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COMPATIBILITY EQUATIONS FOR ISOMETRIC EMBEDDINGS OF RIEMANNIAN MANIFOLDS

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0. Introduction. Let (M, g) be an n-dimensional Riemannian manifold with the Riemannian metric g. A C^1 mapping F of M into a Euclidean space $\mathbf{R}^{n+p}$ is a local isometric embedding if and only if F satisfies

$$\sum_{\alpha=1}^{n+p} \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j} = g_{ij}(x), \qquad 1 \le i, j \le n, \tag{7}$$

where $(x^1, \dots, x^n)$ is a local coordinate system of M and $g_{ij}(x) = g(\partial/\partial x^i, \partial/\partial x^j)$. Since $g_{ij} = g_{ji}$, the number of equations in (7) is $n(n+1)/2$ and thus the system (7) is underdetermined if $p > n(n-1)/2$ and overdetermined if $p < n(n-1)/2$.

In this paper we study a method of prolongation of (7) and conditions on g_{ij} under which (7) can be prolonged to an elliptic system, and discuss some of their geometric consequences. We restrict our interest to the case $p \le n(n-1)/2$, which is a necessary condition for an isometric embedding to be elliptic.

A *compatibility equation* of (7) is an equation obtained by prolongation, that is, a process of differentiation and algebraic operations on (7).

In Section 1 of this paper, we construct compatibility equations of (7) by a method due to A. Finzi [**5**] and show that the classical equations of Gauss are compatibility equations of this type. These equations, which will be called compatibility equations of Finzi type, are the consequences of the cancellation of the principal parts in the process of prolongations of the original system. Thus they reveal properties of the solutions that are not exposed in the principal part.

In Section 2 we prove that a hypersurface H of M is characteristic for a certain system of compatibility equations if and only if H is an

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asymptotic hypersurface (Corollary 10). A related result by Jacobowitz [**9**] states that in the cases $p \leq n(n-1)/2$, an analytic submanifold M^n of $\mathbf{R}^{n+p}$ admits an isometric deformation leaving a hypersurface H^{n-1} fixed only if H^{n-1} is asymptotic. On the other hand, Tenenblat [**21**] showed that if M^n is an analytic submanifold of $\mathbf{R}^{n+p}$, $p = n(n-1)/2$, there exists locally an analytic nontrivial infinitesimal isometric deformation on a neighborhood of a nonasymptotic hypersurface H^{n-1} of M^n. These results are based on the observation that if a hypersurface H of M is characteristic for the equations for infinitesimal isometric deformations then H is asymptotic. Using Corollary 10 we prove an interesting result (Corollary 11) that if a C^2 isometric embedding of M into a Euclidean space is C^∞ except at a hypersurface H of M, then H must be asymptotic. This means that we can fold a submanifold of a Euclidean space twice continuously differentially only along asymptotic manifolds. We do not yet know whether the C^2 assumption can be weakened to C^1 or C^0. For the case of codimension 1, this question gives rise to a problem of determining singular sets of a real Monge-Ampère equation, which will be discussed elsewhere [**4**].

In Section 3 we study ellipticity of isometric embeddings. The notion of ellipticity of embeddings was first introduced by N. Tanaka for the purpose of studying the rigidity of isometric embeddings. He defined an embedding to be elliptic if the second fundamental form for each normal has two nonzero eigenvalues of the same sign. In [**19**] he showed that if F is an elliptic embedding, then a first order linear system of partial differential equations associated with infinitesimal isometric deformations of F is elliptic, and thus there exists a neighborhood U of F with respect to the C^3-topology in the class of embeddings, that the dimension of infinitesimal isometric deformations of any embedding in U is less than or equal to that of F. It follows that any two isometric embeddings contained in a neighborhood of an infinitesimally rigid elliptic embedding F are congruent (Theorem 13). The question remains of whether an elliptic isometric embedding of a compact Riemannian manifold into an Euclidean space is rigid (cf. [**22**]).

Tanaka's ellipticity is a geometric property of an embedding F expressed in terms of the second fundamental form of F. We show that it is equivalent to the ellipticity of a certain system of compatibility equations of Finzi type (Theorem 15). Then a consequence is that if M is an analytic Riemannian manifold and F an isometric embedding of M into an Euclidean space which is elliptic in Tanaka's sense, then F is analytic provided F is twice continuously differentiable. This is a generalization of the result of one of the authors [**7**]. Finally, we prove a local rigidity theorem for the class of elliptic isometric embeddings of real analytic Riemannian manifolds into an Euclidean space.

Our method in this paper is a jet theoretic approach to partial differential equations. For the general theory, we refer to [**12, 16, 17**].

Another approach to the isometric embedding problems is E. Cartan's method of exterior differential systems, found in various references [**1, 2, 3, 6, 20**].

All manifolds in this paper are assumed to be smooth(C^∞) and all embeddings are assumed to be C^2, unless otherwise stated.

1. Compatibility equations of Finzi type. In this section we adopt the definitions and notations of Olver [**16**] : Let X be an open subset of $\mathbf{R}^p$ and let $\mathbf{R}^{(n)}$ be a Euclidean space whose coordinates represent all the partial derivatives of a smooth map $u = (u^1, \dots, u^q)$ from X into $\mathbf{R}^q$ of all orders from 0 to n. A multi-index of order r is an unordered r-tuple of integers $J = (j_1, \dots, j_r)$, with $1 \le j_s \le p$. The order of a multi-index J is denoted by $|J|$. A typical point in $\mathbf{R}^{(n)}$ is denoted by $u^{(n)}$, so that

$$u^{(n)} = (u^\alpha_J)_{1\le\alpha\le q,\, 0\le|J|\le n}.$$

The product space $X \times \mathbf{R}^{(n)}$ is called the n^{th} order jet space of the underlying space $X \times \mathbf{R}^q$. For example, in case $p = 2$ and $q = 1$, $X \times \mathbf{R}^{(2)} =$

$$\{(x^1, x^2, u, u_1, u_2, u_{11}, u_{12}, u_{22}) \mid (x^1, x^2) \in X,\ (u, u_1, u_2, u_{11}, u_{12}, u_{22}) \in \mathbf{R}^6\}.$$

A C^n map from X into $\mathbf{R}^q$ defines a section of $X \times \mathbf{R}^{(n)}$: Let $F = (f^1, \dots, f^q)$ be a C^n map from X into $\mathbf{R}^q$. For each $x \in X$, $j^n_x F$ denote the n-jet of F at x, namely F and all the partial derivatives of F up to order n at x. That is,

$$j^n_x F = (\partial_J f^\alpha(x))_{1\le\alpha\le q,\, 0\le|J|\le n}.$$

Then the map

$$\jmath^n F : X \longrightarrow X \times \mathbf{R}^{(n)}$$

defined by $x \mapsto (x, \jmath^n_x F)$, is a section of $X \times \mathbf{R}^{(n)}$. This section $\jmath^n F$ is called the n-graph of F.

Let $\mathcal{A}$ denote the set of real valued smooth functions $a(x, u^{(n)})$ depending on x, u and derivatives of u up to some finite, but unspecified order n, defined on $X \times \mathbf{R}^{(n)}$. An element of $\mathcal{A}$ is called a differential function. The order of a differential function is the order of the highest derivative that occurs. It is easy to see that $\mathcal{A}$ becomes an algebra and the subset $\mathcal{A}^{(n)}$ of $\mathcal{A}$ consisting of the differential functions of order less than or equal to n is a subalgebra. For $m > n$, we define a projection map

$$\mathbf{proj}\,^m_n : \mathcal{A}^{(m)} \longrightarrow \mathcal{A}^{(n)}$$

by

$$\mathbf{proj}\,^m_n(a(x, u^{(m)})) = \begin{cases} a(x, u^{(m)}), & \text{if the order of } a \leq n \\ 0, & \text{if the order of } a > n. \end{cases}$$

For a finite set $\{a_\nu(x, u^{(n)})\}_{1 \leq \nu \leq k}$ of $\mathcal{A}^{(n)}$, we denote by $(\{a_\nu\})$ the ideal in $\mathcal{A}^{(n)}$ generated by $\{a_\nu\}$. An element of $(\{a_\nu\})$ is of the form

$$\sum_{\nu=1}^{k} c_\nu(x, u^{(n)}) a_\nu(x, u^{(n)}), \qquad c_\nu(x, u^{(n)}) \in \mathcal{A}^{(n)}.$$

Let $\{a_\nu\}_{1 \leq \nu \leq k}$ be a finite subset of $\mathcal{A}^{(n)}$ and let $\{b_\nu\}_{1 \leq \nu \leq r}$ be a finite subset of $\mathcal{A}^{(m)}$ containing $\{a_\nu\}_{1 \leq \nu \leq k}$. Then it is clear that $\mathbf{proj}\,^m_n((\{b_\nu\})) \supset (\{a_\nu\})$ and the following example shows that $\mathbf{proj}\,^m_n((\{b_\nu\})) \neq (\{a_\nu\})$, in general.

Example 1. For case $m = 2$ and $n = 1$

$$\mathbf{proj}\,^2_1((\{u_1, u_{11}, u_2 - u_{11}\})) \neq (\{u_1\}).$$

Now consider a system of n^{th} order differential equations

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad 1 \leq \nu \leq l, \tag{1}$$

for unknown functions $u = (u^1, \ldots, u^q)$ of p variables $x = (x^1, \ldots, x^p) \in X$. Each Δ_ν is assumed to be a differential function of order n, namely, $\Delta(x, u^{(n)})$ is smooth in their arguments. So $\Delta = (\Delta_1, \ldots, \Delta_l)$ can be viewed as a smooth map $\Delta : X \times \mathbf{R}^{(n)} \longrightarrow \mathbf{R}^l$ and (1) describes a subset

$$\mathcal{S}_\Delta := \{(x, u^{(n)}) \in X \times \mathbf{R}^{(n)} \mid \Delta_\nu(x, u^{(n)}) = 0,\ 1 \leq \nu \leq l\}$$

of $X \times \mathbf{R}^{(n)}$, called the solution subvariety of (1), on which the map Δ vanishes. A solution of (1) is a C^n map $F : X \longrightarrow \mathbf{R}^q$ whose n-graph is contained in $\mathcal{S}_\Delta$, that is, $\{(x, j^n_x F) | x \in X\} \subset \mathcal{S}_\Delta$. From this point of view, the system (1) is equivalent to the system consisting of the equations

$$(2) \qquad a(x, u^{(n)}) = 0, \qquad a(x, u^{(n)}) \in (\{\Delta_\nu\}_{1 \leq \nu \leq l})$$

since both (1) and (2) have the same zero set in $X \times \mathbf{R}^{(n)}$.

For each nonnegative integer m, the m^{th} prolongation of (1) is the $(n+m)^{\text{th}}$ order system of differential equations $\Delta^{(m)}(x, u^{(n+m)}) = 0$ which consists of

$$(3) \qquad D_J \Delta_\nu(x, u^{(n+m)}) = 0, \qquad 1 \leq \nu \leq l, \quad 0 \leq |J| \leq m,$$

where $D_J = D_{(j_1, \ldots, j_r)} = D_{j_1} \circ \cdots \circ D_{j_r}$ is a composition of total differential operators. The 0^{th} prolongation is the original system itself. As previously stated, the system (3) is equivalent to the system consisting of the equations

$$(4) \quad a(x, u^{(n+m)}) = 0, \qquad a(x, u^{(n+m)}) \in (\{D_J \Delta_\nu\}_{1 \leq \nu \leq l,\, 0 \leq |J| \leq m})$$

so that we call (4) also the m^{th} prolongation of (1). It is easy to see that (4) must be satisfied by any C^{n+m} solution of (1). So the m^{th} prolongation $\Delta^{(m)}(x, u^{(n+m)}) = 0$ describes the subset $\mathcal{S}_{\Delta^{(m)}} \subset X \times \mathbf{R}^{(n+m)}$ in which all $(n+m)$-jets of C^{n+m} solutions of (1) are to be contained. By a *compatibility equation* of (1) we shall mean a differential equation $b(x, u^{(n+s)}) = 0$ that is contained in some prolongation of (1). The equations $\Delta^{(s)}(x, u^{(n+s)}) = 0$ as well as $\mathbf{proj}^{n+r}_{n+s}(\Delta^{(r)}) = 0$ are to be satisfied by any C^{n+s} solutions, for all $r \geq s$. A compatibility equation $b(x, u^{(n+s)}) = 0$, where

$b(x,u^{(n+s)}) \in \mathbf{proj}\,^{n+r}_{n+s}(\Delta^{(r)}) \setminus \Delta^{(s)}$, has information on the properties of C^{n+s} solutions that the s^{th} prolongations $\Delta^{(s)}(x,u^{(n+s)}) = 0$ do not show explicitly. It is called a *compatibility equation of Finzi type*. In particular, suppose that there are homogeneous differential operators

$$\mathcal{L}_\nu = \sum_{|J|=m} a^J_\nu(x,u^{(n+m)}) D_J, \qquad \nu = 1,\dots,l,$$

which are not all zero at any $(x,u^{(n+m)}) \in \mathcal{S}_{\Delta^{(m)}}$, such that the combination $\sum_{\nu=1}^{l} \mathcal{L}_\nu \Delta_\nu$ depends only on the derivatives of order at most $m+n-1$ (the principal part of each $\mathcal{L}_\nu\Delta_\nu$ being cancelled out in the process of summation). We then obtain a compatibility equation $\sum_{\nu=1}^{l} \mathcal{L}_\nu \Delta_\nu = 0$, which reveal the properties of solutions that are due to the lower order terms of (1).

An invariant for (1) is a differential function $a(x,u^{(m)})$ which does not change under the choice of solutions of (1). That is, if both F and $\widetilde{F}$ are C^m solutions of (1), then $a(x, \jmath^m_x F) = a(x, \jmath^m_x \widetilde{F})$, for each $x \in X$. Given a compatibility equation $c(x,u^{(m)}) = 0$, by extracting a function of independent variables from $c(x,u^{(m)})$, we obtain a trivial invariant for (1): Let $c(x,u^{(m)}) = a(x,u^{(m)}) - b(x)$. Then $c(x,u^{(m)}) = 0$ implies that $a(x, \jmath^m_x F) = b(x)$, for any C^m solution F of (1). So $a(x,u^{(m)})$ is an invariant for (1).

If (1) is determined or overdetermined, that is $l \geq q$, we form a matrix $\mathbf{M}(\xi) = \mathbf{M}_\Delta(\xi; x, u^{(n)})$ of size $l \times q$ whose entries are the homogeneous polynomials

$$\mathbf{M}_{\nu\alpha}(\xi) = \sum_{|J|=n} \left\{ \frac{\partial \Delta_\nu}{\partial u^\alpha_J}(x,u^{(n)}) \right\} \cdot \xi_J, \tag{5}$$

of degree n of $\xi = (\xi_1,\dots,\xi_p)$, where $\xi_J := \xi_{j_1} \cdots \xi_{j_n}$. We call this matrix the principal symbol of (1).

Definition 2. Given a point $(x_0, u^{(n)}_0) \in \mathcal{S}_\Delta$,

(a) a non-zero vector $\xi = (\xi_1,\dots,\xi_p) \in \mathbf{R}^p$ is a noncharacteristic direction (respectively characteristic direction) to (1) at $(x_0, u^{(n)}_0)$ if $\mathbf{M}_\Delta(\xi; x_0, u^{(n)}_0)$ is of maximal rank (respectively not of maximal rank).

(b) A hypersurface $V := \{x \in X \mid \psi(x) = c\}$ of X is noncharacteristic (respectively characteristic) to (1) at $(x_0, u_0^{(n)})$ if $\xi = \mathrm{grad}\psi(x_0)$ is noncharacteristic (respectively characteristic) to (1) at $(x_0, u_0^{(n)})$.

(c) The system (1) is elliptic at $(x_0, u_0^{(n)})$ if there is no characteristic direction at $(x_0, u_0^{(n)})$.

Definition 3. Given a solution F of (1),

(a) A hypersurface $V := \{x \in X \mid \psi(x) = c\}$ of X is noncharacteristic (respectively characteristic) to (1) at F if $\xi = \mathrm{grad}\,\psi\,(x)$ is noncharacteristic (respectively characteristic) to (1) at $(x, \jmath_x^n F)$ for each $x \in V$.

(b) The system (1) is elliptic at a solution F, if it is elliptic at $(x, \jmath_x^n F)$, for all $x \in X$.

A sufficient condition on a determined system $\Delta = 0$ for a Finzi type compatibility equation to exist is found in the following

Theorem 4 (Finzi [**5**] or [**16**]). *Suppose*

$$\Delta_\nu(x, u^{(n)}) \ = 0\ , \qquad 1 \le \nu \le q,$$

is an n^{th} order determined system of differential equations for functions $u = (u^1, \ldots, u^q)$ of $x = (x^1, \ldots, x^p) \in X$, where X is an open subset of $\mathbf{R}^p$. Suppose that Δ has no noncharacteristic directions at $(x_0, u_0^{(n)}) \in \mathcal{S}_\Delta$. Then there exist homogeneous m^{th} order differential operators

$$\mathcal{L}_\nu = \sum_{|J|=m} a_\nu^J(x, u^{(n)}) D_J, \qquad 1 \le \nu \le q,$$

which are not all zero at $(x_0, u_0^{(n)})$, such that at $(x_0, u_0^{(n)})$ the combination $\sum_{\nu=1}^{l} \mathcal{L}_\nu \Delta_\nu$ depends only on derivatives of u of order at most $m + n - 1$, that is,

$$\sum_{\nu=1}^{l} \mathcal{L}_\nu \Delta_\nu(x_0, u_0^{(m+n)}) = b(x_0, u_0^{(k)}),$$

for some differential function $b(x, u^{(k)})$ of order $k \le m + n - 1$.

Moreover, if there are no noncharacteristic directions for Δ *for all* $(x, u^{(n)})$ *in some relatively open subset* $\mathcal{S}_\Delta \cap V$*, with* V *open in* $X \times \mathbf{R}^{(n)}$*, then the differential operators* $\mathcal{L}_\nu$ *depend smoothly on* $(x, u^{(n)})$*.*

The key to the proof is the observation of the fact that Δ has no noncharacteristic directions at $(x_0, u_0^{(n)})$ if and only if $\det[\mathbf{M}_\Delta(\xi; x_0, u_0^{(n)})] = 0$, for all $\xi \in \mathbf{R}^p$. Here, we observe that the above theorem holds for overdetermined systems also. For then any determined subsystem will satisfy the hypothesis of the Theorem 4. So, for the existence of Finzi type compatibility equations we have the following

Corollary 5. *Suppose*

$$\Delta_\nu(x, u^{(n)}) = 0 , \qquad \nu = 1, \dots, l,$$

is an n^{th} *order determined or overdetermined system of differential equations for functions* $u = (u^1, \dots, u^q)$ *of* $x = (x^1, \dots, x^p) \in X$*, where* X *is an open subset of* $\mathbf{R}^p$*. If, for each* $(x, u^{(n)}) \in \mathcal{S}_\Delta$*,* $\mathbf{M}_\Delta(\xi; x, u^{(n)})$ *is not of maximal rank, for any* $\xi \in \mathbf{R}^p$*, then there is a compatibility equation of Finzi type.*

Since the proof of Theorem 4 is constructive, if the hypothesis of Corollary 5 is satisfied, we can construct compatibility equations of Finzi type.

Let (M, g) be an n-dimensional Riemannian manifold, and $F : M \longrightarrow \mathbf{R}^{n+p}$ be an isometric embedding of M into a Euclidean space $\mathbf{R}^{n+p}$. Let $\widetilde{M} = F(M)$. Given a reference point $P \in M$, we assume $F(P) = \widetilde{O}$, for simplicity, where $\widetilde{O}$ is the origin of $\mathbf{R}^{n+p}$. Let $T_{\widetilde{O}}\widetilde{M}$ be the tangent space of $\widetilde{M}$ at $\widetilde{O}$, and $[T_{\widetilde{O}}\widetilde{M}]^\perp$ be the orthogonal complement of $T_{\widetilde{O}}\widetilde{M}$ in $T_{\widetilde{O}}\mathbf{R}^{n+p}$. That is $T_{\widetilde{O}}\widetilde{M} = F_*(T_P M)$ and $[T_{\widetilde{O}}\widetilde{M}]^\perp = [F_*(T_P M)]^\perp$.

Let $(y^1, \dots, y^{n+p})$ be the standard coordinates of $\mathbf{R}^{n+p}$ and let $\partial_{y^1}, \dots, \partial_{y^{n+p}}$ denote the coordinate vector fields. There is no loss of generality in assuming that $T_{\widetilde{O}}\widetilde{M}$ is spanned by $\{\partial_{y^1}, \dots, \partial_{y^n}\}$, and that $\{\partial_{y^{n+1}}, \dots, \partial_{y^{n+p}}\}$ forms a basis of $[T_{\widetilde{O}}\widetilde{M}]^\perp$. Let $(x^1, \dots, x^n)$ be a Riemannian normal coordinate system at P with the coordinate vector fields $\partial_{x^1}, \dots, \partial_{x^n}$. For convenience, we assume that $F_*(\partial_{x^i}|_P) = \partial_{y^i}|_{\widetilde{O}}$,

$i = 1, \ldots, n$. With these coordinates, F is represented by a map from an open neighborhood Ω of the origin O in $\mathbf{R}^n$ into $\mathbf{R}^{n+p}$, $F = (f^1, \ldots, f^{n+p})$, such that $F(O) = \widetilde{O}$ and

$$\frac{\partial f^\alpha}{\partial x^i}(O) \;=\; \begin{cases} 1 & \text{if } i = \alpha \\ 0 & \text{if } i \neq \alpha, \end{cases} \tag{6}$$

At each $x \in \Omega$, let $g_{ij}(x) = g(\partial/\partial x^i, \partial/\partial x^j)$ be the Riemannian metric. Then the map $F : \Omega \longrightarrow \mathbf{R}^{n+p}$ satisfies the following system:

$$\sum_{\alpha=1}^{n+p} u_i^\alpha u_j^\alpha \;=\; g_{ij}(x), \qquad 1 \leq i, j \leq n. \tag{7}$$

(7) is called local isometric embedding equations. Rewrite (7) as

$$\sum_{\alpha=1}^{n+p} u_i^\alpha u_j^\alpha - g_{ij}(x) \;:=\; \Delta_{\mathtt{ij}}(x, u^{(1)}) = 0, \qquad 1 \leq \mathtt{i}, \mathtt{j} \leq n, \tag{8}$$

The number of equations is $n(n+1)/2$ (note that $\Delta_{\mathtt{ij}} = \Delta_{\mathtt{ji}}$), and (8) is determined if $p = n(n-1)/2$ and underdetermined if $p > n(n-1)/2$. In this paper, we consider only the cases $1 \leq p \leq n(n-1)/2$. Then the system (8) is determined or overdetermined. The principal symbol matrix $\mathbf{M}_\Delta(\xi; x, u^{(n)})$ for (8) is of dimension $n^2 \times (n+p)$. The $(n(i-1)+j)$-th row corresponds to the equation $\Delta_{\mathtt{ij}} = 0$. Then the $(n(i-1)+j, \alpha)$ component of $\mathbf{M}_\Delta(\xi; x, u^{(n)})$ is

$$\sum_{k=1}^{n} \left\{ \frac{\partial \Delta_{\mathtt{ij}}}{\partial u_k^\alpha}(x, u^{(1)}) \right\} \cdot \xi_k \;=\; u_j^\alpha \xi_i \;+\; u_i^\alpha \xi_j. \tag{9}$$

It is easy to verify that (8) satisfies the hypothesis of the Corollary 5. In fact, the matrix (9) is of rank n, for all $(x, u^{(n)}) \in \mathcal{S}_\Delta$ and for all $\xi \in \mathbf{R}^n$. Thus there exist Finzi type compatibility equations for (8).

Theorem 6. *For each 4-tuple of integers* $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} = 1, \ldots, n$, *the equation*

$$\sum_{\alpha=1}^{n+p} [u_{ik}^\alpha u_{jl}^\alpha - u_{il}^\alpha u_{jk}^\alpha] - G_{\mathtt{ijkl}}(x) \;:=\; \mathcal{Z}_{\mathtt{ijkl}}(x, u^{(2)}) \;=\; 0, \tag{10}$$

where

$$G_{\mathtt{ijkl}}(x) = \frac{1}{2}\left[\frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}\right],$$

is a compatibility equation of Finzi type for (8).

Proof. For each fixed i, j, k, l, we define second order differential operators by

$$\mathcal{L}_{ab} := \begin{cases} -\frac{1}{2}D_{(j,l)} & \text{if } (a,b) = (i,k) \\ -\frac{1}{2}D_{(i,k)} & \text{if } (a,b) = (j,l) \\ \frac{1}{2}D_{(j,k)} & \text{if } (a,b) = (i,l) \\ \frac{1}{2}D_{(i,l)} & \text{if } (a,b) = (j,k) \\ 0 & \text{otherwise} \end{cases}.$$

Then consider the compatibility equation for (8)

$$\sum_{a=1}^{n}\sum_{b=1}^{n} \mathcal{L}_{ab}\Delta_{\mathtt{ab}} = 0. \tag{11}$$

Substitute $\Delta_{\mathtt{ab}}$ in (11) by (8). Then all the third order partial derivatives of u's in $\mathcal{L}_{ab}\Delta_{\mathtt{ab}}$ cancel out in the process of summation and the left hand side of (11) becomes $\mathcal{Z}_{\mathtt{ijkl}}(x, u^{(2)})$. □

From (10), we obtain Finzi type invariants

$$a_{\mathtt{ijkl}}(x, u^{(2)}) := \sum_{\alpha=1}^{n+p} \left[u^{\alpha}_{ik}u^{\alpha}_{jl} - u^{\alpha}_{il}u^{\alpha}_{jk}\right] \tag{12}$$

for (8). The compatibility equations $\{\mathcal{Z}_{\mathtt{ijkl}}(x, u^{(2)}) = 0\}$ and the Finzi type invariants $\{a_{\mathtt{ijkl}}(x, u^{(2)})\}$ have geometrical meaning, as we shall see. First, we will show that (10) is in fact the Gauss equations. We fix some notations: Let ∇' be the Euclidean connection in $\mathbf{R}^{n+p}$, ∇ the Riemannian connection on M, and α the second fundamental form of M. $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product and R denotes the Riemann curvature tensor of M. Then we have the Gauss formula:

$$(\nabla'_X Y)_{F(P)} = (\nabla_X Y)_P + \alpha(X, Y), \tag{13}$$

for any pair of vector fields X, Y which are tangent to M at $P \in M$. The Gauss equations state that

$$R(X,Y,Z,W) = \langle \alpha(X,Z), \alpha(Y,W)\rangle - \langle \alpha(X,W), \alpha(Y,Z)\rangle, \tag{14}$$

for any $X, Y, Z, W \in T_P(M)$. In local coordinates, using (13), we obtain

$$\langle \alpha(\partial_{x^i}, \partial_{x^k}), \alpha(\partial_{x^j}, \partial_{x^l})\rangle = \sum_{\alpha=1}^{n+p} \left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^k} \frac{\partial^2 f^\alpha}{\partial x^j \partial x^l} \right] - \langle \nabla_{\partial_{x^i}} \partial_{x^k}, \nabla_{\partial_{x^j}} \partial_{x^l}\rangle.$$

Similarly

$$\langle \alpha(\partial_{x^i}, \partial_{x^l}), \alpha(\partial_{x^j}, \partial_{x^k})\rangle = \sum_{\alpha=1}^{n+p} \left[\frac{\partial^2 f^\alpha}{\partial x^i \partial x^l} \frac{\partial^2 f^\alpha}{\partial x^j \partial x^k} \right] - \langle \nabla_{\partial_{x^i}} \partial_{x^l}, \nabla_{\partial_{x^j}} \partial_{x^k}\rangle.$$

Thus if we write

$$T_{\mathtt{ijkl}}(x) := \langle \nabla_{\partial_{x^i}} \partial_{x^k}, \nabla_{\partial_{x^j}} \partial_{x^l}\rangle - \langle \nabla_{\partial_{x^i}} \partial_{x^l}, \nabla_{\partial_{x^j}} \partial_{x^k}\rangle$$

then (14) becomes

$$R_{\mathtt{ijkl}}(x) = \sum_{\alpha=1}^{n+p} [u^\alpha_{ik} u^\alpha_{jl} - u^\alpha_{il} u^\alpha_{jk}] - T_{\mathtt{ijkl}}(x)$$

which is equal to (10) with equality $G_{\mathtt{ijkl}}(x) = T_{\mathtt{ijkl}}(x) + R_{\mathtt{ijkl}}(x)$. And the relationship between the Finzi type invariants $a_{\mathtt{ijkl}}$ and the components $R_{\mathtt{ijkl}}$ of the Riemann curvature tensor of M is given by

$$a_{\mathtt{ijkl}}(x, u^{(2)}) = T_{\mathtt{ijkl}}(x) + R_{\mathtt{ijkl}}(x).$$

In particular, at the reference point P, the Finzi type invariants $a_{\mathtt{ijkl}}$ are the same as the Riemann curvatures $R_{\mathtt{ijkl}}$, since x is a normal coordinate system at P.

In [**1, 2, 3, 6, 17**], the Gauss equations are derived as integrability conditions for the isometric embedding system, using E. Cartan's theory of exterior differential systems.

We consider the following 2^{nd} order system of compatibility equations for (8) consisting of the 1^{st} prolongation $\Delta^{(1)}(x, u^{(2)}) = 0$ and Finzi type compatibility equations (10):

$$(15)\qquad \begin{cases} \Delta_{\mathtt{ij}}\left(x, u^{(1)}\right) = 0, & 1 \le \mathtt{i},\mathtt{j} \le n, \\ \mathcal{F}_{\mathtt{ijk}}\left(x, u^{(2)}\right) = 0, & 1 \le \mathtt{i},\mathtt{j},\mathtt{k} \le n, \\ \mathcal{Z}_{\mathtt{ijkl}}\left(x, u^{(2)}\right) = 0, & 1 \le \mathtt{i},\mathtt{j},\mathtt{k},\mathtt{l} \le n, \end{cases}$$

where

$$\mathcal{F}_{\mathtt{ijk}}(x, u^{(2)}) := D_{(k)}\Delta_{\mathtt{ij}} = \sum_{\alpha=1}^{n+p} [u_j^\alpha u_{ik}^\alpha + u_i^\alpha u_{jk}^\alpha] - (\partial g_{ij}/\partial x^k).$$

Observe that there are duplicates in (15), for example, $\Delta_{\mathtt{ij}} = \Delta_{\mathtt{ji}}$ and $\mathcal{F}_{\mathtt{ijk}} = \mathcal{F}_{\mathtt{jik}}$.

Since (8) has no noncharacteristic direction at any point $(x, u^{(1)}) \in \mathcal{S}_\Delta$ and the principal parts of $\Delta^{(1)}$ come directly from the principal parts of (8) the 1^{st} prolongation $\Delta^{(1)}(x, u^{(2)}) = 0$ has no noncharacteristic direction. But the principal part of the Finzi type equations $\mathcal{Z}_{\mathtt{ijkl}}(x, u^{(2)}) = 0$ comes from the lower order terms so that they put further restrictions on the principal symbol of $\Delta^{(1)}(x, u^{(2)}) = 0$. In fact, in Section 3, we shall show that (15) is elliptic at the solutions satisfying certain geometric conditions.

2. Asymptotic submanifolds. We are considering the isometric embedding of an n-dimensional smooth Riemannian manifold (M, g) into an Euclidean space $\mathbf{R}^{n+p}$, $1 \le p \le n(n-1)/2$. In this section we characterize asymptotic hypersurfaces of M using the compatibility system (15). We first recall the definitions.

Definition 7. A linear subspace W of the tangent space $T_P M$ is asymptotic if there exists a vector N normal to $T_{F(P)}\widetilde{M}$ such that

$$\langle \alpha(X, Y), N \rangle = 0, \qquad \text{for all} \quad X, Y \in W.$$

Definition 8. A submanifold V of M is asymptotic at $z \in V$ if $T_z V$ is asymptotic, and asymptotic if it is asymptotic at each point $z \in V$.

The notion of asymptotic submanifold, which generalizes the idea of the asymptotic lines of a surface in $\mathbf{R}^3$, was used by Jacobowitz [**9, 10**] and Tenenblat [**21**] to study deformation of isometric embeddings. We are interested in asymptotic hypersurfaces, namely, $(n-1)$-dimensional asymptotic submanifolds.

Now consider the following compatibility system for (8) which is a subsystem of (15).

$$\begin{cases} \Delta_{\mathtt{ij}}(x, u^{(1)}) &= 0, \quad \mathtt{i}, \mathtt{j} = 1, \dots, n, \\ \mathtt{H}_{\mathtt{ij}}(x, u^{(2)}) &= 0, \quad \mathtt{i}, \mathtt{j} = 1, \dots, n, \\ \mathtt{L}_{\mathtt{ijk}}(x, u^{(2)}) &= 0, \quad \mathtt{i}, \mathtt{j}, \mathtt{k} = 1, \dots, n, \end{cases} \tag{16}$$

where

$$\mathtt{H}_{\mathtt{ij}}(x, u^{(2)}) := \sum_{\alpha=1}^{n+p} [u_j^\alpha u_{ii}^\alpha + u_i^\alpha u_{ij}^\alpha] - (\partial g_{ij}/\partial x^i)(x),$$

$$\mathtt{L}_{\mathtt{ijk}}(x, u^{(2)}) := \sum_{\alpha=1}^{n+p} [u_{ii}^\alpha u_{jk}^\alpha - u_{ij}^\alpha u_{ik}^\alpha] - \frac{1}{2}\left[\frac{\partial^2 g_{ij}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^i} - \frac{\partial^2 g_{ii}}{\partial x^j \partial x^k}\right].$$

Notice that $\mathtt{H}_{\mathtt{ij}} := \mathcal{F}_{\mathtt{iji}}$ and $\mathtt{L}_{\mathtt{ijk}} := -\mathcal{Z}_{\mathtt{ijki}}$.

Theorem 9. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M \longrightarrow \mathbf{R}^{n+p}$, $1 \le p \le n(n-1)/2$, an isometric embedding of M into a Euclidean space $\mathbf{R}^{n+p}$. Then a hyperplane W of $T_P M$ is characteristic for (16) at $(P, j_P^2 F)$ if and only if it is asymptotic.*

Proof. We may assume that P is the reference point, as in Section 1. Let a nonzero vector $\omega = \omega_1 \partial_{x^1} + \cdots + \omega_n \partial_{x^n}$ define a characteristic hyperplane W for (16) at $(O, j_O^2 F)$. That is, W is a set of vectors which is normal to ω. Then the principal symbol matrix $\mathbf{M}(\omega)$ of (16) at $(O, j_O^2 F)$ is not of maximal rank. We decompose $\mathbf{M}(\omega)$ into $2n+1$

blocks as

$$(17)\qquad \mathbf{M}(\omega;O,j^2_OF)=\begin{pmatrix}\mathbf{M}_\Delta(\omega;O,j^2_OF)\\ \mathbf{M}_{\mathtt{H}}(\omega;O,j^2_OF)\\ \mathbf{M}_{\mathtt{L}}(\omega;O,j^2_OF)\end{pmatrix}=\begin{pmatrix}\mathbf{M}_\Delta(\omega;O,j^2_OF)\\ \mathbf{M}^{\mathtt{1}}_{\mathtt{H}}(\omega;O,j^2_OF)\\ \vdots\\ \mathbf{M}^{\mathtt{n}}_{\mathtt{H}}(\omega;O,j^2_OF)\\ \mathbf{M}^{\mathtt{1}}_{\mathtt{L}}(\omega;O,j^2_OF)\\ \vdots\\ \mathbf{M}^{\mathtt{n}}_{\mathtt{L}}(\omega;O,j^2_OF)\end{pmatrix}$$

where $\mathbf{M}_\Delta(\omega)$ is the principal symbol of the system consisting of n^2 equations, $\Delta_{\mathtt{ij}}(x,u^{(1)})=0$, $\mathtt{i},\mathtt{j}=1,\dots,n$, $\mathbf{M}^{\mathtt{i}}_{\mathtt{H}}(\omega)$ is that of the system consisting of n equations, $\mathtt{H}_{\mathtt{ij}}(x,u^{(2)})=0$, $\mathtt{j}=1,\dots,n$, and $\mathbf{M}^{\mathtt{i}}_{\mathtt{L}}(\omega)$ is that of the system consisting of n^2 equations, $\mathtt{L}_{\mathtt{ijk}}(x,u^{(2)})=0$, $\mathtt{j},\mathtt{k}=1,\dots,n$. Then, $\mathbf{M}_\Delta(\omega;O,j^2_OF)$ is a zero matrix and, noting (6), for each $\mathtt{i}=1,\dots,n$,

$$\mathbf{M}^{\mathtt{i}}_{\mathtt{H}}(\omega;O,j^2_OF)=\begin{pmatrix}\omega_i^2 & 0 & \cdots & 0 & \omega_1\omega_i & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ 0 & \omega_i^2 & \cdots & 0 & \omega_2\omega_i & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ & & & & \vdots & & & & & & & \\ 0 & 0 & \cdots & 0 & \omega_n\omega_i & 0 & \cdots & 0 & \omega_i^2 & 0 & \cdots & 0\end{pmatrix}.$$

$$\begin{array}{cc}\uparrow & \uparrow\\ i\text{-th column} & n\text{-th column}\end{array}$$

The $(n(j-1)+k,\alpha)$ component of $\mathbf{M}^{\mathtt{i}}_{\mathtt{L}}(\omega;O,j^2_OF)$ (the $(n(j-1)+k)$-th row corresponds to the equation $\mathtt{L}_{\mathtt{ijk}}=0$) is

$$\left\{\frac{\partial^2 f^\alpha}{\partial x^j\partial x^k}(O)\right\}\omega_i^2-\left\{\frac{\partial^2 f^\alpha}{\partial x^i\partial x^k}(O)\right\}\omega_i\omega_j\\ -\left\{\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}(O)\right\}\omega_i\omega_k+\left\{\frac{\partial^2 f^\alpha}{(\partial x^i)^2}(O)\right\}\omega_j\omega_k.$$

For $\mathbf{M}_{\mathtt{H}}(\omega;O,j^2_OF)$, the first n columns are linearly independent and the rest ones are zero vectors. Thus the last p columns of $\mathbf{M}_{\mathtt{L}}(\omega;O,j^2_OF)$ must be linearly dependent, for (17) is not of maximal rank. Therefore, if we write the columns of $\mathbf{M}_{\mathtt{L}}(\omega;O,j^2_OF)$ as

$A_1, \ldots, A_{n+p}$, there exist real numbers $a_1, \ldots, a_p$ which are not all zero such that

$$a_1 A_{n+1} + \cdots + a_p A_{n+p} = 0. \tag{18}$$

With these a_i's, we associate a nonzero vector $N_a = a_1\partial_{y^{n+1}} + \cdots + a_p\partial_{y^{n+p}}$ which is normal to $\widetilde{M}$ at $F(P)$. Then (18) holds if and only if for all $1 \le i, j, k \le n$,

$$0 = \langle \alpha(\omega_i \partial_{x^j} - \omega_j \partial_{x^i}\, ,\, \omega_i \partial_{x^k} - \omega_k \partial_{x^i}), N_a \rangle.$$

It is easy to verify that $\{\omega_i \partial_{x^j} - \omega_j \partial_{x^i}\}_{1 \le i,j \le n}$ generate W. In fact, if $\omega_k \ne 0$, $\{\omega_k \partial_{x^j} - \omega_j \partial_{x^k}\}_{1 \le j \le n}$ form a basis of W. Hence we have

$$\langle \alpha(X, Y)\, ,\, N_a \rangle = 0,$$

for all vectors X, Y contained in W. Therefore W is asymptotic at P.

The converse can be proved similarly. □

Corollary 10. *Suppose that (M, g) is a Riemannian manifold of dimension n and that $F : M \longrightarrow \mathbf{R}^{n+p}$, $1 \le p \le n(n-1)/2$, be an isometric embedding of M into a Euclidean space $\mathbf{R}^{n+p}$. Then a hypersurface of M is characteristic to* (16) *at F if and only if it is asymptotic.*

In cases of isometric embeddings $M^n \longrightarrow \mathbf{R}^{n(n+1)/2}$, Tenenblat [**20**] shows that the characteristic $(n-1)$-dimensional submanifolds in Cartan's sense are the asymptotic hypersurfaces.

Corollary 11. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M^n \longrightarrow \mathbf{R}^{n+p}$, $1 \le p \le n(n-1)/2$, an isometric embedding which is twice continuously differentiable. Let a hypersurface H of M divide M into two parts M_1 and M_2. Suppose that both $F|_{M_1 \cup H}$ and $F|_{M_2 \cup H}$ are smooth and that F is not smooth at any point of H. Then H must be an asymptotic hypersurface.*

Proof. Let H be nonasymptotic at P. We may assume that P is the reference point, as in Section 1. Then by Theorem 9, it is noncharacteristic for (16) at $(O, \jmath^2_O F)$, hence noncharacteristic for (16) in an open subset of the solution subvariety of (16). Then, on a neighborhood U of O in X, the values and the 1^{st} derivatives of F uniquely determine all the higher derivatives of F, hence F must be smooth at P. It contradicts the hypothesis. $\square$

3. The ellipticity of isometric embeddings. In this section, we adopt the definitions and notation of Tanaka [**19**]: Let M be an n-dimensional smooth manifold. Let $\Gamma(M,m)$ denote the vector space of all smooth maps of M to $\mathbf{R}^m$ and $\Lambda(M,m)$ the subset of $\Gamma(M,m)$ consisting of all embeddings of M in $\mathbf{R}^m$. Given a positive integer r, we introduce the C^r-topology in $\Gamma(M,m)$ and denote by $\Lambda(M,m)_{C^r}$ the subset $\Lambda(M,m)$ of $\Gamma(M,m)$ equipped with the subspace topology induced by the C^r-topology of $\Gamma(M,m)$. S^2T^* will denote the vector bundle of symmetric tensors of type (0,2) on M.

Let F be a smooth embedding of M into $\mathbf{R}^m$ with $\widetilde{M} = F(M)$. We induce a Riemannian metric g on M by this embedding. Let $P \in M$. Then for any normal vector $N \in [T_{F(P)}\widetilde{M}]^\perp$, we define an element θ_N of $S^2T^*_P$ by

$$\theta_N(X,Y) = \langle \nabla_X \nabla_Y F, N\rangle, \quad \text{for } X, Y \in T_P M,$$

where ∇ is the Levi-Cività connection on M associated with the Riemannian metric g and $\langle \cdot,\cdot\rangle$ denotes the Euclidean inner product, as before. Here

$$\nabla_X \nabla_Y F := \nabla^2 F(X,Y) := (\nabla^2 f^1(X,Y), \dots, \nabla^2 f^m(X,Y)).$$

It is easy to see that $\nabla_X\nabla_Y F = \alpha(X,Y)$. So we call θ_N the second fundamental form of F corresponding to the normal vector N.

Definition 12 (Tanaka [**19**]). A smooth embedding F of M into $\mathbf{R}^m$ is called elliptic at $P \in M$ if it satisfies one of the following equivalent conditions :

(a) For any nonzero normal vector $N \in [T_{F(P)}\widetilde{M}]^\perp$, θ_N has at least two eigenvalues of the same sign.

(b) The subbundle of $S^2T^*_P$ consisting of the second fundamental forms of F contains no nonzero elements of the form $\zeta \cdot \eta$, where ζ and

η are covectors with the same origin. Here $\cdot$ denote the symmetric product, that is, $(\zeta \cdot \eta)(X, Y) = \{\zeta(X)\eta(Y) + \zeta(Y)\eta(X)\}/2$, for $X, Y \in T_P M$.

Now we define F to be elliptic if it is elliptic at each point $P \in M$. The following is a global rigidity theorem for elliptic isometric embeddings.

Theorem 13. [**19**]. *Let F_0 be an embedding of M into $\mathbf{R}^m$. We assume that F_0 is elliptic, $\rho(F_0) = m(m+1)/2$ and that M is compact. Then there exists a neighborhood $U(F_0)$ of F_0 in $\Lambda(M, m)_{C^3}$ having the following property: If $F, F' \in U(F_0)$ and if F and F' induce the same Riemannian metric on M, then there is a Euclidean transformation a of $\mathbf{R}^m$ such that $F' = aF$.*

In Theorem 11, $\rho(F_0)$ denotes the dimension of the space of infinitesimal isometric deformations of F_0 We notice that $\rho(F_0) \geq m(m+1)/2$, in general, and that $\rho(F_0) = m(m+1)/2$ means that the embedding F_0 is infinitesimally rigid.

We observe that Definition 12 makes sense also for the embeddings of differentiability class C^2. In what follows, an elliptic embedding is assumed to be twice continuously differentiable, unless otherwise stated.

On the other hand, we may define ellipticity for a C^2 isometric embedding as follows : An isometric embedding $F : M \longrightarrow \mathbf{R}^{n+p}$ is called elliptic if the system (16) of compatibility equations is elliptic at F, namely, the principal symbol matrix (17) is of maximal rank at any point of two-jets of F.

Now we have two concepts of ellipticity for isometric embeddings, one defined geometrically and the other analytically. We shall show that these two concepts are equivalent.

Lemma 14. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M \longrightarrow \mathbf{R}^{n+p}$, $1 \leq p \leq n(n-1)/2$, is an isometric embedding. Then F is elliptic at P in Tanaka's sense if and only if M has no asymptotic hyperplane at P.*

Proof. Suppose that a hyperplane W of $T_P M$ is asymptotic. Then there exists a nonzero vector $N \in [T_{F(P)}\widetilde{M}]^\perp$ such that $\langle \alpha(X,Y), N\rangle = 0$, for all $X, Y \in W$. We choose an orthonormal basis $\{v_1, \ldots, v_n\}$ of $T_P M$ such that W is spanned by $\{v_1, \ldots, v_{n-1}\}$. With respect to this basis the second fundamental form θ_N of F at P is represented by

$$\begin{aligned}
\theta_N &= \sum_{i=1}^{n}\sum_{j=1}^{n} \theta_N(v_i, v_j)\, v_i^* \otimes v_j^* \\
&= \sum_{i=1}^{n} \theta_N(v_i, v_n)\, v_i^* \otimes v_n^* + \sum_{j=1}^{n} \theta_N(v_n, v_j)\, v_n^* \otimes v_j^* \\
&= \left[\sum_{i=1}^{n} \theta_N(v_i, v_n) v_i^*\right] \otimes v_n^* + v_n^* \otimes \left[\sum_{i=1}^{n} \theta_N(v_i, v_n) v_i^*\right] \\
&= \left[2\sum_{i=1}^{n} \theta_N(v_i, v_n) v_i^*\right] \cdot v_n^*.
\end{aligned}$$

This contradicts the ellipticity of F at P.

Conversely, suppose that F is not elliptic at P in Tanaka's sense. Then there exists a nonzero vector $N = \sum_{r=1}^{p} a_r \partial_{y^{n+r}} \in [T_{\widetilde{O}}\widetilde{M}]^\perp$ and two nonzero covectors $\zeta, \eta \in T_P^* M$ such that $\theta_N = \zeta \cdot \eta$. Let $\zeta = \sum_{r=1}^{n} b_r dx^r$ with $b_\delta \neq 0$ and $\eta = \sum_{s=1}^{n} c_s dx^s$. Then

$$\begin{aligned}
\theta_N &= \left[\sum_{r=1}^{n} b_r dx^r\right] \cdot \left[\sum_{s=1}^{n} c_s dx^s\right] \\
&= \sum_{r=1}^{n}\sum_{s=1}^{n} \left[\frac{1}{2}(b_r c_s + b_s c_r)\right] dx^r \otimes dx^s.
\end{aligned}$$

Let W be an $(n-1)$-dimensional subspace $T_P M$ spanned by

$$\{b_\delta \partial_{x^k} - b_k \partial_{x^\delta} \mid k = 1, \ldots, \delta-1, \delta+1, \ldots, n\}.$$

Then, for each $i, j \in \{1, \ldots, \delta-1, \delta+1, \ldots, n\}$,

$$\begin{aligned}\langle \alpha(b_\delta \partial_{x^i} - b_i \partial_{x^\delta}, b_\delta \partial_{x^j} - b_j \partial_{x^\delta}), N\rangle \\ &= \theta_N(b_\delta \partial_{x^i} - b_i \partial_{x^\delta}, b_\delta \partial_{x^j} - b_j \partial_{x^\delta}) \\ &= \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} \left[\frac{1}{2}(b_r c_s + b_s c_r) \right] dx^r \otimes dx^s \right\} \\ &\qquad \cdot (b_\delta \partial_{x^i} - b_i \partial_{x^\delta}, b_\delta \partial_{x^j} - b_j \partial_{x^\delta}) \\ &= \frac{1}{2}\{(b_i c_j + b_j c_i) b_\delta^2 - (b_i c_\delta + b_\delta c_i) b_\delta b_j \\ &\qquad - (b_\delta c_j + b_j c_\delta) b_i b_\delta + 2 b_\delta c_\delta b_i b_j\} = 0\end{aligned}$$

Thus the subspace W is asymptotic, which contradicts the hypothesis. □

Theorem 15. *Suppose that (M, g) is a Riemannian manifold of dimension n and $F : M \longrightarrow \mathbf{R}^{n+p}$, $1 \leq p \leq n(n-1)/2$, is an isometric embedding. Then F is elliptic in Tanaka's sense if and only if* (16) *is elliptic at F.*

Proof. This follows immediately from Theorem 9 and Lemma 14. □

Now from the analyticity of solutions of elliptic partial differential equations (See [**15**]), it follows that

Corollary 16. *Suppose that (M, g) is a real analytic Riemannian manifold of dimension n and $F : M \longrightarrow \mathbf{R}^{n+p}$, $1 \leq p \leq n(n-1)/2$, is an isometric embedding which is elliptic in Tanaka's sense. Then F is real analytic.*

Proof. Since (M, g) is real analytic, (16) is a real analytic system of partial differential equations. And, by Theorem 15, (16) is elliptic at F which is a C^2 solution. So by the regularity theorem for an elliptic system of partial differential equations, the solution F is real analytic. □

We notice that the ellipticity is not an intrinsic property of manifolds in general but depends on the embedding. But in the cases of codimension 1, ellipticity is invariant under the choice of embeddings. For an isometric embedding F of an n-dimensional Riemannian manifold (M, g) into $\mathbf{R}^{n+1}$, the following are equivalent:

(i) F is elliptic.

(ii) $\widetilde{M}$ has at least two nonzero principal curvatures of the same sign at each point of $\widetilde{M}$.

(iii) $\widetilde{M}$ has a plane section with positive curvature at each point of $\widetilde{M}$.

But the sectional curvatures are invariant under the choice of isometric embedding. Thus for the isometric embeddings of codimension 1, we have

Corollary 17. *Suppose that (M, g) is a real analytic Riemannian manifold of dimension n and $F : M \longrightarrow \mathbf{R}^{n+1}$ is an isometric embedding. If $\widetilde{M}$ has at least two nonzero principal curvatures of the same sign, then F is real analytic. Furthermore, any C^2 isometric embedding of M into $\mathbf{R}^{n+1}$ is real analytic.*

Finally, for the class of elliptic embeddings, we have a local rigidity theorem. In [**9**], using the linearization of the local isometric embedding equation, Jacobowitz proved that if there exists a local analytic isometric deformation of a real analytic isometric embedding $F : M^n \longrightarrow \mathbf{R}^{n+p}$, $1 \leq p \leq n(n-1)/2$, leaving a hypersurface fixed, then this hypersurface is necessarily asymptotic. Thus, an elliptic embedding does not admit such a deformation. Using the compatibility system (16) instead of the linearization, we obtain the following

Theorem 18. *Let (M, g) be a real analytic Riemannian manifold of dimension n and let F_1 and F_2 be two elliptic isometric embeddings of M into $\mathbf{R}^{n+p}$, $1 \leq p \leq n(n-1)/2$. Suppose that $F_1|_H = F_2|_H$ and $dF_1|_H = dF_2|_H$ on a hypersurface H of M. Then $F_1 = F_2$ on a neighborhood of H.*

Proof. Since the embeddings are elliptic, H must be noncharacteristic to (16) at F_1 and at F_2. Then $F_1|_H$ and $dF_1|_H$ determine all derivatives of F_1 on H and $F_2|_H$ and $dF_2|_H$ determine all derivatives of F_2 on H. But then, since $F_1|_H = F_2|_H$ and $dF_1|_H = dF_2|_H$, the values and all the derivatives of F_1 and F_2 coincide on H. Hence $F_1 = F_2$ on a neighborhood of H, by the real analyticity of F_1 and F_2 (Corollary 16). □

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