ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 23, Number 4, Fall 1993

GENERALIZED SCHWARZ CONSTANTS ON THE CLASSICAL CARTAN DOMAINS

SHOZO MATSUURA

1. Introduction. The classical Schwarz lemma for the unit disc has been previously generalized to the cases of several complex variables. The Ahlfors type generalizations for some manifolds are treated by many authors, for example, Chen-Cheng-Look and Royden [**12**]. Yau [**14**] has recently given

$$f^*S_Q \le (\underline{K}_P/\overline{K}_Q)^{1/2}S_P, \qquad f \in \mathrm{Hol}\,(P,Q), \tag{1.1}$$

where (P, S_P^2) is a complete Kähler manifold with Ricci curvature bounded from below by some constant $\underline{K}_P(\le 0)$ and (Q, S_Q^2) is an Hermitian manifold with holomorphic bisectional curvature bounded from above by $\overline{K}_Q(< 0)$. In particular, for any irreducible classical Cartan domain $M_j(j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ (see Section 4) and any self-mapping $f \in \mathrm{Hol}\,(M_j, M_j)$ Look [**6**] gave

$$f^*S_{M_j} \le c(M_j, M_j)S_{M_j}, \tag{1.2}$$

where $c(M_j, M_j)$ denotes the so-called *Schwarz constant* which is the best possible constant depending only on M_j and further $c(M_j, M_j) = (\underline{K}_{M_j}/\overline{K}_{M_j})^{1/2}$ $(j = \mathrm{I} \sim \mathrm{IV})$ are concretely given. Here f^* denotes the pullback by f and $\underline{K}_{M_j}$ and $\overline{K}_{M_j}$ are inf and sup of the holomorphic sectional curvature of M_j, respectively.

First, from another point of view, we shall give a generalized Schwarz-Pick type lemma for $\mathrm{Hol}\,(H, H')$ with respect to the Bergman metric, where H and H' are arbitrary homogeneous bounded domains.

As an application, for two arbitrary (irreducible or reducible) classical Cartan domains M and N, we should like to generalize (1.2) and to give the explicit value of the generalized Schwarz constant $c(M, N)$. Further, in this case we will show that each value $c(M, N)$ (best possible constant) coincides with the value $(\underline{K}_M/\overline{K}_N)^{1/2}$.

Received by the editors on October 11, 1991, and in revised form on March 2, 1992.

Incidentally, (1) we give a generalization of the Schwarz lemma of Ozaki-Matsuno [**11**] in Example 4.1 as a Corollary of Theorem 4.1 and (2) the Schwarz constant $c(M_{ij}, M_{ij}), M_{ij} = M_i \times M_j$, in [**6**] is slightly amended in Lemma 4.2. Through the argument in this paper an interesting canonical condition Prop (A) plays an important role.

Secondly, in Section 5, by combining our expression of the Schwarz constant $c(M_j, M_j)$ with the theorem recently obtained by Kubota [**5**] we shall give a geometrically simple characterization of M_j with Prop (A).

Finally, in an appendix we shall show that Prop (A) is deeply related to some extremal problems in some holomorphic equivalent class of domains.

2. Schwarz lemma on the Bergman metric. For a bounded domain D in $\mathbf{C}^n$, it is known that

$$C_D(t,u) := \sup\{|\partial_{z,u} f(t)|\,|\,f \in \mathrm{Hol}\,(D), f(t) = 0, |f(z)|_D \le 1\} \tag{2.1}$$

and

$$S_D(t,u) := \sup\{|\partial_{z,u} f(t)| k_D(t,t)^{-1/2} | f \in \mathrm{Hol}\,(D) \cap L_2(D), f(t) = 0, ||f||_D \le 1\} \tag{2.2}$$

define the Carathéodory metric and the Bergman metric of D, respectively, where $k_D(z,t)$, $|f|_D$, $||f||_D$ and $\partial_{z,u}$ denote the Bergman kernel of D, $\sup\{|f(z)|\,|z \in D\}$, the L_2-norm of f on D and

$$\partial_{z,u} \cdot := (\partial \cdot / \partial z) u := (\partial/\partial z_1, \ldots, \partial/\partial z_n)(\cdot) u$$

for $z = {}^t(z_1, \ldots, z_n) \in D$ and $u = {}^t(u_1, \ldots, u_n) \in \mathbf{C}^n - \{0\}$, respectively [**2**]. These two metrics are biholomorphically invariant.

For the Bergman tensor

$$T_D(z,t) := \partial^2 \log k_D(z,t) / \partial z^* \partial z \tag{2.3}$$

it is well known that the Bergman metric $S_D(t,u)^2$ coincides with $u^* T_D(t,t) u$, where A^* denotes the transposed-conjugate ${}^t\overline{A}$ of a matrix A.

Lemma 2.1. *Let $B_n(t,\rho)$ be a ball of radius ρ with center at $t \in \mathbf{C}^n$, and put $B = B_n(0,r)$; then we have*

$$S_B(z,u)^2 = (n+1)C_B(z,u)^2, \qquad z \in B, \tag{2.4}$$

and

$$C_B(0,u) = |u|/r \qquad \text{(see [3])}. \tag{2.5}$$

We shall often use $\mu(t,D)$ and $\nu(t,D)$ as the maximum and the minimum characteristic values of $T_D(t,t)$, respectively.

For the extremal radii

$$R(a,D) := \inf\{\rho | B_n(a,\rho) \supset D\}, \qquad r(a,\rho) := \sup\{\rho | B_n(a,\rho) \subset D\} \tag{2.6}$$

at $a \in D$ and the values

$$\Gamma(\beta,D') := R(\beta,D')\mu(\beta,D')^{1/2}, \qquad \gamma(\alpha,D) := r(\alpha,D)\nu(\alpha,D)^{1/2}, \tag{2.7}$$

at $\beta \in D'$ and $\alpha \in D$, put

$$\hat{c}_{\alpha\beta}(D,D') := \Gamma(\beta,D')/\gamma(\alpha,D), \tag{2.8}$$

then we have

Theorem 2.1. *Let $H \subset \mathbf{C}^r$ and $H' \subset \mathbf{C}^s$ be arbitrary homogeneous bounded domains; then we have a generalized Schwarz-Pick type lemma*

$$f^*S_{H'} \le \hat{c}(H,H')S_H, \qquad f \in \mathrm{Hol}\,(H,H'), \tag{2.9}$$

where

$$\hat{c}(H,H') := \inf\{\hat{c}_{\alpha\beta}(H,H') \mid \alpha \in H, \beta \in H'\}.$$

Proof. Equation (2.9) denotes $S_{H'}(w,v) \le \hat{c}(H,H')S_H(z,u)$, $z \in H$, $u \in \mathbf{C}^r - \{0\}$, with $v = (df(z)/dz)u$ for $w = f(z) \in \mathrm{Hol}\,(H,H')$. Let

$f_0 \in \mathrm{Hol}\,(H, H')$ with $f_0(\alpha) = \beta$, then from the monotonicity of the Carathéodory metric [**3**] and Lemma 2.1, we have

$$|u_0|/R(\alpha, H) = C_{B_r(\alpha, R(\alpha,H))}(\alpha, u_0) \le C_H(\alpha, u_0) \\ \le C_{B_r(\alpha, r(\alpha,H))}(\alpha, u_0) = |u_0|/r(\alpha, H).$$

Since $\nu(\alpha, H)|u_0|^2 \le S_H(\alpha, u_0)^2 \le \mu(\alpha, H)|u_0|^2$ holds, then we have

$$(2.10) \qquad \Gamma(\alpha, H)^{-1} S_H(\alpha, u_0) \le C_H(\alpha, u_0) \le \gamma(\alpha, H)^{-1} S_H(\alpha, u_0).$$

On the other hand, the decreasing property of C_H gives

$$(2.11) \quad f_0^* C_{H'} \le C_H, \qquad f_0 \in \mathrm{Hol}\,(H, H'), \qquad f_0(\alpha) = \beta, \qquad [\mathbf{3}].$$

From (2.10) and (2.11), we have

$$(2.12) \qquad S_{H'},(f_0(\alpha), v_0) \le (\Gamma(\beta, H')/\gamma(\alpha, H)) S_H(\alpha, u_0),$$

where $v_0 = (df_0(\alpha)/dz)u_0$. For any $f \in \mathrm{Hol}\,(H, H')$ and any $(t, \tau) = (t, f(t))$ there exist transitive mappings $h_{H,t}$ of H with $h_{H,t}(t) = \alpha$ and $h_{H',\tau}$ of H' with $h_{H',\tau}(\tau) = \beta$. Put $f_0 = h^{-1}_{H',\tau} \circ f \circ h_{H,t}$, $f_0(\alpha) = \beta$ and $(dh_{H,t}(\alpha)/dz)u_0 = u$, then from the relative invariance of the Bergman kernel and the chain rule of derivatives we have $S_H(\alpha, u_0) = S_H(t, u)$ and $S_{H'}(\beta, v_0) = S_{H'}(f(t), (df(t)/dz)u)$. Therefore, from (2.12) we have the result. □

3. Canonical domains and Prop **(A).** The Schwarz constant is a biholomorphical invariant [**6**]. Therefore, in order to get the Schwarz constant we should treat certain canonical domains rather than use arbitrary ones in the biholomorphically equivalent class of domains.

Definition 3.1. A bounded domain D in $\mathbf{C}^n$ with a canonical condition

$$(3.1) \qquad T_D(\alpha, \alpha) = a^2 E_n$$

is called a domain with Prop (A) at $\alpha \in D$, where E_n denotes the unit matrix of order n.

Any bounded domain D is biholomorphically equivalent to a domain with Prop (A) at 0 under a suitable linear transformation since the Bergman tensor $T_D(z,z)$ is positive definite and relatively invariant under any biholomorphic mapping [**1**].

4. Generalized Schwarz constant between arbitrary two classical Cartan domains. Let H and H' be homogeneous bounded domains with Prop (A) at $0 \in H \subset \mathbf{C}^r$ and $0 \in H' \subset \mathbf{C}^s$, respectively; then from Theorem 2.1 we have

$$f^* S_{H'} \leq \hat{c}_{00}(H,H') S_H, \qquad f \in \mathrm{Hol}\,(H,H'), \tag{4.1}$$

where

$$\hat{c}_{00}(H,H') = \Gamma(0,H')/\gamma(0,H). \tag{4.2}$$

Now we may set a conjecture that *for arbitrary homogeneous bounded domains H and H' with* Prop (A) *at* 0, $\hat{c}_{00}(H,H')$ *gives the generalized Schwarz constant* $c(H,H')$.

We will prove that the conjecture is true at least between two arbitrary (irreducible or reducible) classical domains M and N or their biholomorphic images.

In order to treat our cases uniformly, the irreducible classical Cartan domains $M_j (j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ with Prop (A) at 0 are defined as follows:

$M_{\mathrm{I}} = M_{\mathrm{I}(m\times n)} = \{Z_{\mathrm{I}} = (z_{rs}) : m \times n \text{ matrix} | E_n - Z_{\mathrm{I}}^* Z_{\mathrm{I}} > 0\}$, $m \geq n$.

$M_{\mathrm{II}} = M_{\mathrm{II}(n)} = \{Z_{\mathrm{II}} = (z_{rs}) : n \times n \text{ symmetric matrix} | E_n - X_{\mathrm{II}}^* X_{\mathrm{II}} > 0,\ X_{\mathrm{II}} = (x_{rs}) \text{ with } x_{rr} = z_{rr} \text{ and } x_{rs} = z_{rs}/\sqrt{2},\ r \neq s\}$.

$M_{\mathrm{III}} = M_{\mathrm{III}(n)} = \{Z_{\mathrm{III}} = (z_{rs}) : n \times n \text{ skew symmetric matrix} | E_n - Z_{\mathrm{III}}^* Z_{\mathrm{III}} > 0\}$.

$M_{\mathrm{IV}} = M_{\mathrm{IV}(n)} = \{Z_{\mathrm{IV}} = {}^t(z_1,\dots,z_n) | 1 - 2X_{\mathrm{IV}}^* X_{\mathrm{IV}} + |{}^t X_{\mathrm{IV}} X_{\mathrm{IV}}|^2 > 0,\ 1 - |{}^t X_{\mathrm{IV}} X_{\mathrm{IV}}| > 0,\ X_{\mathrm{IV}} = {}^t(x_1,\dots,x_n) = Z_{\mathrm{IV}}/\sqrt{2}\}$, (cf. [**6**]).

The Bergman kernels of M_j's were given by Hua [**4**] so that

$$k_{M_j}(Z_j, Z_j) = c_j (\det Y_j)^{-n_j},$$

where $Y_j = E_n - Z_j^* Z_j$ $(j = \mathrm{I}, \mathrm{III})$, $Y_{\mathrm{II}} = E_n - X_{\mathrm{II}}^* X_{\mathrm{II}}$, $Y_{\mathrm{IV}} = 1 - 2X_{\mathrm{IV}}^* X_{\mathrm{IV}} + |{}^t X_{\mathrm{IV}} X_{\mathrm{IV}}|^2$, $n_{\mathrm{I}} = m + n$, $n_{\mathrm{II}} = n + 1$, $n_{\mathrm{III}} = n - 1$, $n_{\mathrm{IV}} = n$ and each c_j is a positive constant depending only on M_j.

Let the contraction column vectors $\tilde{Z}_j$ with respect to Z_j $(j = \mathrm{I} \sim \mathrm{IV})$ be $\tilde{Z}_{\mathrm{I}} = {}^t(((z_{rs})_{s=1}^n)_{r=1}^m) \in \mathbf{C}^{mn}$, $\tilde{Z}_{\mathrm{II}} = {}^t(((z_{rs})_{s=r}^n)_{r=1}^n) \in \mathbf{C}^{n(n+1)/2}$, $\tilde{Z}_{\mathrm{III}} = {}^t(((z_{rs})_{s=r+1}^n)_{r=1}^{n-1}) \in \mathbf{C}^{n(n-1)/2}$ and $\tilde{Z}_{\mathrm{IV}} = Z_{\mathrm{IV}} \in \mathbf{C}^n$, then the Bergman metrics of M_j's are defined by

$$S_{M_j}(Z_j, U_j)^2 = \partial_j^* \partial_j \log k_{M_j}(Z_j, Z_j) = \tilde{U}_j^* T_{M_j}(Z_j, Z_j) \tilde{U}_j, \tag{4.3}$$

where $\partial_j A = ((\partial a_{rs}/\partial \tilde{Z}_j)\tilde{U}_j)$ for $A = (a_{rs})$ and $\tilde{U}_j = d\tilde{Z}_j$.

Lemma 4.1. *For* M_j $(j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ *we have the table*

j	I	II	III	IV
$\mu(0, M_j) = \nu(0, M_j)$	$m+n$	$n+1$	$2(n-1)$	n
$R(0, M_j)$	$n^{1/2}$	$n^{1/2}$	$[n/2]^{1/2}$	$2^{1/2}$
$r(0, M_j)$	1	1	1	1

Proof. We shall use the formulas $\partial_j(AB) = (\partial_j A)B + A(\partial_j B)$, $\partial_j Y^{-1} = -Y^{-1}(\partial_j Y)Y^{-1}$, $\partial_j \mathrm{tr}\, C = \mathrm{tr}\,(\partial_j C)$ and $\partial_j \log\det Y = \mathrm{tr}\,(Y^{-1}\partial_j Y)$, where AB and Y denote a matrix product and a regular Hermitian matrix function of Z_j and $\overline{Z}_j$. Note that $\partial_j Y_j|_{z_j=0} = 0$ $(j = \mathrm{I} \sim \mathrm{IV})$, then we have

$$\begin{aligned} \partial_j^* \partial_j \log\det Y_j|_{Z_j=0} &= \mathrm{tr}\,(Y_j^{-1}(\partial_j^*\partial_j Y_j - (\partial_j^* Y_j)Y_j^{-1}(\partial_j Y_j))|_{Z_j=0} \\ &= \mathrm{tr}\,(\partial_j^*\partial_j Y_j)|_{Z_j=0} = \partial_j^*\partial_j \mathrm{tr}\ \ Y_j|_{Z_j=0}. \end{aligned} \tag{4.4}$$

For example, $S_{M_{\mathrm{II}}}(0, U_{\mathrm{II}})^2 = -(n+1)\partial_{\mathrm{II}}^*\partial_{\mathrm{II}} \mathrm{tr}\, Y_{\mathrm{II}}|_{Z_{\mathrm{II}}=0} = \tilde{U}_{\mathrm{II}}^* T_{M_{\mathrm{II}}}(0,0)\tilde{U}_{\mathrm{II}}$ and thus $T_{M_{\mathrm{II}}}(0,0) = -(n+1)\partial^2(n - \sum_{r,s=1}^n |x_{rs}|^2)/\partial\tilde{Z}_{\mathrm{II}}^*\partial\tilde{Z}_{\mathrm{II}} = (n+1)E_{n(n+1)/2}$ which shows that $\mu(0, M_{\mathrm{II}}) = \nu(0, M_{\mathrm{II}}) = n+1$ and M_{II} has Prop (A). Similarly, we have the other T_{M_j} $(j = \mathrm{I}, \mathrm{III}, \mathrm{IV})$.

We have the canonical representations as $Z_j = P_j D_j(Z_j) Q_j$ $(j = \mathrm{I}, \mathrm{III}, \mathrm{IV})$ and $X_{\mathrm{II}} = P_{\mathrm{II}} D_{\mathrm{II}}(X_{\mathrm{II}}) Q_{\mathrm{II}}$, where P_j and Q_j are suitable unitary matrices and

$$D_1(Z_{\mathrm{I}}) = \begin{pmatrix} \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) \\ 0 \end{pmatrix},$$

$m \times n$, $1 > \lambda_1 \geq \cdots \geq \lambda_n \geq 0$; $D_{\mathrm{II}}(X_{\mathrm{II}}) = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n)$, $n \times n$, $1 > \lambda_1 \geq \cdots \lambda_n \geq 0$; $D_{\mathrm{III}}(Z_{\mathrm{III}}) = (\hat{E} \times \mathrm{diag}\,(\lambda_1, \dots, \lambda_p))$, $n = 2p$ or $= (\hat{E} \times \mathrm{diag}\,(\lambda_1, \dots, \lambda_p) \dotplus 0)$, $n = 2p + 1$,

$$n \times n, \qquad 1 > \lambda_1 \geq \cdots \geq \lambda_p \geq 0, \qquad p = [n/2], \qquad \hat{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $D_{\mathrm{IV}}(Z_{\mathrm{IV}}) = \sqrt{2} D'_{\mathrm{IV}}(X_{\mathrm{IV}}) = \sqrt{2}\ {}^t(\lambda_1, \sqrt{-1}\lambda_2, 0, \dots, 0)$, $1 > \lambda_1 \geq \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 < 1$ [**4**]. Put

$$(4.5) \qquad \begin{aligned} &D_{\mathrm{I}} = \begin{pmatrix} E_n \\ 0 \end{pmatrix}, \qquad D_{\mathrm{II}} = E_n, \qquad D_{\mathrm{IV}} = {}^t(\sqrt{2}, 0, \dots, 0), \\ &D_{\mathrm{III}} = \hat{E} \times E_p, n = 2p, \quad \text{or} \quad = (\hat{E} \times E_p \dotplus 0), n = 2p + 1, \end{aligned}$$

then we have $D_j \in \partial M_j$, $j = \mathrm{I} \sim \mathrm{IV}$, and $R(0, M_j) = |\tilde{D}_j| = \sqrt{n}$, $j = \mathrm{I}, \mathrm{II}$; $= \sqrt{p}$, $j = \mathrm{III}$, and $= \sqrt{2}$, $j = \mathrm{IV}$, and further $r(0, M_j) = 1$ ($j = \mathrm{I} \sim \mathrm{IV}$), since M_j's are strongly starlike and $n > \mathrm{tr}\ (Z_{\mathrm{I}}^* Z_{\mathrm{I}}) = |\tilde{Z}_{\mathrm{I}}|^2 > 1$, $n > \mathrm{tr}\,(X_{\mathrm{II}}^* X_{\mathrm{II}}) = |\tilde{Z}_{\mathrm{II}}|^2 > 1$, $2p > \mathrm{tr}\,(Z_{\mathrm{III}}^* Z_{\mathrm{III}}) = 2|\tilde{Z}_{\mathrm{III}}|^2 > 2$ and $2 > 2\mathrm{tr}\,(X_{\mathrm{IV}}^* X_{\mathrm{IV}}) = |\tilde{Z}_{\mathrm{IV}}|^2 > 1$ hold. In particular, for $j = \mathrm{IV}$ put $M'_{\mathrm{IV}} = \{X_{\mathrm{IV}} = {}^t(x_1, \dots, x_n) | 1 - 2X_{\mathrm{IV}}^* X_{\mathrm{IV}} + |{}^t X_{\mathrm{IV}} X_{\mathrm{IV}}|^2 > 0,\ 1 - |{}^t X_{\mathrm{IV}} X_{\mathrm{IV}}| > 0\}$, then $X_{\mathrm{IV}} = P'_{\mathrm{IV}}(\lambda_1, \sqrt{-1}\lambda_2, 0, \dots, 0) = Z_{\mathrm{IV}}/\sqrt{2}$ in $\partial M'_{\mathrm{IV}}$ satisfies $\lambda_1 + \lambda_2 = 1$ and $0 \leq \lambda_2 \leq \lambda_1 \leq 1$, i.e., $R(0, M'_{\mathrm{IV}}) = 1$ ($\lambda_1 = 1$, $\lambda_2 = 0$) and $r(0, M'_{\mathrm{IV}}) = 1/\sqrt{2}$ ($\lambda_1 = \lambda_2 = 1/2$). Therefore we have the results. Here $D_k \to \tilde{D}_k$ corresponds to $Z_k \to \tilde{Z}_k$. □

Theorem 4.1. *Between M_i and M_j, $i, j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, we have the generalized Schwarz-Pick type lemma*

$$(4.6) \qquad f_{ij}^* S_{M_j} \leq c(M_i, M_j) S_{M_i}, \qquad f_{ij} \in \mathrm{Hol}\,(M_i, M_j),$$

where the Schwarz constants $c(M_i, M_j)$ are given by

$$(4.7) \qquad c(M_i, M_j) = \hat{c}_{00}(M_i, M_j) = \Gamma(0, M_j)/\gamma(0, M_i).$$

In particular, for $\mathrm{Hol}\,(M_j, M_j)$ *we have the usual Schwarz constants*

$$(4.8) \qquad c(M_j, M_j) = \hat{c}_{00}(M_j, M_j) = R(0, M_j), \qquad [\mathbf{6}].$$

Proof. From (4.1) and (4.2), it is sufficient to show that the equality in $f_{ij}^* S_{M_j} \leq \hat{c}_{00}(M_i, M_j) S_{M_i}$ holds at $Z_i = 0 \in M_i$ for some $f_{ij} \in \mathrm{Hol}\,(M_i, M_j)$ with $f_{ij}(0) = 0$ and some $\tilde{U}_i = d\tilde{Z}_i \in \mathbf{C}^{\dim M_t} - \{0\}$ since $M_k (k = i, j)$ are homogeneous. Noting that $S_{M_k}(0, U_k)^2 = \mu(0, M_k)|\tilde{U}_k|^2$, $k = i, j$, hold from Prop (A) at 0, we may only show that

$$|\partial_{\tilde{Z}_t, \tilde{U}_t} \tilde{f}_{ij}(0)| \equiv |(d\tilde{f}_{ij}(0)/d\tilde{Z}_i)\tilde{U}_i| = R(0, M_j)|\tilde{U}_i| \tag{4.9}$$

holds for some $f_{ij} \in \mathrm{Hol}\,(M_i, M_j)$ and $\tilde{U}_i = d\tilde{Z}_i \in \mathbf{C}^{\dim M_t} - \{0\}$.

Put

$$\tilde{f}_{ij}(Z_i) = A_{ij}\tilde{Z}_i = \begin{cases} P_{ij}\tilde{Z}_i & \text{for } i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \\ P_{ij}Q\tilde{Z}_i & \text{for } i = \mathrm{IV}, \end{cases} \tag{4.10}$$

where

$$Q = \begin{pmatrix} 1 & -\sqrt{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} / \sqrt{2}, \quad \dim M_{\mathrm{IV}} \times \dim M_{\mathrm{IV}} \text{ matrix},$$

and $P_{ij} = (\tilde{D}_j 0)$ $(\dim M_j \times \dim M_i)$ for $j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, then f_{ij} belongs to $\mathrm{Hol}\,(M_i, M_j)$. Indeed, for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$ we have $A_{ij}\tilde{Z}_i = (\tilde{D}_j 0)\tilde{Z}_i = \tilde{D}_j t \in M_j$ since M_j is strongly starlike or more precisely convex and $D_j \in \partial M_j$ and $|t| < 1$ for the first element t of $\tilde{Z}_i$, $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$. And further, we have

$$|\partial_i f_{ij}(0)| = |\tilde{D}_j| = R(0, M_j)|\tilde{U}_i|, \qquad \tilde{U}_i = {}^t(1, 0, \dots, 0),$$

which give (4.9) for $i = \mathrm{I}, \mathrm{II}, \mathrm{III}$.

Next for $i = \mathrm{IV}$ and $M'_{\mathrm{IV}} \ni X_{\mathrm{IV}} = {}^t(x_1, \dots, x_n) = Z_{\mathrm{IV}}/\sqrt{2}$, $A_{\mathrm{IV},j}\tilde{Z} = (\tilde{D}_j 0)Q\tilde{Z}_{\mathrm{IV}} = (\tilde{D}_j 0)Q\sqrt{2}\tilde{X}_{\mathrm{IV}} = \tilde{D}_j t$, where $|t| = |x_1 - \sqrt{-1}x_2| < 1$ [**6**, p. 485] and $D_j \subset \partial M_j$, $j = \mathrm{I} \sim \mathrm{IV}$. Therefore $f_{\mathrm{IV},j} : M_{\mathrm{IV}} \to M_j$ and further we get $|\partial_{\mathrm{IV}} \tilde{f}_{\mathrm{IV},j}(0)| = |(\tilde{D}_j 0)Q\tilde{U}_{\mathrm{IV}}| = |\tilde{D}_j| = R(0, M_j)|\tilde{U}_{\mathrm{IV}}|$ for $\tilde{U}_{\mathrm{IV}} = {}^t(1, \sqrt{-1}, 0, \dots, 0)/\sqrt{2}$. This completes the proof. □

Example 4.1. Let $M = M_{\mathrm{I}(m \times n)}$ (in particular $M_{\mathrm{I}(k \times 1)} = B_k(0,1) = B$), then from (4.7) and Lemma 4.1 we have

$$c(M, B) = \hat{c}_{00}(M, B) = \frac{R(0, B)}{r(0, M)} \left(\frac{\mu(0, B)}{\nu(0, M)} \right)^{1/2} = \left(\frac{k+1}{m+n} \right)^{1/2}$$

and the generalized Schwarz-Pick type lemma

$$\begin{aligned}(4.11)\qquad ||df(Z)/d\tilde{Z}||^2 &\le (1-|f(Z)|^2)^N/(1-||Z||^2)^2,\\ Z\in M,&\quad f\in \mathrm{Hol}\,(M,B),\end{aligned}$$

where $N=2$ for $k=1$, $N=1$ for $k\ge 2$ and $||X||^2$ denotes the maximum characteristic value of X^*X.

In particular, for $f\in \mathrm{Hol}\,(B,B)$ we have Ozaki-Matsuo's theorem

$$(4.12)\qquad ||df(Z)/dZ||^2 \le (1-|f(Z)|^2)/(1-|Z|^2)^2,\qquad Z\in B\qquad [\mathbf{11}],$$

and for $f\in \mathrm{Hol}\,(\Delta,\Delta)$ (Δ: unit disc) we have the usual Schwarz lemma $|df(z)/dz|\le(1-|f(z)|^2)/(1-|z|^2)$, $z\in\Delta$.

Lemma 4.2. *Let c_{kk} and μ_k denote the Schwarz constant $c(M_k,M_k)$ and $\mu(0,M_k)$, $k=i,j$, respectively, then we have the Schwarz constant $c_{ij,ij}=c(M_{ij},M_{ij})$ for $M_{ij}=M_i\times M_j$ as*

$$(4.13)\qquad c_{ij,ij}^2=(\mu_i c_{ii}^2+\mu_j c_{jj}^2)/\min(\mu_i,\mu_j).$$

In particular, if μ_i equals μ_j, then we have

$$(4.14)\qquad c_{ij,ij}^2=c_{ii}^2+c_{jj}^2\qquad (\text{cf. } [\mathbf{6}, \text{pp. } 492\text{–}494]).$$

Proof. If $\mu_i\le\mu_j$, put $Z_i'=(\mu_i/\mu_j)^{1/2}Z_i$: $M_i\to M_i'$, then we have $M_{ij}'=M_i'\times M_j$ with $\mathrm{Prop}\,(A)$ at 0, i.e., $\mu(0,M_{ij}')=\nu(0,M_{ij}')=\mu_j$ since we have $T_{M_{ij}'}(0,0)=\mathrm{diag}\,(T_{M_i'}(0,0),T_{M_j}(0,0))=\mu_j E_{\dim M_{ij}}$ from $k_{M_i\times M_j}=k_{M_i}\times k_{M_j}$. Hence we have

$$\begin{aligned}(4.15)\qquad R(0,M_{ij}')^2&=R(0,M_i')^2+R(0,M_j)^2\\ &=(\mu_i/\mu_j)R(0,M_i)^2+R(0,M_j)^2\end{aligned}$$

and

$$(4.16)\qquad r(0,M_{ij}')^2=r(0,M_i')^2=(\mu_i/\mu_j)r(0,M_i)^2=\mu_i/\mu_j.$$

Therefore, from (4.8) we have

$$(4.17)\qquad (R(0,M_{ij}')/r(0,M_{ij}'))^2=(\mu_i c_{ii}^2+\mu_j c_{jj}^2)/\min(\mu_i,\mu_j).$$

In order to show that $R(0, M'_{ij})/r(0, M'_{ij})$ gives the Schwarz constant $c(M'_{ij}, M'_{ij})(= c(M_{ij}, M_{ij}))$, from (2.9) it is sufficient to prove

$$|(d\tilde{f}_0(0)/d\tilde{Z}')\tilde{U}| = (R(0, M'_{ij})/r(0, M'_{ij}))|\tilde{U}|, \tag{4.18}$$

$\tilde{Z}' = {}^t({}^t\tilde{Z}'_i, {}^t\tilde{Z}_j)$, $Z' \in M'_{ij}$, for some $f_0 \in \mathrm{Hol}\,(M'_{ij}, M'_{ij})$ and some $U \in \mathbf{C}^{\dim M'_{ij}} - \{0\}$ (see (4.9)). We can construct a required extremal mapping f_0 in the following. Set

$$\tilde{f}_0(Z') = A\tilde{Z}' = \begin{cases} P'\tilde{Z}' & \text{for } i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \\ P'Q'\tilde{Z}' & \text{for } i = \mathrm{IV}, \end{cases}$$

$Z' \in M'_{ij}$, where

$$P' = \begin{pmatrix} \tilde{D}_i & 0 \\ (\mu_j/\mu_i)^{1/2}\tilde{D}_j & 0 \end{pmatrix} \quad \text{and} \quad Q' = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

are square matrices of order $\dim M_{ij}$ and further $\tilde{D}_i$ $(i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ and Q are defined in (4.5) and (4.10).

Let z_1 (respectively z'_1) be the first element of $\tilde{Z}_i$ (respectively $\tilde{Z}'_i$), then we have

$$|z'_1| = (\mu_i/\mu_j)^{1/2}|z_1| \begin{cases} < (\mu_i/\mu_j)^{1/2}, & i = \mathrm{I}, \mathrm{II}, \mathrm{III}, \\ = (\mu_i/\mu_j)^{1/2}\sqrt{2}x_1 < (\mu_i/\mu_j)^{1/2}\sqrt{2}, & i = \mathrm{IV}, \end{cases}$$

where x_1 denotes the first element of $X_{\mathrm{IV}} \in M_{\mathrm{IV}}(X_{\mathrm{IV}})$ with $z_1 = \sqrt{2}x_1$ and $|x_1| < 1$. Further put $D'_i = (\mu_i/\mu_j)^{1/2}D_i$, then $D'_i \in \partial M'_i$.

Now, for each $i\,(= \mathrm{I}, \mathrm{II}, \mathrm{III})$ and $Z' \in M'_{ij}$ we have $P'\tilde{Z}' = {}^t({}^t\tilde{D}_i, (\mu_j/\mu_i)^{1/2}\,{}^t\tilde{D}_j)z'_1 = {}^t({}^t\tilde{D}'_i, {}^t\tilde{D}_j)z_1 \in M'_{ij}$, $\tilde{Z}' = {}^t({}^t\tilde{Z}'_i, {}^t\tilde{Z}_j) = {}^t((\mu_i/\mu_j)^{1/2}\,{}^t\tilde{Z}_i, {}^t\tilde{Z}_j)$ and for $i = \mathrm{IV}$ we have

$$\begin{aligned} P'Q'\tilde{Z}' &= {}^t({}^t\tilde{D}_i, (\mu_j/\mu_i)^{1/2}\tilde{D}_j)(z'_1 - \sqrt{-1}z'_2)/\sqrt{2} \\ &= {}^t({}^t\tilde{D}'_i, {}^t\tilde{D}_j)(x_1 - \sqrt{-1}x_2) \in M'_{ij} \end{aligned}$$

since $Z_{\mathrm{IV}} = {}^t(z_1, z_2, \ldots, z_n) = \sqrt{2}\ {}^t(x_1, x_2, \ldots, x_n) = \sqrt{2}X_{\mathrm{IV}}$ (see the definition of $M_{\mathrm{IV}}(Z_{\mathrm{IV}})$) and $|x_1 \pm \sqrt{-1}x_2| < 1$ [**6**, p. 485].

On the other hand, for each $i(= \mathrm{I} \sim \mathrm{IV})$ we have, from (4.15) and (4.16),

$$\begin{aligned} |(df_0(0)/d\tilde{Z}')\tilde{U}|^2 &= |{}^t({}^t\tilde{D}_i, (\mu_j/\mu_i)^{1/2}\,{}^t\tilde{D}_j|^2 \\ &= R(0, M_i)^2 + (\mu_j/\mu_i)R(0, M_j)^2 \\ &= (R(0, M'_{ij})/r(0, M'_{ij}))^2|\tilde{U}|^2, \end{aligned}$$

where $\tilde{U} = {}^t({}^t\tilde{U}_i, {}^t\tilde{U}_j)$ with $\tilde{U}_i = {}^t(1, 0, \ldots, 0)$ $(i = \mathrm{I}, \mathrm{II}, \mathrm{III})$, $\tilde{U}_{\mathrm{IV}} = (1, \sqrt{-1}, 0, \ldots, 0)$ $(i = \mathrm{IV})$ and $\tilde{U}_j = \tilde{0}$ $(j = \mathrm{I} \sim \mathrm{IV})$. This completes the proof. □

Theorem 4.2. *Let M and N be arbitrary* (*irreducible or reducible*) *classical Cartan domains and $\underline{K}_D$ (respectively $\overline{K}_D$) be the* inf (*respectively* sup) *of the holomorphic sectional curvature with respect to the Bergman metric for a bounded domain D, then the generalized Schwarz constant $c(M, N)$ is given by*

$$c(M, N) = \hat{c}(M', N') = (\underline{K}_M/\overline{K}_N)^{1/2}, \tag{4.19}$$

where M' and N' are suitable linear images of M and N, respectively, and have Prop (A) *at* 0.

Proof. Since the Bergman metrics, the Schwarz constants and the holomorphic sectional curvatures are biholomorphically invariant, then, by the same way as in the proof of Theorem 4.1, we have

$$c(M, N) = \hat{c}(M', N') = \Gamma(0, N')/\gamma(0, M').$$

On the other hand, using $\underline{K}_{M_i \times M_j} = \min\{\underline{K}_{M_i}, \underline{K}_{M_j}\}$, $\overline{K}^{-1}_{M_i \times M_j} = \overline{K}^{-1}_{M_i} + \overline{K}^{-1}_{M_j}$ [**6**], (4.15) and (4.16), we have

$$c_{ij} = c(M_i, M_j) = \Gamma(0, M_j)/\gamma(0, M_i) = (\underline{K}_{M_i}/\overline{K}_{M_j})^{1/2}, \quad i, j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV},$$

from (4.7), Lemma 4.1 and the following table [**6**]

j	I	II	III	IV
$\overline{K}_{M_j}$	$-2/n(m+n)$	$-2/n(n+1)$	$-1/[n/2](n-1)$	$-1/n$
$\underline{K}_{M_j}$	$-2/(m+n)$	$-2/(n+1)$	$-1/(n-1)$	$-2/n$

.

In general, $\Gamma(0,N')/\gamma(0,M') = (\underline{K}_{M'}/\overline{K}_{N'})^{1/2} = (\underline{K}_M/\overline{K}_N)^{1/2}$. This completes the proof. □

Example 4.2. (1) $c(M_{\mathrm{I}(m\times n)}, M_{\mathrm{II}(k)})^2 = k(k+1)/(m+n) = \underline{K}_{M_{\mathrm{I}(m\times n)}}/\overline{K}_{M_{\mathrm{II}(k)}}$.

(2) For $M = M_{\mathrm{I}(m\times n)} \times M_{\mathrm{II}(n)}$ and $N = M_{\mathrm{III}(n)} \times M_{\mathrm{IV}(n)}$, set $M' = M_{\mathrm{I}(m\times n)} \times M'_{\mathrm{II}(n)}$ and $N' = M_{\mathrm{III}(n)} \times M'_{\mathrm{IV}(n)}$, where $M'_{\mathrm{II}(n)} = ((n+1)/(m+n))^{1/2} M_{\mathrm{II}(n)}$ and $M'_{\mathrm{IV}(n)} = (n/2(n-1))^{1/2} M_{\mathrm{IV}(n)}$, then by direct calculations we get

$$c(M',N') = \hat{c}(M',N') = 2((n-1)/(n+1))([n/2] + n/(n-1)) = \underline{K}_M/\overline{K}_N$$

since $R(0,N')^2 = R(0,M_{\mathrm{II}(n)})^2 + R(0,M'_{\mathrm{IV}(n)})^2 = [n/2] + n/(n-1)$, $\mu(0,N') = \mu(0,M_{\mathrm{III}(n)}) = \mu(0,M'_{\mathrm{IV}(n)}) = 2(n-1)$, $r(0,M')^2 = (n+1)/(m+n)$ and $\mu(0,M') = m+n$ hold.

5. Geometric meaning of the Schwarz constants and Prop **(A).** First we need a slight modification of Theorem 1 of Kubota [**5**].

Lemma 5.1. *For any* M_j $(j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ *with* Prop (A) *and any* $\lambda > 0$, *set*

$$\rho_{M_j}(\lambda) = \sup\{\rho_{f,M_j} = r(0, f(M_j)) \mid f \in \mathrm{Hol}\,(M_j, B_n(0,\lambda)), \; 0 \in f(M_j)\}, n = \dim M_j, \tag{5.1}$$

then we have

$$\begin{aligned}\rho_{M_j} &= \lambda/R(0,M_j) \qquad [\mathbf{5}, \text{Lemma 1}] \\ &= \lambda/c_{jj} \qquad (\text{see } (4.7)).\end{aligned} \tag{5.2}$$

Theorem 5.1. *For any* $M_j (j = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV})$ *with* Prop (A) *we have*

(5.3)
$$\begin{aligned} c_{jj} &= R(0,M_j)/r(0,M_j) = (\underline{K}_{M_j}/\overline{K}_{M_j})^{1/2} \\ &= \inf_{f,t}\{R(t,f(M_j))/r(t,f(M_j)) \mid f \in \mathrm{Hol}^{\,n}(M_j), \; f(0) = t\}.\end{aligned}$$

Proof. Equation (4.18) for $M = N = M_j$ gives the first part of (5.3). Without loss of generality, we may treat the case of $f(0) = 0$. For any $\lambda(> 0)$ put

$$\mathrm{Hol}_0(M_j : \lambda) = \{f \in \mathrm{Hol}^n(M_j) \mid R(0, f(M_j)) = \lambda,\ f(0) = 0\},$$

then from Theorem 4.1 and Lemma 5.1, we have

$$\frac{R(0, f(M_j))}{r(0, f(M_j))} = \frac{\lambda}{r(0, f(M_j))} \geq \frac{\lambda}{\rho_{M_j}(\lambda)}$$
$$= R(0, M_j) = \frac{R(0, M_j)}{r(0, M_j)} = c_{jj}$$

for $f \in \mathrm{Hol}_0(M_j : \lambda)$. Since $\lambda(> 0)$ is arbitrarily chosen, then we obtain (5.3). □

Remark 5.1. By Theorem 5.1 for $f \in \mathrm{Hol}^n(M_j)$ with $f(0) = t$ we have

$$R(0, M_j)/r(0, M_j) \leq R(t, f(M_j))/r(t, f(M_j)),$$

which shows that any classical Cartan domain M_j satisfying Prop (A) at 0 is most close to a ball in the $\mathrm{Hol}^n(M_j)$-equivalent class of domains.

Appendix

6. On Prop **(A).** Many canonical domains are more or less related to the L_2-minimum problems for some classes of holomorphic mappings (see [**9**]).

A bounded domain D in $\mathbf{C}^n$ with $k_D(z, 0) = k_D(0, 0)$ for $z \in D$ is called a *Bergman minimal domain* (with center at 0) and it has the minimum volume in the $F = \{f \in \mathrm{Hol}^n(D) \mid f(P) = 0, \det(df(P)/dz) = 1\}$-equivalent class of domains [**1**].

Any bounded complete Carathéodory circular domain (shortly a bounded balanced domain) M, say a classical Cartan domain, is a Bergman minimal and is also a *representative* ($\Leftrightarrow T_M(z, 0) = T_M(0, 0)$ for $z \in M$) *domain* with the same center at 0 (shortly an $m-r$ domain) [**7**].

For the class $G=\{f\in H^n(D)\cap L_2(D)|f(0)=0,\ |\det(df(0)/dz)|=1\}$, an extremal mapping w_0 in this class, which minimizes the moment of inertia $I(D)=\int_{w(D)}|w|^2\omega_w$, defines the *Mitchell minimal domain* $w_0(D)$ with center at 0 [**9, 10, 13**].

Prop (A) has the following extremal properties:

Theorem 6.1. *Let D be an $m-r$ domain in $\mathbf{C}^n$; then the following are equivalent.*

(1) *D has* Prop (A) *at* $0 : T_D(0,0)=a^2E_n,\ a>0$.

(2) *D is a Mitchell minimal domain with center at* 0.

In particular, if D is a bounded balanced domain, then (1), (2) *and*

(3) *D has* Prop (A') *at* $0 : \int_D zz^*\omega_z = a'^2E_n, \quad a'>0$ [**9**] *are equivalent.*

Further, for a balanced domain D with one of (1) $\sim$ (3) *satisfies*

$$a^2{a'}^2=\mathrm{vol}\,(D). \tag{6.1}$$

Proof. Let $m_D^1(z,0)$ and $M_D^{0\ E_n}(z,0)$ be the minimal mappings of $\int_D|J_f(z)|^2\omega_z$ for $f\in F$ and $\int_D|g(z)|^2\omega_z$ for $g\in H=\{f\in \mathrm{Hol}^n(D)|\ f(0)=0,\ df(0)/dz=E_n\}$, respectively, then we have $m_D^1(z,0)=k_D(z,0)/k_D(0,0)=1$ and $M_D^{0\ E_n}(z,0)/m_D^1(z,0)=T_D^{-1}(0,0)\int_0^z T_D(t,0)\,dt=z$ for $z\in D$ (see details in [**9**]). Therefore, for an $m-r$ domain D we have

$$M_D^{0\ E_n}(z,0)=z, \qquad z\in D. \tag{6.2}$$

On the other hand, D is a Mitchell minimal domain if and only if D satisfies

$$M_D^{0\ E_n}(z,0)=z \quad \text{and} \quad T_D(0,0)=a^2E_n, \qquad a>0, \qquad [\mathbf{9,\ 13}]. \tag{6.3}$$

This shows (1) $\Leftrightarrow$ (2).

Suppose that D is a balanced domain, then D is an $m-r$ domain and the linear part of an orthonormal system of D is given by $B_1z=$

$\partial k_D(z,0)/\partial z^*$, where B_1 is a constant $n \times n$ matrix [**1, 4**]. Hence, we have

$$T_D(0,0) = (\partial^2 k_D(0,0)/\partial z^* \partial z)/k_D(0,0) = B_1^* B_1 \mathrm{vol}\,(D)$$

since $k_D(0,0) = \mathrm{vol}\,(D)^{-1}$ [**1**]. Therefore, we obtain

$$\int_D zz^* \omega_z = (B_1^* B_1)^{-1} = \mathrm{vol}\,(D) T_D^{-1}(0,0).$$

This shows (3) $\Leftrightarrow$ (2). (6.1) is clear. □

For a bounded simply connected Lu-Qi Keng domain ($\Leftrightarrow k_D(z,t) \neq 0$ for $(z,t) \in D \times D$), the image domain N of D under the *normal mapping*

$$\eta(z) = T_D^{-1/2}(0,0) \int_0^z T_D(t,0)\,dt, \qquad t \in D, \tag{6.4}$$

is called a *normal domain* with center at 0. It is known that the normal domain N of D with center at 0 is unique up to unitary linear transformations in $\mathrm{Bihol}\,(D)$-equivalent class of domains and satisfies

$$T_N(z,0) = T_N(0,0) = E_n \tag{6.5}$$

and vice versa (see details in [**8**]).

Lemma 6.1. *A formally defined normal domain N of a bounded balanced domain D is similarly equivalent to a Mitchell minimal domain M of D with the same center at 0 up to unitary linear and similar transformations.*

Proof. N is a representative domain with $\mathrm{Prop}\,(A)$ from (6.5).

Since D is an $m - r$ domain with center at 0, then from (6.4) the normal mapping $\eta(z)$ equals $T_D^{1/2}(0,0)z = V^*\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)Vz$, $V^*V = E_n$, where $T_D(0,0) = V^*\mathrm{diag}\,(\lambda_1^2, \ldots, \lambda_n^2)V$ $(\lambda_1 \geq \cdots \geq \lambda_n > 0)$. Take a linear mapping $u(z) = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)Vz/(\prod_{i=1}^n \lambda_i)^{1/n} \in G$ and put $M = u(D)$, then we have $T_M(0,0) = a^2 E_n$ where $a^{2n} = \det T_D(0,0)$. This shows that M is a Mitchell minimal domain of D and $\eta(z) = aV^*u(z)$, i.e., $N = aV^*M$. This completes the proof. □

REFERENCES

1. S. Bergman, *The kernel function and conformal mapping*, Amer. Math. Soc., New York, 1950.

2. J. Burbea, *Inequalities between intrinsic metrics*, Proc. Amer. Math. Soc. **67** (1977), 50–54.

3. I. Graham, *Boundary behavior of the Carathéodory and Kobayashi metrics on strongly pseudo convex domain in* $\mathbf{C}^n$ *with smooth boundary*, Trans. Amer. Math. Soc. **207** (1975), 219–240.

4. L.K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, Amer. Math. Soc. **6**, 1963.

5. Y. Kubota, *A note on holomorphic imbeddings of classical Cartan domains into the unit ball*, Proc. Amer. Math. Soc. **85** (1982), 65–68.

6. K.H. Look, *Schwarz lemma and analytic invariants*, Scientia Cinica, **7** (1958), 453–504.

7. M. Maschler, *Classes of minimal and representative domains and their kernel functions*, Pacific J. Math. **9** (1958), 763–782.

8. S. Matsuura, *On the normal domains and the geodesics in the bounded symmetric spaces and the projective space*, Sci. Rep. Gunma Univ. **15** (1966), 1–21.

9. ———, *Bergman kernel functions and the three types of canonical domains*, Pacific J. Math. **33** (1970), 363–384.

10. J. Mitchell, *Area and moment of inertia theorems for circular domains in* $\mathbf{C}^n$, Duke Math. J. **33** (1966), 209 –221.

11. S. Ozaki and T. Matsuno, *Note on bounded functions of several complex variables*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **5** (1955), 130–136.

12. H.L. Royden, *The Ahlfors-Schwarz lemma in several complex variables*, Comment. Math. Helvetici **55** (1980), 547–558.

13. T. Tsuboi and S. Matsuura, *Some canonical domains in* $\mathbf{C}^n$ *and moment of inertia theorems*, Duke Math. J. **36** (1969), 517–536.

14. S.T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math **100** (1978), 197–203.

Department of Mathematics, Fukui Institute of Technology, Gakuen 3-6-1, Fukui, 910, Japan