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ALMOST CR STRUCTURES, f-STRUCTURES, ALMOST PRODUCT STRUCTURES AND ASSOCIATED CONNECTIONS

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ABSTRACT. A nondegenerate annihilating frame for a partially integrable almost CR structure determines an affine connection; a single nondegenerate annihilating form determines a pair of partial connections.

0. Introduction. In [**5**] and [**8**], Tanaka and Webster independently showed that a nondegenerate integrable pseudo-hermitian structure determines a canonical affine connection. A number of authors have used this connection to study analytical problems on codimension 1 CR manifolds. In [**6**], by considering contact Riemannian structures, Tanno proved that the integrability condition can be relaxed to that of partial integrability provided that the pseudo-hermitian structure is strongly pseudo-convex. It turns out (see the Remark following Proposition 7.1) that the assumption of strong pseudo-convexity is not necessary for the relaxation of the integrability condition—nondegeneracy suffices.

In fact, we show that a partially integrable almost CR structure of arbitrary codimension, together with a nondegenerate, globally defined frame for the annihilator of the holomorphic tangent bundle, determines a canonical affine connection (see Theorem 7.1). Moreover, if the hypothesis of a nondegenerate annihilating frame is weakened to that of a single, nondegenerate annihilating form, then although we can no longer define a canonical affine connection, we can define two canonical partial connections, including a connection on the holomorphic tangent bundle (see Theorem 7.2). These connections should be useful in the study of almost CR submanifolds of partially integrable pseudo-hermitian manifolds.

These results all follow from a more general result on affine connections that parallelize an almost product structure (see Theorem 6.1).

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The idea of relating almost CR structures to almost product structures is motivated by the intermediary position occupied by f-structures. In his recent book [**1**], Bejancu has developed some of the relations between f-structures and almost CR structures; as far back as 1961, Walker [**7**] observed that f-structures (he did not use this term) constitute a special case of almost product structures. However, it does not seem to be a matter of common knowledge that these three notions are interrelated. Therefore, although the primary purpose of this paper is to present the results formulated in Theorems 2.1, 6.1, 7.1, and 7.2, a secondary purpose is to collect in compact form some basic facts about almost CR structures, f-structures, almost product structures, and their linkage.

In Section 1 we define the basic objects in the theory of almost CR structures. In Section 2 we introduce annihilating forms and frames, use them to define two higher-codimensional analogues of pseudohermitian structures, and give several concrete examples. In Section 3 we consider an auxiliary geometric object, adumbrated in Section 2, which is equivalent to an f-structure. Because of this equivalence, we recall the definition of f-structures and briefly discuss their relations to almost CR structures. In Section 4 we introduce almost product structures, concentrating on a subclass that includes f-structures as a special case. In Section 5 we introduce a notion of a partial connection in a vector bundle. In Section 6 we use partial connections to study affine connections associated to almost product structures. Finally, in Section 7 we consider connections related to almost CR structures.

Throughout this paper, M denotes a fixed C^∞ manifold of dimension m. All bundles are vector bundles (real or complex) over M. If F is a bundle, then $\Gamma(F)$ is the module of global C^∞ sections of F.

The complexification of a real bundle E is denoted by $\mathbf{C}E$; at times E is identified with the real subbundle of $\mathbf{C}E$ that comprises all real (i.e., self-conjugate) vectors. In particular, TM is often considered as a subbundle of $\mathbf{C}TM$. Also, a real bundle map is frequently not distinguished from its complexification.

A bilinear bundle map $b : E_1 \times E_2 \to E_3$ is *nondegenerate* if the associated bundle map $\hat{b} : E_1 \to \operatorname{Hom}(E_2, E_3)$, determined by the requirement that $\hat{b}(X)(Y) = b(X,Y)$ for all $X \in \Gamma(E_1)$, and $Y \in \Gamma(E_2)$, is an isomorphism. On occasion we shall think of b as a section of the bundle Hom $(E_1, E_2; E_3)$.

For simplicity, two standard notational conventions are employed. The first is summation convention: unless otherwise indicated, repeated indices, one raised and the other lowered, are summed over. The second is a conjugation convention: complex conjugation may be carried out in the indices. For example, the conjugate of a^i_j may be written as $\bar{a}^i_j$ or $a^{\bar{\imath}}_{\bar{\jmath}}$ (or even $a^i_{\bar{\jmath}}$ or $a^{\bar{\imath}}_j$ if one of the indices is associated to a real object). Finally, the symbol I_k denotes the index set $\{1, 2, \dots, k\}$; I_0 denotes the empty set.

1. Almost CR structures. We begin by recalling some elementary facts about almost complex structures. An almost complex structure on M can be given in a number of ways:

(1) a decomposition $\mathbf{C}TM = T_{1,0} \oplus T_{0,1}$ with $T_{0,1} = \overline{T}_{1,0}$;

(2) a decomposition $\mathbf{C}T^*M = T^{1,0} \oplus T^{0,1}$ with $T^{0,1} = \overline{T}^{1,0}$;

(3) an endomorphism $J : TM \to TM$ with $J^2 = -1$;

(4) an endomorphism $J^* : T^*M \to T^*M$ with $(J^*)^2 = -1$.

The decompositions in (1) and (2) are related by duality, as are the endomorphisms in (3) and (4). The endomorphism in (3) extends by complex linearity to an endomorphism of $\mathbf{C}TM$ that acts on $T_{1,0}$ (respectively, $T_{0,1}$) as multiplication by i (respectively, $-i$).

An almost complex structure is *integrable* if the bundles $T_{1,0}, T_{0,1}$, $T^{1,0}$, and $T^{0,1}$ are involutive, or equivalently, if for all $X, Y \in \Gamma(TM)$

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

A *complex structure* is an integrable almost complex structure.

Generalizing (1) and (3) leads to the definition of an *almost CR structure* as a complex subbundle $\mathcal{H} \subset \mathbf{C}TM$ such that $\mathcal{H} \cap \overline{\mathcal{H}} = 0$, or equivalently, as a pair (H, J), where H is a subbundle of TM and $J : H \to H$ is an endomorphism with the property that $J^2 = -1$. The bundle $\mathcal{H}$ and the pair (H, J) are called, respectively, the complex and real forms of the almost CR structure. They are related as follows: $\mathcal{H} \oplus \overline{\mathcal{H}} = \mathbf{C}H$, and the complex linear extension of J to $\mathbf{C}H$ acts on $\mathcal{H}$ (respectively, $\overline{\mathcal{H}}$) as multiplication by i (respectively, $-i$).

The *annihilator* of $\mathcal{H}$ is the subbundle $H^a \subset T^*M$ determined by the following condition:

$$\theta \in H^a \Longleftrightarrow X \rfloor \theta = 0 \qquad \text{for all } X \in H.$$

The *dimension* (respectively, *codimension*) of $\mathcal{H}$ is the complex (respectively, real) fiber dimension of $\mathcal{H}$ (respectively, H^a). Clearly, a codimension 0 almost CR structure is an almost complex structure, and vice versa.

$\mathcal{H}$ is *partially integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \oplus \overline{\mathcal{H}}$, or equivalently, if for all $X, Y \in \Gamma(H)$

$$[X, Y] - [JX, JY] \in \Gamma(H).$$

$\mathcal{H}$ is *integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$, or equivalently, if it is partially integrable and for all $X, Y \in \Gamma(H)$

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

A CR *structure* is an integrable almost CR structure.

The *Levi form* of $\mathcal{H}$ is the map

$$\mathcal{L} : \mathcal{H} \times \mathcal{H} \to \mathbf{C}TM/\mathbf{C}H$$

determined by the requirement that

$$\mathcal{L}(X, Y) = i\pi[X, \overline{Y}], \qquad \text{for all } X, Y \in \Gamma(\mathcal{H}),$$

where $\pi : \mathbf{C}TM \to \mathbf{C}TM/\mathbf{C}H$ is the natural projection. It is easy to verify that $\mathcal{L}$ is hermitian symmetric, i.e., that $\overline{\mathcal{L}(X, Y)} = \mathcal{L}(Y, X)$.

Throughout this paper, (H, J) and $\mathcal{H}$ will denote the real and complex forms of an almost CR structure on M of dimension n and codimension c with annihilator H^a and Levi form $\mathcal{L}$.

2. Generalizations of pseudo-hermitian structures. An *annihilating form* θ is a nowhere-zero section of H^a; it is *nondegenerate* (respectively, *definite*) if $\theta \circ \mathcal{L}_p$ is a nondegenerate (respectively, definite) hermitian form on $\mathcal{H}_p$ for each $p \in M$. If $c = 1$, then the pair $(\mathcal{H}, \theta)$ is called a *pseudo-hermitian structure*. It is *nondegenerate* (respectively, *strongly pseudo-convex*) if θ is nondegenerate (respectively, definite);

it is *partially integrable* (respectively, *integrable*) if $\mathcal{H}$ is partially integrable (respectively, integrable). If $c > 1$, then the pair $(\mathcal{H}, \theta)$ is an obvious higher-codimensional analogue of a pseudo-hermitian structure. However, since if $c = 1$ an annihilating form is also a frame for H^a, there is an equally obvious analogue, namely a pair $(\mathcal{H}, \varphi)$ where φ is an *annihilating frame*, i.e., a frame for H^a. Each nonempty subset $S \subset I_c$ determines a class of annihilating frames: $\varphi = (\varphi^1, \varphi^2, \dots, \varphi^c)$ is *nondegenerate of type* S if the annihilating form φ^α is nondegenerate for each $\alpha \in S$. Clearly, there are topological obstructions to the existence of annihilating forms and frames. However, in general, by restricting to a sufficiently small generic neighborhood in M, one can find both kinds of objects.

Example 1. Let $(\hat{H}, \hat{J})$ be an almost CR structure on a manifold $\hat{N}$, and let $\tilde{N}$ be a codimension k submanifold of $\hat{N}$. Let $\tilde{H} = (T\tilde{N} \cap \hat{H}) \cap \hat{J}(T\tilde{N} \cap \hat{H})$, and denote the restriction of $\hat{J}$ to $\tilde{H}$ by $\tilde{J}$. Suppose that $\tilde{H}$ is a subbundle of $T\tilde{N}$. Then $(\tilde{H}, \tilde{J})$ is easily seen to be an almost CR structure on $\tilde{N}$, and $\tilde{N}$ is called an *almost CR submanifold* of $\hat{N}$. Clearly, $(\tilde{H}, \tilde{J})$ inherits any integrability property enjoyed by $(\hat{H}, \hat{J})$.

(a) Suppose that $\hat{\theta}$ is a definite annihilating form for $(\hat{H}, \hat{J})$. By restriction, $\hat{\theta}$ induces a nondegenerate (indeed definite) annihilating form $\tilde{\theta}$ for $(\tilde{H}, \tilde{J})$. In particular, an almost CR submanifold of a strongly pseudo-convex pseudo-hermitian manifold always admits a nondegenerate annihilating form.

(b) Suppose that the codimension of $(\hat{H}, \hat{J})$ is 0, i.e., that $\hat{J}$ is an almost complex structure, and that $\tilde{N}$ is the zero-variety of a submersion $f : \hat{N} \to \mathbf{R}^k$. Let $\varphi = \hat{J}^* df$ and note that, for any $X \in T\hat{N}|_{\tilde{N}}$,

$$\text{(i)} \qquad X \in T\tilde{N} \Longleftrightarrow X \rfloor df = 0$$

and

$$\text{(ii)} \qquad X \in \hat{J}(T\tilde{N}) \Longleftrightarrow \hat{J}X \in T\tilde{N} \Longleftrightarrow \hat{J}X \rfloor df = 0 \Longleftrightarrow X \rfloor \varphi = 0.$$

Therefore,

$$\text{(iii)} \qquad X \in \tilde{H} \Longleftrightarrow X \rfloor df = 0 \quad \text{and} \quad X \rfloor \varphi = 0.$$

Since by assumption $\tilde{H}$ is a subbundle and df has rank k, it follows from (iii) that $\tilde{\varphi}$, the restriction of φ to $T\tilde{N}$, has locally constant rank. If this rank is k (the generic case), then $(\tilde{H}, \tilde{J})$ is a codimension k almost CR structure and $\tilde{\varphi}$ is an annihilating frame.

Returning to more general considerations, we show that if $\mathcal{H}$ is partially integrable, then a nondegenerate annihilating frame determines additional canonical objects.

Theorem 1. *Let S be a nonempty subset of I_c, and let $\eta : I_c \to S$ be a map. If $\mathcal{H}$ is partially integrable and φ is an annihilating frame, nondegenerate of type S, then there is a canonical subbundle $C \subset TM$ such that $TM = H \oplus C$. Moreover, there exists a unique frame $e = (e_1, e_2, \dots, e_c)$ of C with the following properties*:

(a) $$e_\alpha \rfloor \varphi^\beta = \delta_\alpha^\beta \qquad \text{for all } \alpha, \beta \in I_c;$$

(b) $$e_\alpha \rfloor d\varphi^{\eta(\alpha)} \in \Gamma(H^a), \qquad \text{for all } \alpha \in I_c, \quad (\text{summation convention suspended}).$$

Proof. Let U be an open subset of M on which there is given a $\mathbf{C}^n$-valued 1-form ω, such that

(i) the components of $\omega, \bar{\omega}$ and φ frame $\mathbf{C}T^*U$, and for all $X \in \Gamma(\mathbf{C}TU)$

(ii) $X \in \Gamma(\mathcal{H}|_U) \Longleftrightarrow X \rfloor \bar{\omega} = 0$ and $X \rfloor \varphi = 0$.

By duality, there exist unique complex vector fields $W_1, W_2, \dots, W_n$ and real vector fields $Z_1, Z_2, \dots, Z_c$ on U with the following properties:

(iii) $W = (W_1, W_2, \dots, W_n)$ frames $\mathcal{H}|_U$;

(iv) $(W_1, W_2, \dots, W_n, W_{\bar{1}}, W_{\bar{2}}, \dots, W_{\bar{n}}, Z_1, Z_2, \dots, Z_c)$ frames $\mathbf{C}TU$;

(v) $W_i \rfloor \omega^j = \delta_i^j$ for all $i, j \in I_n$;

(vi) $Z_\alpha \rfloor \varphi^\beta = \delta_\alpha^\beta$ and $Z_\alpha \rfloor \omega^j = 0$ for all $\alpha, \beta \in I_c$ and $j \in I_n$.

Since $\mathcal{H}$ is partially integrable and $\varphi = \bar{\varphi}$, there also exist complex functions $h_{j\bar{k}}^\alpha$ and $a_{j\beta}^\alpha$ and real functions $c_{\beta\gamma}^\alpha$, all defined on U, with the following properties:

(vii) $d\varphi^\alpha = ih^\alpha_{j\bar k}\omega^j \wedge \omega^{\bar k} + a^\alpha_{j\beta}\omega^j \wedge \varphi^\beta + a^\alpha_{\bar j\beta}\omega^{\bar j} \wedge \varphi^\beta + c^\alpha_{\beta\gamma}\varphi^\beta \wedge \varphi^\gamma$ for all $\alpha \in I_c$;

(viii) $\bar h^\alpha_{j\bar k} = h^\alpha_{k\bar j}$ for all $\alpha \in I_c$ and $j, k \in I_n$;

(ix) $c^\alpha_{\beta\gamma} = -c^\alpha_{\gamma\beta}$ for all $\alpha, \beta, \gamma \in I_c$.

Note that for all $\alpha \in I_c$ and $j, k \in I_n$

(x) $\varphi^\alpha \circ \mathcal{L}(W_j, W_k) = i\varphi^\alpha([W_j, W_{\bar k}]) = -id\varphi^\alpha(W_j, W_{\bar k})$.

It follows from (vii) and (x) that, for all $\alpha \in I_c$ and $j, k \in I_n$,

$$\varphi^\alpha \circ \mathcal{L}(W_j, W_k) = h^\alpha_{j\bar k}.$$

Therefore, since φ is nondegenerate of type S, for each $\alpha \in S$, there exist complex functions $h^{\alpha,\bar t r}$ defined on U such that, for all $k, t \in I_n$,

$$\text{(xi)} \qquad h^{\alpha,\bar t r} h^\alpha_{r\bar k} = \delta^{\bar t}_{\bar k} \quad \text{and} \quad h^\alpha_{k\bar r} h^{\alpha,\bar r t} = \delta^t_k.$$

A c-tuple $\tilde Z = (\tilde Z_1, \tilde Z_2, \dots, \tilde Z_c)$ of real vector fields on U corresponds to a pair of maps

$$v : U \to \operatorname{Hom}(\mathbf{C}^n, \mathbf{R}^c) \quad \text{and} \quad B : U \to \operatorname{Hom}(\mathbf{R}^c, \mathbf{R}^c)$$

by the formulae

$$\text{(xii)} \qquad \tilde Z_\beta = v^j_\beta W_j + v^{\bar j}_\beta W_{\bar j} + B^\gamma_\beta Z_\gamma \qquad \text{for all } \beta \in I_c.$$

A routine computation shows that, for all $\alpha, \beta \in I_c$,

$$\text{(xiii)} \qquad \begin{gathered} \tilde Z_\beta \rfloor \varphi^\alpha = \delta^\alpha_\beta \quad \text{and} \quad \tilde Z_\alpha \rfloor d\varphi^{\eta(\alpha)} \in \Gamma(H^a|_U) \\ \text{(summation convention suspended)} \end{gathered}$$

if and only if, for all $\alpha, \beta \in I_c$ and $j \in I_n$,

$$\text{(xiv)} \qquad B^\alpha_\beta = \delta^\alpha_\beta \quad \text{and} \quad v^j_\beta = -ih^{\eta(\alpha),\bar k j} a^{\eta(\alpha)}_{\bar k\beta}.$$

Let $\mathcal{U}$ be an open cover of M such that on each $V \in \mathcal{U}$ there is given a $\mathbf{C}^n$-valued 1-form ω^V that satisfies (i) and (ii). (It is easy to verify that such a cover always exists.) It follows from (xiii) and (xiv) that for each $V \in \mathcal{U}$ there exist unique vector fields $\tilde{Z}_1^V, \tilde{Z}_2^V, \dots, \tilde{Z}_c^V$ on V such that, for all $\alpha, \beta \in I_c$,

$$\tilde{Z}_\beta^V \rfloor \varphi^\alpha = \delta_\beta^\alpha \quad \text{and} \quad \tilde{Z}_\beta^V \rfloor d\varphi^{\eta(\beta)} \in \Gamma(H^a|_V)$$
$$\text{(summation convention suspended)}.$$

For each $\beta \in I_c$, let e_β be the unique vector field on M that restricts to $\tilde{Z}_\beta^V$ on each $V \in \mathcal{U}$. It is easy to verify that the definition of e_β is canonical. Finally, let $C \subset TM$ be the subbundle framed by $e = (e_1, e_2, \dots, e_c)$. □

Corollary 1. *If $(\mathcal{H}, \theta)$ is a nondegenerate partially integrable pseudo-hermitian structure on M, then there exists a unique vector field e such that $e \rfloor \theta = 1$ and $e \rfloor d\theta = 0$.*

Proof. By Theorem 1, there exists a canonical subbundle $C \subset TM$ such that $TM = H \oplus C$ and a unique vector field e that frames C and that satisfies the following conditions:

(i) $e \rfloor \theta = 1$;

(ii) $e \rfloor d\theta \in \Gamma(H^a)$.

By (ii), $e \rfloor d\theta$ vanishes on H; since C is framed by e and $d\theta$ is skew-symmetric, $e \rfloor d\theta$ vanishes on C. But $TM = H \oplus C$, so $e \rfloor d\theta = 0$. □

Remark. Clearly, the nondegeneracy of $(\mathcal{H}, \theta)$ implies that θ is a contact form. It is well-known that if μ is any contact form on M, then there exists a unique vector field ξ, called the *canonical transverse vector field of* μ, such that $\xi \rfloor \mu = 1$ and $\xi \rfloor d\mu = 0$. Therefore, the hypothesis of partial integrability can be dropped from Corollary 1.

Although Theorem 1 deals with annihilating frames, it can be used to obtain results about annihilating forms.

Corollary 2. *Suppose that $\mathcal{H}$ is partially integrable and that θ is a nondegenerate annihilating form. There exists a canonical subbundle $C \subset TM$ such that $TM = H \oplus C$.*

Proof. Given $p \in M$, let U be a neighborhood of p, and let

$\varphi = (\varphi^1, \varphi^2, \dots, \varphi^c)$ and $\tilde{\varphi} = (\tilde{\varphi}^1, \tilde{\varphi}^2, \dots, \tilde{\varphi}^c)$ be annihilating frames on U with $\varphi^1 = \tilde{\varphi}^1 = \theta|_U$. Clearly, φ and $\tilde{\varphi}$ are nondegenerate of type $\{1\}$. There exists a smooth map $P : U \to GL(c, \mathbf{R})$ such that

$$\text{(i)} \quad \tilde{\varphi}^\alpha = P^\alpha_\beta \varphi^\beta \qquad \text{for all } \alpha \in I_c \quad \text{and} \quad P^1_j = \delta^1_j \qquad \text{for all } j \in I_n.$$

Let $e = (e_1, e_2, \dots, e_c)$ and $\tilde{e} = (\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_c)$ be the vector fields associated with φ and $\tilde{\varphi}$ by Theorem 1. Since $\tilde{e}_\alpha \rfloor d\varphi^1 \in \Gamma(H^a)$ for all $\alpha \in I_c$,

$$\text{(ii)} \qquad P^\alpha_\beta \tilde{e}_\alpha \rfloor d\varphi^1 \in \Gamma(H^a)$$

for all $\beta \in I_c$. Moreover, for all $\beta, \gamma \in I_c$

$$\text{(iii)} \qquad P^\alpha_\beta \tilde{e}_\alpha \rfloor \varphi^\gamma = \delta^\gamma_\beta.$$

By the uniqueness assertion of Theorem 1, it follows from (ii) and (iii) that $P^\alpha_\beta \tilde{e}_\alpha = e_\beta$ for all $\beta \in I_c$. Hence, $e(p)$ and $\tilde{e}(p)$ frame the same subspace $C_p \subset T_pM$. Let $C = \cup_{p \in M} C_p$. It is clear that C is a canonical subbundle of TM and that $TM = H \oplus C$. □

3. Complemented almost CR structures and f-structures. Motivated by Theorem 2.1 and Corollary 2.2, we define a *complement* for $\mathcal{H}$ to be a subbundle $C \subset TM$ such that $TM = H \oplus C$, and a *framed complement* to be a pair (C, e), where C is a complement and e is a frame for C.

Theorem 2.1 associates a canonical framed complement to a nondegenerate annihilating frame. In the other direction, a framed complement (C, e) determines an annihilating frame φ by the following conditions:

$$X \rfloor \varphi = 0 \qquad \text{for all } X \in H;$$
$$e_\beta \rfloor \varphi^\alpha = \delta^\alpha_\beta \qquad \text{for all } \alpha, \beta \in I_c.$$

Note that φ may well be degenerate.

Corollary 2.2 associates a canonical complement to a nondegenerate annihilating form. Combining this result with Example 2.1, we see that the almost CR structure on an almost CR submanifold of a strongly pseudo-convex partially integrable pseudo-hermitian manifold has a canonical complement. Other examples are easily obtained.

Example 1. Let g be a metric on M. Then $H^\perp$, the orthogonal complement of H with respect to g, is a complement for $\mathcal{H}$.

Example 2. Let $\hat{J}$ be an almost complex structure on a manifold $\hat{N}$, let $\tilde{N} \subset \hat{N}$ be a codimension k almost CR submanifold with a codimension k almost CR structure $(\tilde{H}, \tilde{J})$, and let $\tilde{g}$ be a metric on $\tilde{N}$.

(a) If, as in (b) of Example 2.1, $\tilde{N}$ is given by a global defining map, then the associated annihilating frame $\tilde{\varphi}$ determines a frame $\tilde{e}$ for the complement $\tilde{H}^\perp$ as follows: for all $\alpha \in I_c$ and $X \in T\tilde{N}$

$$\tilde{g}(X, \tilde{e}_\alpha) = X \rfloor \tilde{\varphi}^\alpha.$$

Note that unlike the situation in Theorem 2.1, here the frame $\tilde{\varphi}$ may be degenerate.

(b) Suppose that $\tilde{g}$ is the restriction of a hermitian metric $\hat{g}$ on $\hat{N}$. It is easy to verify that $\hat{J}$ induces an isomorphism of $\tilde{H}^\perp$ with the normal bundle of $\tilde{N}$. Also, it can be shown (see [**1**] or [**12**]) that if $\hat{J}$ is integrable and $\hat{g}$ is Kaehler then $\tilde{H}^\perp$ is involutive. The geometry of CR submanifolds of hermitian and Kaehler manifolds has been investigated recently in a number of papers and at least two books (see [**1**] and [**12**] for references).

It should be noted that some of the objects defined in this section have been studied since the early 1960's under different names.

For instance, an *f-structure* on M is an endomorphism $f : TM \to TM$ that satisfies the identity $f^3 + f = 0$. Stong [**4**] proved that the rank of such an endomorphism is locally constant. For convenience, suppose that M is connected and that f has rank r. Then the kernel of f is a subbundle $K \subset TM$ of fiber dimension $m - r$, and the image of f is a subbundle $R \subset TM$ of fiber dimension r. By restriction, f determines an endomorphism $B : R \to R$, which satisfies the identity $B^2 = -1$. Indeed, if $X \in R$ then $X = fY$ for some $Y \in TM$, so $B^2 X = f^3 Y = -fY = -X$. Hence (R, B) is a codimension $(m - r)$ almost CR structure with complement K. Conversely, if C is a complement for (H, J), then there exists a unique endomorphism $g : TM \to TM$ such that $gX = JX$ for all $X \in H$ and $gX = 0$ for all $X \in C$; clearly, g is an f-structure.

Thus, a complemented almost CR structure is equivalent to an f-structure. Similarly, an almost CR structure with framed complement is equivalent to an *f-structure with complemented frames*; see [**1**] for the appropriate definition and a discussion of this equivalence.

4. Almost product structures. Let r be a positive integer. An *r-fold almost product* structure on M is an r-tuple $\mathcal{E} = (E_1, E_2, \ldots, E_r)$ of subbundles of $\mathbf{C}TM$ such that $\mathbf{C}TM = \oplus_{i\in I_r} E_i$. Each pair of nonnegative integers (d, d') with $2d + d' = r$ determines a class of almost product structures. Namely, $\mathcal{E}$ is of *type* (d, d'), if it satisfies the following conditions:

$$\begin{aligned} \bar{E}_i &= E_{d+i} \qquad \text{for all } i \in I_d; \\ \bar{E}_{2d+i} &= E_{2d+i} \qquad \text{for all } i \in I_{d'}. \end{aligned}$$

If $\mathcal{E}$ is of type (d, d'), then for each $i \in I_{d'}$ let $C_i \subset TM$ be the real part of E_{2d+i}. It is easy to verify that $\oplus_{i\in I_d} E_i$ is an almost CR structure with complement $\oplus_{i\in I_{d'}} C_i$. Conversely, if C is a complement for the almost CR structure $\mathcal{H}$, then $\mathcal{E} = (\mathcal{H}, \overline{\mathcal{H}}, C)$ is an almost product structure of type (1,1); given a frame e for C, one can produce almost product structures of type $(1, d')$ for each $d' \in I_c$. Another example occurs in [**3**], where it is shown how to associate an almost product structure of type $(n, 2)$ to a CR structure of dimension n and codimension 2 whose Levi form satisfies certain generic conditions.

An r-fold almost product structure $\mathcal{E}$ determines r projection maps $\pi_i : \mathbf{C}TM \to E_i$, which in turn determine a skew-symmetric bilinear map $\tau : \mathbf{C}TM \times \mathbf{C}TM \to \mathbf{C}TM$, called the *torsion* of $\mathcal{E}$, by the formula

$$\tau = \frac{1}{2} \sum_{i\in I_r} \pi_i[\pi_i, \pi_i], \tag{1}$$

where $[\pi_i, \pi_i]$ is the Nijenhuis torsion of π_i (see [**7**]). We shall need two simple propositions.

Proposition 1. (a) *For all* $X, Y \in \Gamma(\mathbf{C}TM)$

$$\tau(X, Y) = \sum_{i\in I_r} ([\pi_i X, \pi_i Y]_i + [X, Y]_i - [\pi_i X, Y]_i - [X, \pi_i Y]_i)$$

where $[\ ,\ ]_i$ *stands for* $\pi_i \circ [\ ,\ ]$.

(b) *For all (not necessarily distinct)* $i, j \in I_r, Z_i \in \Gamma(E_i)$ *and* $Z_j \in \Gamma(E_j)$

$$\tau(Z_i, Z_j) = \sum_{k \in I_r - \{i,j\}} [Z_i, Z_j]_k.$$

Proof. (a) follows immediately from the definition of τ; (b) is a special case of (a). □

Proposition 2. *If* $\mathcal{E}$ *is of type* (d, d')*, then* $\bar{\tau} = \tau$.

Proof. It follows from (1) that $\bar{\tau} = (1/2)\sum_{i \in I_r} \bar{\pi}_i[\bar{\pi}_i, \bar{\pi}_i]$. But $\bar{\pi}_i = \pi_{d+i}$ for all $i \in I_d$, and $\bar{\pi}_{2d+i} = \pi_{2d+i}$ for all $i \in I_{d'}$. Therefore, $\bar{\tau} = \tau$. □

Henceforth, any almost product structure that we consider will be of some type (d, d'). Following the convention, mentioned in Section 0, of not distinguishing between a real bundle map and its complexification, we shall occasionally view τ as a skew-symmetric map from $TM \times TM$ to TM. Also, we shall adopt the following notation: if $\mathcal{E}$ is an r-fold almost product structure and $A : \mathbf{C}TM \times \mathbf{C}TM \to \mathbf{C}TM$ is a map, then for all $i, j, k \in I_r$,

$$A_{ij} : E_i \times E_j \to \mathbf{C}TM \quad \text{and} \quad A_{ijk} : E_i \times E_j \to E_k$$

are the maps obtained, respectively, by restricting A, and by composing A_{ij} with π_k.

Starting at least as far back as the 1950's, a number of people have discussed various affine connections related to an almost product structure (see, e.g., [**7**] and references therein). We shall consider such connections in Section 6, but first, in Section 5, we formulate a generalization of the notion of a connection on a vector bundle which helps to simplify the subsequent exposition.

We close this section by observing that our use of the term "almost product structure" is not universal. Yano and Kon [**13**] define an almost product structure to be a direct sum decomposition of TM into two factors; Kobayashi and Nomizu [**2**] allow any number of factors. Walker [**7**] seems to use the term as we do; at any rate, he does include in his examples a structure of the sort that we are calling type (1, 1).

5. Partial connections. Throughout this section, K, and also K', denotes either $\mathbf{R}$ or $\mathbf{C}$, subject to the restriction that in any argument involving both K and K', if $K' = \mathbf{R}$ then $K = \mathbf{R}$ as well. $\mathcal{M}_K$ denotes the ring of C^∞ K-valued functions on M. Recall our convention of considering TM to be a real subbundle of $\mathbf{C}TM$.

Let E be a K-subbundle of $\mathbf{C}TM$, and let F be a K'-bundle over M. An *E-partial connection in F* is a map $\nabla : \Gamma(E) \times \Gamma(F) \to \Gamma(F)$ that satisfies the following conditions for all $X, X' \in \Gamma(E)$, $Y, Y' \in \Gamma(F)$, $f \in \mathcal{M}_k$, and $g \in \mathcal{M}_{K'}$:

$$\begin{aligned} \nabla_{X+X'}Y &= \nabla_X Y + \nabla_{X'}Y; \\ \nabla_{fX}Y &= f\nabla_X Y; \\ \nabla_X(Y+Y') &= \nabla_X Y + \nabla_X Y'; \\ \nabla_X(gY) &= (Xg)Y + g\nabla_X Y. \end{aligned}$$

Clearly, if $E = TM$ then an E-partial connection in F is equivalent to an ordinary connection in F.

If E is a subbundle of TM and F is a complex bundle over M on which there is given a conjugation, then a $\mathbf{C}E$-partial connection ∇ in F is *real* if $\overline{\nabla} = \nabla$, i.e., if $\overline{\nabla_X Y} = \nabla_{\overline{X}}\overline{Y}$ for all $X \in \Gamma(\mathbf{C}E)$ and $Y \in \Gamma(F)$. For future reference, we list some basic facts about partial connections in the following propositions. The proofs of all but the last of them are completely elementary and are therefore omitted.

Proposition 1. *Let E be a K-subbundle of $\mathbf{C}TM$ and F be a K'-bundle over M, and suppose that ∇ is an E-partial connection in F. If $E' \subset E$ is a K-subbundle, then by restriction ∇ induces an E'-partial connection in F.*

By a slight abuse of notation, we denote the induced partial connection by ∇.

Proposition 2. *Let r and r' be positive integers. For each $i \in I_r$ let E_i be a K-subbundle of $\mathbf{C}TM$, for each $j \in I_{r'}$ let F_j be a K'-*

bundle over M, and for each $(i,j) \in I_r \times I_{r'}$ let $\nabla^{i,j}$ be an E_i-partial connection in F_j. There exists a unique $(\oplus_{i\in I_r} E_i)$-partial connection ∇ in $\oplus_{j\in I_{r'}} F_j$ such that $\nabla_X Y = \nabla^{i,j}_X Y$ for all $X \in \Gamma(E_i)$ and $Y \in \Gamma(F_j)$.

We shall denote ∇ by $\oplus \nabla^{i,j}$.

Proposition 3. *Let E be a real subbundle of $\mathbf{C}TM$.*

(a) *Let F be a real bundle over M. An E-partial connection in F extends by complex linearity to a unique real $\mathbf{C}E$-partial connection in $\mathbf{C}F$, and each real $\mathbf{C}E$-partial connection in $\mathbf{C}F$ is the extension of a unique E-partial connection in F.*

(b) *Let F be a complex bundle over M. An E-partial connection in F extends by complex linearity to a unique $\mathbf{C}E$-partial connection in F, and each $\mathbf{C}E$-partial connection in F is the extension of a unique E-partial connection in F.*

In practice, we shall not distinguish between partial connections related in the manner of (a) or (b).

Proposition 4. *Let E be a K-subbundle of $\mathbf{C}TM$, and let F be a trivial K'-bundle over M with frame $(Y_1, Y_2, \dots, Y_s)$. There exists a unique E-partial connection ∇ in F such that $\nabla_X Y_j = 0$ for all $X \in \Gamma(E)$ and $j \in I_s$.*

Proposition 5. *Let E be a K-subbundle of $\mathbf{C}TM$, let F^1, F^2, and F^3 be K'-bundles over M, and let ∇^2 and ∇^3 be E-partial connections in F^2 and F^3, respectively.*

(a) *If ∇^1 is an E-partial connection in F^1, then there exists a unique E-partial connection ∇ in $\mathrm{Hom}\,(F^1, F^2; F^3)$ such that for all $X \in \Gamma(E)$, $Y \in \Gamma(F^1)$, $Z \in \Gamma(F^2)$ and $b \in \Gamma(\mathrm{Hom}\,(F^1, F^2; F^3))$*

$$(1) \qquad (\nabla_X b)(Y,Z) = \nabla^3_X(b(Y,Z)) - b(\nabla^1_X Y, Z) - b(Y, \nabla^2_X Z).$$

(b) *Suppose that $b \in \Gamma(\mathrm{Hom}\,(F^1, F^2; F^3))$ is nondegenerate. There exists a unique E-partial connection ∇^1 in F^1 such that if ∇ is defined as in* (a), *then $\nabla_X b = 0$ for all $X \in \Gamma(E)$.*

Proof. (a) Simply verify that if $(\nabla_X b)(Y, Z)$ is defined by (1), then ∇ is indeed a partial connection.

(b) Note that if $\nabla_X b = 0$, then by (1)

$$b(\nabla^1_X Y, Z) = \nabla^3_X(b(Y, Z)) - b(Y, \nabla^2_X Z). \tag{i}$$

Since b is nondegenerate, (i) determines $\nabla^1_X Y$ uniquely. To prove existence, define ∇^1_X by (i), and verify that ∇^1 is a partial connection. □

6. Parallelizing connections. Let $\mathcal{E} = (E_1, E_2, \dots, E_r)$ be an almost product structure of type (d, d') with torsion τ. A *parallelizing connection for* $\mathcal{E}$ is an affine connection ∇ with respect to which each of the distributions E_i is parallel, i.e., $\nabla_X Y \in \Gamma(E_i)$ for all $i \in I_r$, $X \in \Gamma(TM)$ and $Y \in \Gamma(E_i)$.

Proposition 1. *An affine connection* ∇ *is parallelizing if and only if for each* $i \in I_r$ *there exists a* $\mathbf{C}TM$*-partial connection* ∇^i *in* E_i *with the following properties*:

(a) $\nabla = \oplus \nabla^i$;

(b) $\nabla^{d+i}_{\overline{X}} \overline{Y} = \overline{\nabla^i_X Y}$ *for all* $i \in I_d$, $X \in \Gamma(\mathbf{C}TM)$ *and* $Y \in \Gamma(E_i)$;

(c) ∇^{2d+j} *is real for all* $j \in I_{d'}$.

Proof. Routine verification. □

Corollary 1. *There exists a parallelizing connection for* $\mathcal{E}$.

Proof. Recall the well-known fact that if F is any real or complex bundle over M then there exists a connection in F. For each $i \in I_d$ let ∇^i be the extension to $\mathbf{C}TM$ of a connection in E_i (see Proposition 5.3) and define a $\mathbf{C}TM$-partial connection in E_{d+i} as follows:

$$\nabla^{d+i}_{\overline{X}} \overline{Y} = \overline{\nabla^i_X Y}, \qquad \text{for all } i \in I_d, X \in \Gamma(\mathbf{C}TM) \text{ and } Y \in \Gamma(E_i).$$

For each $i \in I_{d'}$, let C_i be the real part of E_{2d+i}, let ∇^{2d+i} be a connection in C_i, and consider ∇^{2d+i} as a real $\mathbf{C}TM$-partial connection in E_{2d+i} (see Proposition 5.3). Finally, let $\nabla = \oplus \nabla^i$. □

This corollary has been proved repeatedly in the literature, in many different ways (see [**7, 9, 10, 11**]). The chief reason for considering the present proof is that it shows that, in order to specify a canonical parallelizing connection, it suffices to specify a canonical $\mathbf{C}TM$-partial connection in E_j for each j in the index set

$$R = \{1, 2, \ldots, d\} \cup \{2d+1, 2d+2, \ldots, r\}.$$

Proposition 2. *Let ∇ be a parallelizing connection, with torsion T, and let $i, j \in I_r$ be distinct.*

(a) $T_{ijk} = -\tau_{ijk}$ *for all* $k \in I_r - \{i, j\}$.

(b) $T_{ij} = -\tau_{ij}$ *if and only if* $\nabla_X Y = [X, Y]_j$ *for all* $X \in \Gamma(E_i)$ *and* $Y \in \Gamma(E_j)$.

Proof. Let $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$. By definition, $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$, so

(i) $$T(X,Y) = (\nabla_X Y - [X,Y]_j) - (\nabla_Y X - [Y,X]_i) - \sum_{k \in I_r - \{i,j\}} [X,Y]_k.$$

Since ∇ is parallelizing, (i) implies the following equations:

(ii) $T_{iji}(X, Y) = -(\nabla_Y X - [Y, X]_i)$;

(iii) $T_{ijj}(X, Y) = \nabla_X Y - [X, Y]_j$;

(iv) $T_{ijk}(X, Y) = -[X, Y]_k$ for all $k \in I_r - \{i, j\}$.

Proposition 4.1 implies that τ_{ijk} equals 0 if $k \in \{i, j\}$ and equals $[X, Y]_k$ otherwise. Therefore, (a) follows from (iv), and (b) follows from (a), (ii) and (iii). □

For each $j \in R$ let ∇^j be a $\mathbf{C}TM$-partial connection in E_j, and for each $i \in I_r$ let $\nabla^{i,j}$ be the restriction of ∇^j to E_i. By the preceding results, these partial connections determine a parallelizing connection with $T_{\alpha\beta} = -\tau_{\alpha\beta}$ for all distinct $\alpha, \beta \in I_r$ if and only if for each $j \in R$ and $i \in I_r - \{j\}$ the partial connection $\nabla^{i,j}$ is defined as follows:

$$\nabla^{i,j}_X Y = [X, Y]_j \qquad \text{for all } X \in \Gamma(E_i) \quad \text{and } Y \in \Gamma(E_j).$$

Thus, the problem of specifying a canonical parallelizing connection reduces to that of specifying a canonical E_j-partial connection in E_j for each $j \in R$. One might hope to solve this problem by requiring that $T = -\tau$. Unfortunately, it is known (see [**7**]) that there exist many parallelizing connections with this property. The following theorem shows how to solve the problem if certain auxiliary data are available. Despite its appearance of artificiality, this theorem yields results on almost CR structures, which are presented in Section 7.

Theorem 1. *Let $\{Q_1, Q_2\}$ be a partition of R, and let ρ and σ be maps from Q_2 to I_r. Suppose that for each $i \in Q_1$ the bundle E_i is framed by $(e_{i1}, e_{i2}, \dots, e_{is_i})$, where, if $i > 2d$, each vector field e_{ij} is real, and that for each $i \in Q_2$ the numbers $i, \rho(i)$ and $\sigma(i)$ are mutually distinct and the bilinear map $\tau_{i\rho(i)\sigma(i)}$ is nondegenerate. Then there exists a unique parallelizing connection ∇, with torsion T, that satisfies the following conditions*:

(a) $T_{ij} = -\tau_{ij}$ *for all distinct* $i, j \in I_r$;

(b) $\nabla_{e_{i\alpha}} e_{i\beta} = 0$ *for all* $i \in Q_1$ *and* $\alpha, \beta \in I_{s_i}$;

(c) $\nabla_X \tau_{i\rho(i)\sigma(i)} = 0$ *for all* $i \in Q_2$ *and* $X \in \Gamma(E_i)$.

Moreover, ∇ *has the following properties*:

(d) $\nabla_X Y = [X, Y]_j$ *for all distinct* $i, j \in I_r$, $X \in \Gamma(E_i)$ *and* $Y \in \Gamma(E_j)$;

(e) $\tau_{i\rho(i)\sigma(i)}(\nabla_X Y, Z) = [X, [Y, Z]_{\sigma(i)}]_{\sigma(i)} - [Y, [X, Z]_{\rho(i)}]_{\sigma(i)}$ *for all* $i \in Q_2$, $X, Y \in \Gamma(E_i)$ *and* $Z \in \Gamma(E_{\rho(i)})$;

(f) $T_{iij} = -\tau_{iij}$ *for all distinct* $i, j \in I_r$;

(g) $T_{iii}(e_{i\alpha}, e_{i\beta}) = -[e_{i\alpha}, e_{i\beta}]_i$ *for all* $i \in Q_1$ *and* $\alpha, \beta \in I_{s_i}$;

(h) $T_{i\rho(i)\sigma(i)}(T_{iii}(X, Y), Z) = [X, [Y, Z]_{\sigma(i)} + [Y, Z]_{\rho(i)}]_{\sigma(i)} + [Y, [Z, X]_{\sigma(i)} + [Z, X]_{\rho(i)}]_{\sigma(i)} + [Z, [X, Y]_i]_{\sigma(i)}$ *for all* $i \in Q_2$, $X, Y \in \Gamma(E_i)$ *and* $Z \in \Gamma(E_{\rho(i)})$.

Proof. For each pair (i, j) with $j \in R$ and $i \in I_r - \{j\}$ define an E_i-partial connection $\nabla^{i,j}$ in E_j by requiring that $\nabla^{i,j}_X Y = [X, Y]_j$ for all $X \in \Gamma(E_i)$ and $Y \in \Gamma(E_j)$. For each $j \in Q_1$, let $\nabla^{j,j}$ be the unique E_j-partial connection in E_j with the property that $\nabla^{j,j}_{e_{j\alpha}} e_{j\beta} = 0$ for all $\alpha, \beta \in I_{s_j}$ (see Proposition 5.4). For each $j \in Q_2$, let $\nabla^{j,j}$ be the unique

E_j-partial connection in E_j with the property that for all $X, Y \in \Gamma(E_j)$ and $Z \in \Gamma(E_{\rho(j)})$
(i)
$$\tau_{j\rho(j)\sigma(j)}(\nabla_X^{j,j} Y, Z) = \nabla_X^{j,\sigma(j)}(\tau_{j\rho(j)\sigma(j)}(Y, Z)) - \tau_{j\rho(j)\sigma(j)}(Y, \nabla_X^{j,\rho(j)} Z),$$

(see Proposition 5.5) or, equivalently,

$$\text{(ii)} \qquad \tau_{j\rho(j)\sigma(j)}(\nabla_X^{j,j} Y, Z) = [X, [Y, Z]_{\sigma(j)}]_{\sigma(j)} + [Y, [Z, X]_{\rho(j)}]_{\sigma(j)}.$$

Finally, for each $j \in R$ let $\nabla^j = \oplus_{i \in I_r} \nabla^{i,j}$, for each $j \in I_d$ define a $\mathbf{C}TM$-partial connection ∇^{d+j} in E_{d+j} by requiring that $\nabla_{\overline{X}}^{d+j} \overline{Y} = \overline{\nabla_X^j Y}$ for all $X \in \Gamma(\mathbf{C}TM)$ and $Y \in \Gamma(E_j)$, and let $\nabla = \oplus_{j \in I_r} \nabla^j$.

It is easy to verify that ∇ is the unique parallelizing connection that satisfies (a)–(c). Moreover, (d) is obvious, and (e) follows from (ii). Since ∇ is parallelizing, for all $i \in I_r$ and $X, Y \in \Gamma(E_i)$

$$\text{(iii)} \qquad T_{iii}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_i$$

and for all $j \in I_r - \{i\}$

$$\text{(iv)} \qquad T_{iij}(X, Y) = -[X, Y]_j.$$

Clearly, (iv) implies (f), and (b) and (iii) together imply (g).

Finally, it follows from (iii) that for all $i \in Q_2$, $X, Y \in \Gamma(E_i)$ and $Z \in \Gamma(E_{\rho(i)})$

$$\text{(v)} \qquad \tau_{i\rho(i)\sigma(i)}(T_{iii}(X, Y), Z) = \tau_{i\rho(i)\sigma(i)}(\nabla_X Y - \nabla_Y X - [X, Y]_i, Z).$$

Together, (ii) and (v) imply (h). $\square$

7. Applications to almost CR geometry. Suppose that the almost CR structure $\mathcal{H}$ is partially integrable and that φ is an annihilating frame, nondegenerate of some type S; choose $j_0 \in S$. Let $a = (a_1, a_2, \dots, a_c)$ be the canonical frame for the canonical complement of $\mathcal{H}$ (see Theorem 2.1), and for each $j \in I_c$ let E_{2+j} be the complex subbundle of $\mathbf{C}TM$ framed by a_j. Define an almost product structure $\mathcal{E} = (E_1, E_2, \dots, E_{2+c})$ by setting $E_1 = \mathcal{H}$ and $E_2 = \overline{\mathcal{H}}$; denote the torsion of $\mathcal{E}$ by τ. Let $Q_2 = \{1\}$, and define maps ρ and

σ from Q_2 to I_{2+c} by setting $\rho(1) = 2$ and $\sigma(1) = 2 + j_0$. Note that $\tau_{1\rho(1)\sigma(1)}$ is nondegenerate since φ^{j_0} is nondegenerate; this follows from the observation that for all $X, Y \in \Gamma(\mathcal{H})$

$$\tau_{1\rho(1)\sigma(1)}(X, \overline{Y}) = [X, \overline{Y}]_{\sigma(1)} = \varphi^{j_0}([X, \overline{Y}])a_{j_0} = -d\varphi^{j_0}(X, \overline{Y})a_{j_0}.$$

Finally, let $Q_1 = \{3, 4, \dots, c+2\}$, and for each $i \in Q_1$ let $e_{i1} = a_{i-2}$. An invocation of Theorem 6.1 establishes the following theorem.

Theorem 1. *A nondegenerate annihilating frame for a partially integrable almost* CR *structure determines a canonical affine connection.*

Now, in addition, suppose that $c = j_0 = 1$ and that $\mathcal{H}$ is integrable. Let $\theta = \varphi^1$ and $f = a_1$. Then $(\mathcal{H}, \theta)$ is a nondegenerate integrable pseudo-hermitian structure and f is its canonical transverse vector field. Define a hermitian metric g on $\mathcal{H}$ by requiring that for all $X, Y \in \Gamma(\mathcal{H})$

$$g(X, Y) = i\theta([X, \overline{Y}]),$$

and let ∇ be the canonical affine connection given by Theorem 1. We shall examine this special case in some detail.

Lemma 1. $[V, f]_3 = 0$ *for all* $V \in \Gamma(\mathcal{H} \oplus \overline{\mathcal{H}})$.

Proof. $[V, f]_3 = \theta([V, f])f = -d\theta(V, f)f$. Since $f \rfloor d\theta = 0$ by definition, $[V, f]_3 = 0$. $\square$

Lemma 2. $\nabla\tau_{123} = 0$.

Proof. Let $X, Y, Z \in \Gamma(\mathcal{H})$. By Theorem 6.1, $\nabla_X\tau_{123} = 0$. By definition,

(i) $(\nabla_{\overline{X}}\tau_{123})(Y, \bar{Z}) = \nabla_{\overline{X}}(\tau_{123}(Y, \bar{Z})) - \tau_{123}(\nabla_{\overline{X}}Y, \bar{Z}) - \tau_{123}(Y, \nabla_{\overline{X}}\bar{Z})$.

Recall that ∇ is a parallelizing connection for $\mathcal{E}$, that $\overline{\nabla} = \nabla$, and that $\bar{\tau} = \tau$. These facts, together with the definition of τ_{123}, imply the following extended equations:

(ii) $\overline{\nabla_{\overline{X}}(\tau_{123}(Y,\bar{Z}))} = \nabla_X(\overline{\tau(Y,\bar{Z})}) = \nabla_X(\tau(\overline{Y},Z)) = -\nabla_X(\tau_{123}(Z,\overline{Y}))$;

(iii) $\overline{\tau_{123}(\nabla_{\overline{X}}Y,\bar{Z})} = \tau(\overline{\nabla_{\overline{X}}Y},Z) = \tau(\nabla_X\overline{Y},Z) = -\tau_{123}(Z,\nabla_X\overline{Y})$;

(iv) $\overline{\tau_{123}(Y,\nabla_{\overline{X}}\bar{Z})} = \tau(\overline{Y},\overline{\nabla_{\overline{X}}\bar{Z}}) = \tau(\overline{Y},\nabla_X Z) = -\tau_{123}(\nabla_X Z,\overline{Y})$.

Together, (i)–(iv) imply

(v) $\overline{(\nabla_{\overline{X}}\tau_{123})(Y,\bar{Z})} = -\nabla_X(\tau_{123}(Z,\overline{Y})) + \tau_{123}(Z,\nabla_X\overline{Y}) + \tau_{123}(\nabla_X Z,\overline{Y})$.

The right hand side of (v) equals $-(\nabla_X\tau_{123})(Z,\overline{Y})$, which is equal to 0. Therefore, $\nabla_{\overline{X}}\tau_{123} = 0$.

Thus, it suffices to show that $\nabla_f\tau_{123} = 0$. By definition,

(vi) $(\nabla_f\tau_{123})(Y,\bar{Z}) = \nabla_f(\tau_{123}(Y,\bar{Z})) - \tau_{123}(\nabla_f Y,\bar{Z}) - \tau_{123}(Y,\nabla_f\bar{Z})$.

The following extended equations are easily verified:

(vii) $\nabla_f(\tau_{123}(Y,\bar{Z})) = \nabla_f([Y,\bar{Z}]_3) = [f,[Y,\bar{Z}]_3]_3 = [f,[Y,\bar{Z}] - [Y,\bar{Z}]_1 - [Y,\bar{Z}]_2]_3$;

(viii) $\tau_{123}(\nabla_f Y,\bar{Z}) = \tau_{123}([f,Y]_1,\bar{Z}) = [[f,Y]_1,\bar{Z}]_3 = [[f,Y] - [f,Y]_2 - [f,Y]_3,\bar{Z}]_3$;

(ix) $\tau_{123}(Y,\nabla_f\bar{Z}) = \tau_{123}(Y,[f,\bar{Z}]_2) = [Y,[f,\bar{Z}]_2]_3 = [Y,[f,\bar{Z}] - [f,\bar{Z}]_1 - [f,\bar{Z}]_3]_3$.

Since $\mathcal{H}$ is integrable,

(x) $[[f,Y]_2,\bar{Z}]_3 = 0$ and $[Y,[f,\bar{Z}]_1]_3 = 0$.

By Lemma 1,

(xi) $[f,[Y,\bar{Z}]_1]_3 = [f,[Y,\bar{Z}]_2]_3 = [f,Y]_3 = [f,\bar{Z}]_3 = 0$.

Taken together, (vi)–(xi) imply that

(xii) $(\nabla_f\tau_{123})(Y,\bar{Z}) = [f,[Y,\bar{Z}]]_3 - [[f,Y],\bar{Z}]_3 - [Y,[f,\bar{Z}]]_3$.

An application of Jacobi's identity to (xii) shows that $\nabla_f\tau_{123} = 0$. □

Proposition 1. *For all* $X \in \Gamma(TM)$ *and* $Y,Z,W \in \Gamma(\mathcal{H})$,

(a) $\nabla_X Y \in \Gamma(\mathcal{H})$;

(b) $\nabla_X f = 0$;

(c) $\nabla_X g = 0$;

(d) $\nabla_X \theta = 0$;

(e) $T(Y, Z) = 0$;

(f) $T(Y, \bar{Z}) = ig(Y, Z)f$;

(g) $T(Y, f) \in \Gamma(\overline{\mathcal{H}})$.

Proof. Since ∇ is a parallelizing connection for $\mathcal{E}$, (a) is obvious. Consider the following direct consequences of Theorem 6.1:

(i) $\nabla_f f = 0$;

(ii) $\nabla_V f = [V, f]_3$ for all $V \in \Gamma(\mathcal{H} \oplus \overline{\mathcal{H}})$;

(iii) $T_{12} = -\tau_{12}$ and $T_{13} = -\tau_{13}$;

(iv) $T_{112} = -\tau_{112}$ and $T_{113} = -\tau_{113}$;

(v) $\tau_{123}(T_{111}(Y, Z), \overline{W}) = [Y, [Z, \overline{W}]_3 + [Z, \overline{W}]_2]_3 - [Z, [Y, \overline{W}]_3 + [Y, \overline{W}]_2]_3 - [[Y, Z], \overline{W}]_3$.

(b) follows from (i), (ii), and Lemma 1; (d) follows from (a), (b), and the fact that for all $A \in \Gamma(\mathbf{C}TM)$

$$(\nabla_X \theta)(A) = \nabla_X(\theta(A)) - \theta(\nabla_X A);$$

(g) follows from (iii). Observe that

$$\text{(vi)} \qquad g(Y, Z)f = i\theta([Y, \bar{Z}])f = i[Y, \bar{Z}]_3 = i\tau_{123}(Y, \bar{Z}).$$

(f) follows from (iii) and (vi). As a consequence of (b),

$$\text{(vii)} \quad (\nabla_X g)(Y, Z)f = \nabla_X(g(Y, Z)f) - g(\nabla_X Y, Z)f - g(Y, \nabla_X Z)f.$$

Together, (vi) and (vii) imply that
(viii)

$$(\nabla_X g)(Y, Z)f = i\nabla_X(\tau_{123}(Y, \bar{Z})) - i\tau_{123}(\nabla_X Y, \bar{Z}) - i\tau_{123}(Y, \overline{\nabla_X Z}).$$

Since $\overline{\nabla_X Z} = \nabla_{\overline{X}} \bar{Z} = \nabla_X \bar{Z}$, (c) follows from (viii) and Lemma 2.

Finally, note the following consequences of the integrability of $\mathcal{H}$:

(ix) $\tau_{112} = 0$ and $\tau_{113} = 0$;

(x) $[Y, [Z, \overline{W}]_1]_3 = 0$ and $[Z, [Y, \overline{W}]_1]_3 = 0$.

Together, (v) and (x) imply that

(xi) $\tau_{123}(T_{111}(Y,Z),\overline{W}) = [Y,[Z,\overline{W}]]_3 - [Z,[Y,\overline{W}]]_3 - [[Y,Z],\overline{W}]_3.$

Since τ_{123} is nondegenerate, an application of Jacobi's identity to (xi) shows that $T_{111}=0$. Moreover, $T_{112}=0$ and $T_{113}=0$ by (iv) and (ix); hence, (e) is true. $\square$

Remark. The conditions in Proposition 1 include all of those needed to characterize the Webster connection for $(\mathcal{H},\theta)$ (see [**8**] or [**5**]). Thus, as promised in Section 0, we have rederived this connection from a more general point of view and have relaxed the usual integrability assumption in the process.

We conclude with an analogue of Theorem 1 in which the annihilating frame is replaced by a single annihilating form.

Theorem 2. *Suppose that $\mathcal{H}$ is partially integrable and that θ is a nondegenerate annihilating form, and let C be the canonical complement of $\mathcal{H}$ (see Corollary 2.2). There exists a canonical connection $\nabla^{\mathcal{H}}$ in the bundle $\mathcal{H}$, and a canonical $\mathcal{H}$-partial connection $\nabla^{\mathcal{H},C}$ in the bundle $\mathbf{C}C$.*

Proof. Define an almost product structure $\mathcal{E}=(E_1,E_2,E_3)$ by setting $E_1=\mathcal{H}$, $E_2=\overline{\mathcal{H}}$, and $E_3=\mathbf{C}C$. For all $X,Y\in\Gamma(\mathcal{H})$

$$\theta\circ\mathcal{L}(X,Y) = i\theta([X,\overline{Y}]) = i\theta\circ\tau_{12}(X,\overline{Y}).$$

Therefore, the nondegeneracy of θ implies the nondegeneracy of the bilinear form $b=\theta\circ\tau_{12}$, which we view as a bundle map from $E_1\times E_2$ to $M\times\mathbf{C}$.

For all distinct $i,j\in I_3$, define an E_i-partial connection $\nabla^{i,j}$ in E_j by setting $\nabla^{i,j}_X Y = [X,Y]_j$ for all $X\in\Gamma(E_i)$ and $Y\in\Gamma(E_j)$. Define a connection $\tilde{\nabla}$ in $M\times\mathbf{C}$ by requiring $\tilde{\nabla}\mathbf{1}=0$, where $\mathbf{1}$ is the map $x\mapsto 1$, viewed as a trivializing section of $M\times\mathbf{C}$. Since b is nondegenerate, Proposition 5.5 shows that $b,\nabla^{1,2}$, and $\tilde{\nabla}$ together determine a canonical E_1-partial connection $\nabla^{1,1}$ in E_1. Let $\nabla^{\mathcal{H}} = \nabla^{1,1}\oplus\nabla^{2,1}\oplus\nabla^{3,1}$; by Proposition 5.2, $\nabla^{\mathcal{H}}$ is a connection in $\mathcal{H}$. Finally, let $\nabla^{\mathcal{H},C}=\nabla^{1,3}$. $\square$

In light of Example 2.1, this theorem should prove useful in the study of almost CR submanifolds of strongly pseudo-convex partially integrable pseudo-hermitian manifolds.

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