\sqrt{A_{s_{i^*+1}}}) = \mathcal{M}(\sqrt{A})_{c(s_j)} = \mathcal{M}(\sqrt{A})_{c(s_i)}$ where the latter equality holds since if $s_i \notin \mathrm{Im}\,(\sqrt{A})$, then $c(s_i) \notin \mathrm{Im}\,(\mathcal{M}(\sqrt{A}))$. Hence, $\mathcal{M}(A)_{c(s_i)} = \mathcal{M}(\sqrt{A})_{c(s_i)}$. If j doesn't exist and i^* does, then $(\sqrt{A})_{s_{i^*+1}} = R$. Thus $\varnothing = \mathcal{M}(\sqrt{A_{s_{i^*+1}}}) = \mathcal{M}(A_{s_{i^*+1}}) \supseteq \mathcal{M}(A_{s_{i+1}}) = \mathcal{M}(A)_{c(s_i)} \supseteq \mathcal{M}(\sqrt{A})_{c(s_i)} \supseteq \varnothing$. Hence, $\mathcal{M}(A)_{c(s_i)} = \mathcal{M}(\sqrt{A})_{c(s_i)}$. Suppose that j exists and i^* does not. Then $i = m$ and so $\mathcal{M}(A)_{c(s_m)} = L^k = \mathcal{M}(\sqrt{A})_{c(s_m)}$. Hence, $\mathcal{M}(A) = \mathcal{M}(\sqrt{A})$. □

Corollary 1.11. *If P is a fuzzy prime ideal of R belonging to the fuzzy primary ideal Q of R, then $\mathcal{M}(Q) = \mathcal{M}(P)$.*

Proof. $\sqrt{Q} = P$, [**5**]. □

Lemma 1.12. *Suppose that A and B are fuzzy ideals of R such that* $\mathrm{Im}\,(A) = \{s_0, s_1, \dots, s_m\}$ *where* $s_0 < s_1 < \cdots < s_m = 1$ *and* $\mathrm{Im}\,(B) = \{s, 1\}$ *where* $s < 1$. *Then* $\mathcal{M}(A \cap B) = \mathcal{M}(A) \cup \mathcal{M}(B)$.

Proof. (1) Suppose that $s_i \leq s < s_{i+1}$. Then $\{s_0, s_1, \dots, s_i, 1\} \subseteq \mathrm{Im}\,(A \cap B) \subseteq \mathrm{Im}\,(A) \cup \mathrm{Im}\,(B)$.

(1.1) We first show that $\mathcal{M}(A \cap B)_{c(s_j)} = (\mathcal{M}(A \cup \mathcal{M}(B))_{c(s_j)}$ for all $s_j \in \mathrm{Im}\,(A)$. Now $\mathcal{M}(A \cap B)_{c(1)} = L^k = L^k \cup L^k = \mathcal{M}(A)_{c(1)} \cup \mathcal{M}(B)_{c(1)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(1)}$.

(1.1.1) Suppose that $1 > s_j \geq s_{i+1}$.

(1.1.1.1) Suppose that $s_j \notin \mathrm{Im}\,(A \cap B)$. Then $(A \cap B)_{s_j} = (A \cap B)_{s_{j+1}}$ since s_{j+1} is the next largest possible element in $\mathrm{Im}\,(A \cap B)$. Thus $\mathcal{M}(A \cap B)_{c(s_j)} = \mathcal{M}((A \cap B)_{s_j})$ (Proposition 1.4 (3)) $= \mathcal{M}((A \cap B)_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}} \cap B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_{s_{j+1}})$ (crisp case) $= \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B_1)$ (Proposition 1.4 (1) and $s_{j+1} > s$) $= \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s)} = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s_j)}$ (since $c(s_j) > 0$) $= (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_j)}$.

(1.1.1.2) Suppose that $s_j \in \mathrm{Im}\,(A \cap B)$. Then $\mathcal{M}(A \cap B)_{c(s_j)} =$

$\mathcal{M}((A \cap B)_{s_{j^*+1}})$ where j^* is the smallest integer $\geq j$ such that $s_{j^*+1} \in \mathrm{Im}\,(A \cap B)$. Now $(A \cap B)_{s_{j^*+1}} = (A \cap B)_{s_{j+1}}$. Hence $\mathcal{M}(A \cap B)_{c(s_j)} = \mathcal{M}((A \cap B)_{s_{j+1}}) = \mathcal{M}() \cup \mathcal{M}(B))_{c(s_j)}$ as just argued above.

(1.1.2) Suppose now that $s_j < s_i$. Then $\mathcal{M}(A \cap B)_{c(s_j)} = \mathcal{M}((A \cap B)_{s_{j+1}})$ (since $s_j, s_{j+1} \in \mathrm{Im}\,(A \cap B)) = \mathcal{M}(A_{s_{j+1}} \cap B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_{s_{j+1}}) = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(R) = \mathcal{M}(A)_{c(s_j)} \cup \varnothing = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s_j)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_j)}$ since $c(s_j) > c(s)$ and $c(s)$ is the largest element in $\mathrm{Im}\,(B)$.

(1.1.3) We now consider the case $s_j = s_i$. Then $s_j \in \mathrm{Im}\,(A \cap B)$.

(1.1.3.1) Suppose that $s_i < s < s_{i+1}$.

(1.1.3.1.1) Suppose that $s \notin \mathrm{Im}\,(A \cap B)$. Then $c(s) \notin \mathrm{Im}\,(\mathcal{M}(A \cap B))$. Thus $\mathcal{M}(A \cap B)_{c(s_i)} = \mathcal{M}(A \cap B)_{c(s)} = \mathcal{M}((A \cap B)_s) = \mathcal{M}(A_s \cap B_s) = \mathcal{M}(A_s) \cup \mathcal{M}(B_s) = \mathcal{M}(A_{s_{i+1}}) \cup \mathcal{M}(B_s) = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(R) = \mathcal{M}(A)_{c(s_i)} \cup \varnothing = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B)_{c(s_i)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_i)}$.

(1.1.3.1.2) Suppose that $s \in \mathrm{Im}\,(A \cap B)$. Then $\mathcal{M}(A \cap B)_{c(s_i)} = \mathcal{M}((A \cap B)_s) = \mathcal{M}(A_s \cap B_s) = \mathcal{M}(A_{s_{i+1}} \cap R) = \mathcal{M}(A_{s_{i+1}}) \cup \mathcal{M}(R) = \mathcal{M}(A)_{c(s_i)} \cup \varnothing = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B)_{c(s_i)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_i)}$.

(1.1.3.2) Suppose that $s_i = s$. Then for i^* the smallest integer greater than or equal to i such that $s_{i^*+1} \in \mathrm{Im}\,(A \cap B)$, $\mathcal{M}(A \cap B)_{c(s_i)} = \mathcal{M}((A \cap B)_{s_{i^*+1}}) = \mathcal{M}((A \cap B)_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}} \cap B_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}}) \cup \mathcal{M}(B_{s_{i+1}}) = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B_1) = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B)_{c(s)} = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B)_{c(s_i)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_i)}$. Hence we conclude that $\mathcal{M}(A \cap B)_{c(s_j)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_j)}$ for all $s_j \in \mathrm{Im}\,(A)$.

(1.2) We consider s.

(1.2.1) Suppose that $s \in \mathrm{Im}\,(A \cap B)$. Then, for $i^* \geq i$ smallest such that $s_{i^*+1} \in \mathrm{Im}\,(A \cap B)$, $\mathcal{M}(A \cap B)_{c(s)} = \mathcal{M}((A \cap B)_{s_{i^*+1}}) = \mathcal{M}((A \cap B)_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}} \cap B_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}}) \cup \mathcal{M}(B_{s_{i+1}}) = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B_1) = \mathcal{M}(A)_{c(s)} \cup \mathcal{M}(B)_{c(s)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s)}$.

(1.2.2) Suppose that $s \notin \mathrm{Im}\,(A \cap B)$. Then $s_i < s < s_{i+1}$ and $(A \cap B)_s = (A \cap B)_{s_{i+1}}$. Thus, $\mathcal{M}(A \cap B)_{c(s)} = \mathcal{M}((A \cap B)_s) = \mathcal{M}((A \cap B)_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}} \cap B_{s_{i+1}}) = \mathcal{M}(A_{s_{i+1}}) \cup \mathcal{M}(B_{s_{i+1}}) = \mathcal{M}(A)_{c(s_i)} \cup \mathcal{M}(B_1) = \mathcal{M}(A)_{c(s)} \cup \mathcal{M}(B)_{c(s)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s)}$. Thus for all $r \in \mathrm{Im}\,(A) \cup \mathrm{Im}\,(B)$, $\mathcal{M}(A \cap B)_{c(r)} = (\mathcal{M}(A) \cup \mathcal{M}(A))_{c(r)}$ under the assumption that $s_i \leq s < s_{i+1}$.

(2) Now assume that $s < s_0$. Then $s \in \mathrm{Im}\,(A \cap B)$.

(2.1) Let $s_j \in \mathrm{Im}\,(A)$, $s_j < 1$.

(2.1.1) Suppose that $s_j \notin \mathrm{Im}\,(A \cap B)$. Then $(A \cap B)_{s_j} = (A \cap B)_{s_{j+1}}$ since s_{j+1} is the next possible largest value in $\mathrm{Im}\,(A \cap B)$. Thus $\mathcal{M}(A \cap B)_{c(s_j)} = \mathcal{M}((A \cap B)_{s_j}) = \mathcal{M}((A \cap B)_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}} \cap B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_1) = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s)} = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s_j)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_j)}$.

(2.1.2) Suppose that $s_j \in \mathrm{Im}\,(A \cap B)$. Let j^* be the smallest integer $\geq j$ such that $s_{j^*+1} \in \mathrm{Im}\,(A \cap B)$. Then $(A \cap B)_{s_{j+1}} = (A \cap B)_{s^*_{j+1}}$. Thus $\mathcal{M}(A \cap B)_{c(s_j)} = \mathcal{M}((A \cap B)_{s_{j^*+1}}) = \mathcal{M}((A \cap B)_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}} \cap B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_{s_{j+1}}) = \mathcal{M}(A_{s_{j+1}}) \cup \mathcal{M}(B_1) = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s)} = \mathcal{M}(A)_{c(s_j)} \cup \mathcal{M}(B)_{c(s_j)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s_j)}$.

(2.2) We now consider s. Let $j \geq 0$ be the smallest integer such that $s_j \in \mathrm{Im}\,(A \cap B)$. Then $(A \cap B)_{s_j} = (A \cap B)_{s_0}$. Thus, $\mathcal{M}(A \cap B)_{c(s)} = \mathcal{M}((A \cap B)_{s_j}) = \mathcal{M}((A \cap B)_{s_0}) = \mathcal{M}(A_{s_0} \cap B_{s_0}) = \mathcal{M}(R \cap B_1) = \mathcal{M}(B_1) = \mathcal{M}(B)_{c(s)} = \varnothing \cup \mathcal{M}(B)_{c(s)} = \mathcal{M}(A)_{c(s)} \cup \mathcal{M}(B)_{c(s)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(s)}$. Thus for all $r \in \mathrm{Im}\,(A) \cup \mathrm{Im}\,(B)$, $\mathcal{M}(A \cap B)_{c(r)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(r)}$.

Now suppose that $r \notin \mathrm{Im}\,(A) \cup \mathrm{Im}\,(B)$. Then $r \notin \mathrm{Im}\,(A \cap B)$ and so $c(r) \notin \mathrm{Im}\,\mathcal{M}(A \cap B)$. Hence $\mathcal{M}(A \cap B)_{c(r)} = \mathcal{M}((A \cap B)_r) = \mathcal{M}(A_r \cap B_r) = \mathcal{M}(A_r) \cup \mathcal{M}(B_r) = \mathcal{M}(A)_{c(r)} \cup \mathcal{M}(B)_{c(r)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(r)}$. Thus for all $r \in [0,1]$, $\mathcal{M}(A \cap B)_{c(r)} = (\mathcal{M}(A) \cup \mathcal{M}(B))_{c(r)}$. Let $t \in [0,1] - c([0,1])$. Since $c([0,1]) \supseteq c(\mathrm{Im}\,(A) \cup \mathrm{Im}\,(B))$, $t \notin c(\mathrm{Im}\,(A) \cup \mathrm{Im}\,(B))$ and so $c(t) \notin \mathrm{Im}\,(A) \cup \mathrm{Im}\,(B)$. Hence, $c(t) \notin \mathrm{Im}\,(A \cap B)$. Thus, $\mathcal{M}(A \cap B)_t = \mathcal{M}((A \cap B)_{c(t)}) = \mathcal{M}(A_{c(t)} \cap B_{c(t)}) = \mathcal{M}(A_{c(t)}) \cup \mathcal{M}(B_{c(t)}) = \mathcal{M}(A)_t \cup \mathcal{M}(B)_t = (\mathcal{M}(A) \cup \mathcal{M}(B))_t$. Therefore, $\mathcal{M}(A \cap B) = \mathcal{M}(A) \cup \mathcal{M}(B)$. $\square$

Theorem 1.13. *If A and B are finite-valued fuzzy ideals of R such that $A(0) = B(0) = 1$, then $\mathcal{M}(A \cap B) = \mathcal{M}(A) \cup \mathcal{M}(B)$.*

Proof. Suppose that $\mathrm{Im}\,(B) = \{r_0, r_1, \dots, r_q\}$ where $r_0 < r_1 < \cdots < r_q$. Define the fuzzy subsets $B^{(i)}$ of R by $B^{(i)}(x) = r_i$ if $x \in R - B_{r_{i+1}}$ and $B^{(i)}(x) = 1$ if $x \in B_{r_{i+1}}$ for $i = 0, 1, \dots, q-1$. Then $B = \cap_{i=0}^{q-1} B^{(i)}$ and each $B^{(i)}$ is two-valued. Hence $\mathcal{M}(A \cap B) = \mathcal{M}(A \cap B^{(0)} \cap \cdots \cap B^{(q-1)}) = \mathcal{M}(A) \cup \mathcal{M}(B^{(0)}) \cup \cdots \cup \mathcal{M}(B^{(q-1)}) =$

$\mathcal{M}(A) \cup \mathcal{M}(B^{(0)} \cap \cdots \cap B^{(q-1)}) = \mathcal{M}(A) \cup \mathcal{M}(B)$ by Lemma 1.12. $\square$

As we can now see, the proofs of the following two results are entirely similar to the proofs of the preceding two results. Hence, we omit the proofs for the sake of brevity.

Lemma 1.14. *Suppose that M and N are fuzzy algebraic varieties such that $\mathrm{Im}\,(M) = \{t_0, t_1, \dots, t_n\}$ where $0 = t_0 < t_1 < \cdots < t_n$ and $\mathrm{Im}\,(N) = \{0, t\}$ where $0 < t$. Then $\mathcal{J}(M \cup N) = \mathcal{J}(M) \cap \mathcal{J}(N)$.*

Theorem 1.15. *If M and N are fuzzy algebraic varieties such that $0 \in \mathrm{Im}\,(A) \cap \mathrm{Im}\,(B)$, then $\mathcal{J}(M \cup N) = \mathcal{J}(M) \cap \mathcal{J}(N)$.*

Irreducible fuzzy algebraic varieties.

Definition 2.1. Let M be a fuzzy algebraic variety. Then M is *irreducible* if and only if for all fuzzy algebraic varieties M' and M'' such that $M = M' \cup \mathcal{M}''$ either $M = M'$ or $M = M''$; otherwise, M is called *reducible*.

Theorem 2.2. *Let M be a fuzzy algebraic variety. Then M is irreducible and nonconstant if and only if $\mathrm{Im}\,(M) = \{0, t\}, 0 < t$, and M_t is irreducible.*

Proof. Suppose that M is irreducible. Let $\mathrm{Im}\,(M) = \{t_0, t_1, \dots, t_n\}$ where $t_0 < t_1 < \cdots < t_n$ and suppose that $n \geq 2$. Define the fuzzy subsets U and V of L^k by $U(b) = t_n$ if $b \in M_{t_n}$, $U(b) = (t_{n-1} + t_{n-2})/2$ if $b \in M_{t_{n-1}} - M_{t_n}$, $U(b) = M(b)$ otherwise and $V(b) = (t_n + t_{n-1})/2$ if $b \in M_{t_n}$, $V(b) = M(b)$ otherwise. Then U and V are fuzzy algebraic varieties by Theorem 1.8. Now $M = U \cup V$ and $M \supset U$ and $M \supset V$. Hence, M is not irreducible, a contradiction. Thus, $n = 1$ and $\mathrm{Im}\,(M) = \{t_0, t_1\}$. Suppose that $0 < t_0$. Define the fuzzy subsets U and V of L^k by $U(b) = t_1$ if $b \in M_{t_1}$, $U(b) = 0$ otherwise and $V(b) = (t_1 + t_0)/2$ if $b \in M_{t_1}$, $V(b) = t_0$ otherwise. Then U and V are fuzzy algebraic varieties by Theorem 1.8. Now $M = U \cup V$ and $M \supset U$ and $M \supset V$. Hence, M is not irreducible, a contradiction. Thus $t_0 = 0$

and $\mathrm{Im}\,(M) = \{0, t_1\}$. Suppose that $M_{t_1} = G \cup H$ where G and H are algebraic varieties such that $M_{t_1} \supset G$ and $M_{t_1} \supset H$. Define the fuzzy subsets U and V of L^k by $U(b) = t_1$ if $b \in G$, $U(b) = 0$ otherwise and $V(b) = t_1$ if $b \in H$, $V(b) = 0$ otherwise. Then U and V are fuzzy algebraic varieties and $M = U \cup V$. However, this contradicts the irreduciblity of M since $M \supset U$ and $M \supset V$. Thus, M_{t_1} is irreducible. Conversely, suppose that $\mathrm{Im}\,(M) = \{0, t\}$, $0 < t$, and M_t is irreducible. Suppose that $M = U \cup V$ where U and V are fuzzy algebraic varieties. Then $M_t = (U \cup V)_t = U_t \cup V_t$ and U_t, V_t are algebraic varieties by Theorem 1.8. Since M_t is irreducible, either $U_t = M_t$ or $V_t = M_t$, say $U_t = M_t$. Now t is the largest value M takes on and $M \supseteq U$. Thus for all $b \in U_t$, $U(b) = t = M(b)$. Let $b \in L^k - M_t = L^k - U_t$. Then $0 = M(b) \geq U(b) \geq 0$. Hence, $M = U$. Thus, M is irreducible. □

Theorem 2.3. *Let A be a nonconstant finite-valued fuzzy ideal of R. Then $\mathcal{J}(\mathcal{M}(A))$ is prime if and only if $\mathcal{M}(A)$ is irreducible.*

Proof. Suppose that $\mathcal{J}(\mathcal{M}(A))$ is prime. Then $\mathrm{Im}\,(\mathcal{J}(\mathcal{M}(A))) = \{s, 1\}$ where $s < 1$. Hence $\mathrm{Im}\,(\mathcal{M}(A)) = \{0, c(s)\}$ since $\mathcal{M}(\mathcal{J}(\mathcal{M}(A))) = \mathcal{M}(A)$. Now $\mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_{c(s)} = \mathcal{M}(\mathcal{J}(\mathcal{M}(A))_1)$ by Proposition 1.4. Since $\mathcal{J}(\mathcal{M}(A))_1$ is a prime ideal of R, $\mathcal{M}(A)_{c(s)} = \mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_{c(s)}$ is irreducible. Thus, $\mathcal{M}(A)$ is irreducible by Theorem 2.2. Conversely, suppose that $\mathcal{M}(A)$ is irreducible. Then $\mathrm{Im}\,(\mathcal{M}(A)) = \{0, t\}$ where $0 < t$. Hence $\mathrm{Im}\,(\mathcal{J}(\mathcal{M}(A))) = \{c(t), 1\}$. Now $\mathcal{M}(A)_t = \mathcal{M}(\mathcal{J}(\mathcal{M}(A)))_{c(c(t))} = \mathcal{M}(\mathcal{J}(\mathcal{M}(A))_1)$. Since $\mathcal{M}(A)_t$ is irreducible, $\mathcal{J}(\mathcal{M}(A))_1$ is prime. Thus, $\mathcal{J}(\mathcal{M}(A))$ is prime. □

Theorem 2.4. *Let A be a finite-valued fuzzy ideal of R with $A(0) = 1$. Then $\mathcal{J}(\mathcal{M}(A)) = \sqrt{A}$.*

Proof. For all $s \in [0, 1]$, $(\sqrt{A})_s = \sqrt{A_s} = \mathcal{J}(\mathcal{M}(A_s)) = \mathcal{J}(\mathcal{M}(A))_s$ by Proposition 1.6. □

Corollary 2.5. *Let A and B be finite-valued fuzzy ideals of R such that $A(0) = B(0) = 1$. Then $\mathcal{M}(A) \subset \mathcal{M}(B)$ if and only if $\sqrt{A} \supset \sqrt{B}$.*

Proof. By Theorem 2.4, $\mathcal{M}(A) \subset \mathcal{M}(B)$ if and only if $\sqrt{A} = \mathcal{J}(\mathcal{M}(A)) \supset \mathcal{J}(\mathcal{M}(B)) = \sqrt{B}$ where strict containment is preserved by Proposition 1.7. $\square$

Theorem 2.6. *There exists a one-to-one correspondence between fuzzy algebraic varieties M with $0 \in \mathrm{Im}\,(M)$ and fuzzy radical ideals.*

Proof. Let M be a fuzzy algebraic variety. Then there exists a fuzzy ideal A of R such that $M = \mathcal{M}(A)$. Then $A(0) = 1$. Consider the correspondence $M = \mathcal{M}(A) \mapsto \sqrt{A}$. By Corollary 2.5, $\mathcal{M}(A) = \mathcal{M}(B)$ if and only if $\sqrt{A} = \sqrt{B}$. Hence, the correspondence is single-valued and one-to-one. Given $\sqrt{A}$, $\mathcal{M}(A) \mapsto \sqrt{A}$ and so the correspondence is onto. $\square$

Theorem 2.7. *Every fuzzy algebraic variety M with $0 \in \mathrm{Im}\,(M)$ can be uniquely expressed as the union of a finite number of irreducible algebraic varieties no one of which is contained in the union of the others.*

Proof. Now $M = \mathcal{M}(A)$ for some finite-valued fuzzy ideal A of R with $A(0) = 1$. Now $\mathcal{J}(\mathcal{M}(A)) = \sqrt{A}$ and $\sqrt{A}$ has a unique irredundant primary fuzzy representation $\mathcal{J}(\mathcal{M}(A)) = \sqrt{A} = P_1 \cap \cdots \cap P_r$ where P_i is a fuzzy prime ideal of R, $i = 1, \dots, r$, [**6**]. Thus, by Theorem 1.13, $\mathcal{M}(A) = \mathcal{M}(\mathcal{J}(\mathcal{M}(A))) = \mathcal{M}(P_1 \cap \cdots \cap P_r) = \mathcal{M}(P_1) \cup \cdots \cup \mathcal{M}(P_r)$ where $\mathcal{M}(P_i)$ is irreducible, $i = 1, \dots, r$. If $\mathcal{M}(P_i) = \cup\{\mathcal{M}(P_j) \mid j = 1, \dots, r; j \neq i\}$, then $P_i = \mathcal{J}(\mathcal{M}(P_i)) = \mathcal{J}(\cup\{\mathcal{M}(P_j) \mid j = 1, \dots, r; j \neq i\}) = \cap\{\mathcal{J}(\mathcal{M}(P_j)) \mid j = 1, \dots, r; j \neq i\} = \cap\{P_j \mid j = 1, \dots, r; j \neq i\}$ contradicting the irredundancy of $\sqrt{A} = P_1 \cap \cdots \cap P_r$. Now suppose that $M = M_1 \cup \cdots \cup M_w$ where M_i is an irreducible fuzzy algebraic variety and no M_i is contained in the union of the others, $i = 1, \dots, w$. Then $\mathcal{J}(M_i)$ is a fuzzy prime ideal and the representation $\mathcal{J}(\mathcal{M}(A)) = \sqrt{A} = \mathcal{J}(M_1) \cap \cdots \cap \mathcal{J}(M_w)$ must be irredundant. Thus $r = w$ and $P_i = \mathcal{J}(M_i)$, $i = 1, \dots, r$. Hence, $\mathcal{M}(P_i) = \mathcal{M}(\mathcal{J}(M_i)) = M_i$, $i = 1, \dots, r$. $\square$

Given a finite-valued fuzzy ideal A of R with $A(0) = 1$, let $\sqrt{A} = P_1 \cap \cdots \cap P_r$ be a fuzzy irredundant primary representation. Then the P_i

are the minimal fuzzy prime ideals belonging to A, [**6**, Theorem 3.17]. We thus may obtain $\mathcal{M}(A)$ as the union of fuzzy algebraic varieties of the minimal fuzzy prime ideals among the fuzzy prime ideals belonging to the fuzzy primary ideals in an irredundant primary representation of A.

Theorem 2.8 [**9**]. *Let S be a fuzzy subset of R. Then for all $x \in R$, $\langle S\rangle(x) = \sup\{(\sum_{i=1}^{n}(r_i)_1(x_i)_{t_i})(x) \mid r_i,\ x_i \in R,\ t_i \leq S(x_i),$ $i = 1, \dots, n;\ n \in N\}$ where N denotes the positive integers.*

Example 2.9. Let $R = F[x,y,z]$ where F is the field of complex numbers and x,y,z are algebraically independent indeterminants over F. Define the fuzzy subset A of R by $A(0) = 1$, $A(f) = 1/2$ if $f \in \langle x^2z\rangle - \langle 0\rangle$, $A(f) = 1/4$ if $f \in \langle x^2+y^2-1, x^2z\rangle - \langle x^2z\rangle$, and $A(f) = 0$ if $f \in R - \langle x^2+y^2-1, x^2z\rangle$. Then A is a fuzzy ideal of R. Now $\sqrt{A}$ is such that $\sqrt{A}(0) = 1$, $\sqrt{A}(f) = 1/2$ if $f \in \langle xz\rangle - \langle 0\rangle$, $\sqrt{A}(f) = 1/4$ if $f \in \langle x^2+y^2-1, xz\rangle - \langle xz\rangle$, and $\sqrt{A}(f) = 0$ if $f \in R - \langle x^2+y^2-1, xz\rangle$. Hence,

$$\begin{aligned} A_0 &= R, & \sqrt{A_0} &= R \\ A_{1/4} &= \langle x^2+y^2-1, x^2z\rangle, & \sqrt{A_{1/4}} &= \langle x^2+y^2-1, xz\rangle \\ A_{1/2} &= \langle x^2z\rangle, & \sqrt{A_{1/2}} &= \langle xz\rangle \\ A_1 &= \langle 0\rangle, & \sqrt{A_1} &= \langle 0\rangle. \end{aligned}$$

Since $F^k = \mathcal{M}(\langle 0\rangle)$,

$$\mathcal{M}(A)(b) = \begin{cases} c(1/2) & \text{if } b \in \mathcal{M}(\langle 0\rangle) - \mathcal{M}(\langle xz\rangle), \\ c(1/4) & \text{if } b \in \mathcal{M}(\langle xz\rangle) - \mathcal{M}(\langle x^2+y^2-1, xz\rangle), \\ c(0) = 1 & \text{if } b \in \mathcal{M}(\langle x^2+y^2-1, xz\rangle). \end{cases}$$

Consider the fuzzy subsets W, X, Y of R defined by $W(f) = 1$ if $f \in \langle x^2+y^2-1, x^2z\rangle$, $W(f) = 0$ otherwise; $X(f) = 1$ if $f \in \langle x^2z\rangle$, $X(f) = 1/4$ otherwise; $Y(f) = 1$ if $f \in \langle 0\rangle$, $Y(f) = 1/2$ otherwise. Then W, X, Y are fuzzy ideals of R and $A = W \cap X \cap Y$. Define the fuzzy subsets $Q^{(i)}$ of R, $i = 1, \dots, 6$ by $Q^{(1)}(f) = 1$ if $f \in \langle x^2, y-1\rangle$, $Q^{(1)}(f) = 0$ otherwise; $Q^{(2)}(f) = 1$ if $f \in \langle x^2, y+1\rangle$, $Q^{(2)}(f) = 0$

otherwise; $Q^{(3)}(f) = 1$ if $f \in \langle x^2 + y^2 - 1, z\rangle$, $Q^{(3)}(f) = 0$ otherwise; $Q^{(4)}(f) = 1$ if $f \in \langle x^2\rangle$, $Q^{(4)} = 1/4$ otherwise; $Q^{(5)}(f) = 1$ if $f \in \langle z\rangle$, $Q^{(5)}(f) = 1/4$ otherwise, $Q^{(6)}(f) = 1$ if $f \in \langle 0\rangle$, $Q^{(6)}(f) = 1/2$ otherwise. Then $Q^{(i)}$ is a fuzzy ideal of R, $i = 1, \dots, 6$, such that $W = Q^{(1)} \cap Q^{(2)} \cap Q^{(3)}$ since $\langle x^2 + y^2 - 1, x^2 z\rangle = \langle x^2, y - 1\rangle \cap \langle x^2, y + 1\rangle \cap \langle x^2 + y^2 - 1, z\rangle$, $X = Q^{(4)} \cap Q^{(5)}$ since $\langle x^2 z\rangle = \langle x^2\rangle \cap \langle z\rangle$, $Y = Q^{(6)}$. Thus $A = \cap_{i=1}^{6} Q^{(i)}$ and in fact this is an irredundant fuzzy primary representation of A. Now $\sqrt{Q}^{(1)}(f) = 1$ if $f \in \langle x, y - 1\rangle$, $\sqrt{Q}^{(1)}(f) = 0$ otherwise; $\sqrt{Q}^{(2)}(f) = 1$ if $f \in \langle x, y - 1\rangle$, $\sqrt{Q}^{(2)}(f) = 0$ otherwise; $\sqrt{Q}^{(3)} = Q^{(3)}$, $\sqrt{Q}^{(4)}(f) = 1$ if $f \in \langle x\rangle$, $\sqrt{Q}^{(4)}(f) = 0$ otherwise; $\sqrt{Q}^{(5)} = Q^{(5)}$, $\sqrt{Q}^{(6)} = Q^{(6)}$. Hence $\sqrt{A} = \cap_{i=1}^{6} P^{(i)}$ where $P^{(i)} = \sqrt{Q}^{(i)}$ is a fuzzy prime ideal of R, $i = 1, \dots, 6$. We have the following fuzzy algebraic varieties:

$$\begin{cases} \mathcal{M}(P^{(1)})(b) = 1 & \text{if } b \in \mathcal{M}(\langle x, y-1\rangle), \\ & \mathcal{M}(P^{(1)})(b) = 0 \text{ otherwise;} \\ \mathcal{M}(P^{(2)})(b) = 1 & \text{if } b \in \mathcal{M}(\langle x, y+1\rangle), \\ & \mathcal{M}(P^{(2)}(b) = 0 \text{ otherwise;} \\ \mathcal{M}(P^{(3)})(b) = 1, & \text{if } b \in \mathcal{M}(\langle x^2 + y^2 - 1, z\rangle), \\ & \mathcal{M}(P^{(3)})(b) = 0 \text{ otherwise;} \\ \mathcal{M}(P^{(4)})(b) = c(1/4) & \text{if } b \in \mathcal{M}(\langle x\rangle), \quad \mathcal{M}(P^{(4)}(b) = 0 \text{ otherwise;} \\ \mathcal{M}(P^{(5)})(b) = c(1/4) & \text{if } b \in \mathcal{M}(\langle z\rangle), \quad \mathcal{M}(P^{(5)})(b) = 0 \text{ otherwise;} \\ \mathcal{M}(P^{(6)})(b) = c(1/2), & \forall b \in L^k. \end{cases}$$

Then $\mathcal{M}(A) = \cup_{i=1}^{6} \mathcal{M}(P^{(i)})$ and in fact $\mathcal{M}(P^{(i)})$ is irreducible and no $\mathcal{M}(P^{(i)})$ is contained in the union of the others, $i = 1, \dots, 6$.

Consider the nonlinear system of equations of fuzzy singletons,

$$\begin{aligned} (x_s)^2 + (y_t)^2 - 1_{1/4} &= 0_{1/4}, \\ (x_s)^2 z_u &= 0_{1/2}. \end{aligned}$$

Then a solution is given by $t \geq 1/4$ and $\min\{s, u\} = 1/2$ and the solution of $x^2 + y^2 - 1 = 0$ and $x^2 z = 0$. Note also that $A = \langle (x^2 + y^2 - 1)_{1/4},\ (x^2 z)_{1/2}\rangle$ by Theorem 2.8. If we let $c(0) = 1$, $c(1/4) = 1/2$, $c(1/2) = 1/4$, $c(1) = 0$, then the above representation of $\mathcal{M}(A)$ seems to better represent the solution of the above nonlinear system of equations of fuzzy singletons. The $\mathcal{M}(P^{(i)})$ for $i = 1, 2, 3$, yield the crisp part of the solution while the $\mathcal{M}(P^{(i)})$ for $i = 4, 5, 6$ yield the fuzzy part.

Proposition 2.10. *Let S be a fuzzy subset of R. Then $\langle S\rangle^* = \langle S^*\rangle$.*

Proof. $\langle S\rangle \supseteq S$. Thus $\langle S\rangle^* \supseteq S^*$ and so $\langle S\rangle^* \supseteq \langle S^*\rangle$. Let $f \in \langle S\rangle^*$. Then $\langle S\rangle(f) > 0$ and so, by Theorem 2.8, $f = \sum_{i=1}^n r_i f_i$ for $r_i \in R$ and $f_i \in S^*$. Thus $f \in \langle S^*\rangle$. Hence, $\langle S\rangle^* \subseteq \langle S^*\rangle$. $\square$

Corollary 2.11. *Let S be a fuzzy subset of R. Then $b \in L^k$ is a zero of S^* if and only if b is a zero of $\langle S\rangle^*$.*

Proof. $f(b) = 0$ for all $f \in S^*$ if and only if $f(b) = 0$ for all $f \in \langle S^*\rangle$ (from the crisp case) $= \langle S\rangle^*$. $\square$

Let W be a finite subset of S^* such that $\langle W\rangle = \langle S^*\rangle$ where S is a fuzzy subset of R. Define the fuzzy subset T of R by $T(b) = S(b)$ if $b \in W$ and $T(b) = 0$ otherwise. Then $T^* = W$. Hence, $\langle T\rangle^* = \langle T^*\rangle = \langle S^*\rangle = \langle S\rangle^*$. Thus b is a zero of $\langle S\rangle^*$ if and only if b is a zero of $\langle T\rangle^*$.

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