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WELL-POSED OPTIMIZATION PROBLEMS AND A NEW TOPOLOGY FOR THE CLOSED SUBSETS OF A METRIC SPACE

GERALD BEER AND ROBERTO LUCCHETTI

ABSTRACT. We provide a further analysis of the bounded proximal topology, recently defined in the setting of minimization problems and then studied by the present authors in a unifying article on hyperspace topologies. We exhibit its main topological properties, and we compare it with other hyperspace topologies. We further consider this topology in the context of minimization problems, specifically with respect to problems that are well-posed in the generalized sense (g.w.p.). It is shown that the solution set of such a minimum problem can be recovered from a sequence of level sets of approximating functions and that nearby problems to a given g.w.p. convex function will necessarily have a solution if and only if the underlying space is reflexive. On the other hand, nearby problems need not be g.w.p., even if they have unique minimizers.

1. Introduction. When dealing with minimization problems we have to consider sets that represent not only constraint sets but also functions, as identified with their epigraphs. Thus, topologies on the closed sets of a metric space (called *hyperspace topologies* [**30**]), are a fundamental tool in some aspects of optimization, as for instance in stability analysis. But the best known one—the Hausdorff metric topology [**18, 26**]—is not usually well-suited for this analysis because it fails to work well when sets under analysis are unbounded. The first attempt at overcoming this difficulty was made by using the notions of topological Lim sup and Lim inf of a sequence (or net) of sets [**27**, Section 29]. A sequence $\langle A_n \rangle$ of closed sets is declared *Painlevé-Kuratowski convergent* to the set A if, at the same time, $A = \operatorname{Lim\,sup} A_n$ and $A = \operatorname{Lim\,inf} A_n$. When the metric space X

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is locally compact, the corresponding hyperspace topology is the so-called Fell topology. This is, for most purposes, *the* right hyperspace topology, when X is a finite dimensional linear space. But it turns out that Painlevé-Kuratowski convergence is often too weak in infinite dimensional spaces and, in this setting, convergence of *nets* of sets is not topological.

An important step in defining a good convergence notion in spaces of infinite dimensions was developed by U. Mosco [**31**] for closed convex sets, who considered Kuratowski limits with respect to both the natural topologies with which we can endow a Banach space: the strong and weak topologies. In other words, a sequence $\langle A_n \rangle$ converges in this sense to a set A provided A is at the same time the Lim sup and the Lim inf of the sequence $\langle A_n \rangle$, when X is given the norm topology *and* when X is given the weak topology. In the reflexive setting, this convergence notion, called *Mosco convergence* [**1, 17**], and the corresponding *Mosco topology* τ_M [**7**] have for a long time been considered optimal, especially for their beautiful properties in best approximation problems, convex duality, and in the study of the solutions to variational inequalities. Nevertheless, Mosco convergence has some (unavoidable) weakness; for instance, (i) it behaves badly without reflexivity [**11**]; (ii) it is not stable with respect to certain standard operations on convex sets and functions; and (iii) it fails to reduce to the Hausdorff metric topology on the closed and bounded convex sets.

Partially in response to these shortcomings, a new topology, called usually the *Attouch-Wets topology*, the *bounded Hausdorff topology*, or, in the case of functions identified with their epigraphs, the *epi-distance topology*, has attracted considerable attention. Convergence of a sequence of sets in this (metrizable) topology means uniform convergence of the associated sequence of distance functionals for the sequence on bounded subsets of the underlying space. The initial motivation for investigating it was to obtain Hölder continuity results in stability analysis [**4**]. It has been shown that the bounded Hausdorff topology behaves well in any normed linear space, particularly with respect to constrained problems [**5, 9, 14, 15, 33, 37, 36**].

Since this topology, which we denote by τ_{AW} in the sequel, is much stronger (in infinite dimensions) than the Mosco topology τ_M, it is of interest to search for intermediate convergence notions or topologies that have possibly a broader range of application than τ_{AW}, and that behave better than τ_M in some circumstances. In his recent Ph.D. dissertation, P. Shunmugaraj [**36**] investigated a particular convergence notion introduced in [**2**] which maintains some of the strong stability properties previously proved for τ_{AW} [**15**]. This convergence notion was subsequently shown to be compatible with a fundamental topology on closed sets from the perspective of an overall theory of hyperspaces of a metric space [**16**] and was called the *bounded proximal topology* σ_d therein. The bounded proximal topology is significantly weaker than τ_{AW}; for instance, an increasing sequence of finite dimensional subspaces X_n in a separable Hilbert space H with $H = \mathrm{cl}\,(\cup X_n)$ converges to H in the σ_d sense, but not in the bounded Hausdorff sense. Thus, all Riesz type methods for minimizing a function over an infinite dimensional constraint set can be applied with the former topology, but not with the latter topology.

The goals of this paper are a careful study of the topological character of σ_d in the context of a general metric space and a deeper investigation of its properties in optimization. We determine (i) when equivalent metrics on the space X give rise to the same hyperspace; (ii) which conditions on the space are necessary and sufficient to guarantee that the hyperspace is first countable, second countable and metrizable; and (iii) the relationship of σ_d to nearby hyperspace topologies. The second part of the paper is devoted to applications in optimization, specifically to problems that are well-posed in the generalized sense: the set of minimizers is nonempty and compact and minimizing sequences contain subsequences convergent to a minimizer of the objective function. We show that, if a function which is well-posed in the generalized sense is the σ_d-limit of a sequence of lower semicontinuous functions f_n, then the set of its minimizers can be recovered as the σ_d-limit of certain sublevel sets of the f_n. Then we analyze the properties of unconstrained and constrained minimization problems that are close, in various senses, to a given problem that is well-posed in the generalized sense. In the convex case, it is shown that even existence of a solution for nearby problems cannot be guaranteed, the only exception being when considering problems in reflexive spaces where nearness in the sense of the bounded proximal topology will guarantee existence. On the other hand, existence and uniqueness of nearby problems in the much stronger bounded Hausdorff topology does not guarantee their well-posedness.

2. Preliminaries and presentations of σ_d. Let $\langle X, d\rangle$ be a metric space and x_0 a given point of X that will be (arbitrarily) fixed throughout the paper. $\mathrm{CL}\,(X)$ will indicate the nonempty closed subsets of X, $\mathrm{CLB}\,(X)$ the nonempty closed and bounded subsets, and if X is a linear space, $\mathrm{CLC}\,(X)$ will be the closed nonempty convex sets. In case the specification of the distance d is essential, we shall use the notations $\mathrm{CL}_d(X)$, etc. Similarly, we shall subscript hyperspace topologies when appropriate, e.g., we might write τ_{AW_d} for τ_{AW}. Often, X will be a normed linear space, and in this case, we will denote the origin by θ and the closed solid unit ball by U.

Given a set $F \in \mathrm{CL}\,(X)$, $S_\varepsilon[F]$ denotes the (open) *ε-enlargement* of F, $\{x \in X : d(x,F) < \varepsilon\}$. The *gap* between two elements A and B of $\mathrm{CL}\,(X)$ is defined by $D_d(A,B) = \inf\{d(a,b) : a \in A, b \in B\} = \inf\{d(b,A) : b \in B\}$.

One basic class of hyperspace topologies are the "hit-and-miss" topologies that we shall now describe. For $E \subset X$, we specify the following subsets of $\mathrm{CL}\,(X)$:

$$E^- = \{F \in \mathrm{CL}\,(X) : F \cap E \neq \varnothing\}, \qquad E^+ = \{F \in \mathrm{CL}\,(X) : F \subset E\},$$
$$E^{++} = \{F \in \mathrm{CL}\,(X) : \exists\, \varepsilon > 0 \text{ such that } S_\varepsilon[F] \subset E\}.$$

Observe that another description of E^{++} is $E^{++} = \{F \in \mathrm{CL}\,(X) : D_d(F, E^c) > 0\}$ where E^c denotes the complement of the set E.

The classical *Vietoris topology* [**26, 30**], for instance, can be described as the topology having as a subbase all sets of the form G^+, where G is open, and all sets of the form V^-, where V is open; whereas the *Mosco topology* τ_M [**31, 7, 11**] on the nonempty closed convex subsets $\mathrm{CLC}\,(X)$ of a normed linear space X, compatible with Mosco convergence of sequences of closed convex sets [**7**, Theorem 3.1], has as a subbase all sets of the form V^-, where V is norm open, and all sets of the form $(K^c)^+$, where K is a weakly compact set. Observe that $(K^c)^+ = (K^c)^{++}$ for weakly compact sets.

Hyperspace topologies on a given subfamily of $\mathrm{CL}\,(X)$ can also be presented as weak topologies, i.e., as topologies that are the *weakest* ones making continuous a given family of functionals. Probably the most important and natural example in this sense is the *Wijsman topology* τ_{W_d} [**40, 24, 28, 19, 17, 10**] on $\mathrm{CL}\,(X)$, which is the weakest one such that for each $x \in X$ the function $A \to d(x,A)$ is continuous on CL(X). Also, the Hausdorff metric topology, the Vietoris topology, the Attouch-Wets topology, and the topology of Mosco convergence can be described in this way, and, more generally, it has been shown that this approach leads to a unified theory for hyperspace topologies [**16**].

Finally, a lattice theoretic approach can sometimes be useful in describing these topologies [**24**]. All the usual hyperspace topologies can be defined as the supremum of a lower topology and an upper topology. The lower (respectively, upper) topologies share the property that a neighborhood of a closed set F is automatically a neighborhood of each superset (respectively, subset) of F. Thus, for instance, a subbase for the *lower Vietoris topology* is given by the sets V^- where V ranges over the open sets of X, and a base for the *upper Vietoris topology* is given by all sets of the form G^+ where G ranges over the open sets of X.

Most of the attention of this paper is devoted to the bounded proximal topology σ_d, and the rest of this section is dedicated to its formal definition and to a summary of its descriptions as presented in Section 3 of [**9**]. There are many ways to describe this topology; we choose as a definition the presentation which seems to us the simplest and which, at the same time, is consistent with the point of view of [**9**] (see also [**38**]).

Definition. Let $\langle X, d\rangle$ be a metric space. The *bounded proximal topology* σ_d is the weakest topology τ on CL(X) such that, for each closed and bounded subset B of X, the gap functional $A \to D_d(B, A)$ is τ-continuous on CL(X).

Thus, σ_d is completely regular, being a weak topology [**23**]. Since singleton subsets are bounded sets, σ_d is finer than the Wijsman topology τ_{W_d} so that σ_d is also Hausdorff. Evidently, the weak topology determined by $\{D_d(F, \cdot) : F \in \mathrm{CL}(X)\}$ is finer than σ_d and, following [**13, 16**], we call this stronger topology the *proximal topology*, retaining the notation τ_{δ_d} for this hyperspace. As a hit-and-miss topology, the bounded proximal topology is described as follows ([**16**, Theorem 3.6] or [**38**, Proposition 6]):

Theorem 2.1. *Let $\langle X, d\rangle$ be a metric space. A subbase for σ_d*

consists of all sets of the form V^-, where V is open in X, and all sets of the form $(B^c)^{++}$, where B is a closed and bounded subset of X.

From Theorem 2.1 we immediately see that σ_d is finer than the Mosco topology τ_M on $\mathrm{CLC}(X)$ for any normed linear space X because weakly compact sets are norm bounded. As for local presentations of this topology, we have [**16**, Lemma 3.1]:

Theorem 2.2. *Let $\langle X, d\rangle$ be a metric space, and let $A \in \mathrm{CL}(X)$. Then each of the following families constitutes a local base for the bounded proximal topology at A:*

(1) *All sets of the form*

$$\Phi_A[n; a_1, a_2, \dots, a_k] \equiv \{F \in \mathrm{CL}(X) : F \cap S_n[x_0] \subset S_{1/n}[A], \\ \text{and } \forall\, i \le k, d(a_i, F) < 1/n\}$$

where $\{a_1, a_2, \dots, a_k\}$ is a finite subset of A and $n \in Z^+$;

(2) *All sets of the form*

$$\Theta_A[B; \varepsilon; a_1, a_2, \dots, a_k] \equiv \{F \in \mathrm{CL}(X) : F \cap B \subset S_\varepsilon[A], \\ \text{and } \forall\, i \le k, d(a_i, F) < \varepsilon\}$$

where $\{a_1, a_2, \dots, a_k\} \subset A$, $\varepsilon > 0$, and B is a bounded subset of X;

(3) *All sets of the form*

$$\Lambda_A[B; \varepsilon; x_1, x_2, \dots, x_k] \equiv \{F \in \mathrm{CL}(X) : \forall\, x \in B, d(x, A) - \varepsilon \\ < d(x, F), \text{ and } \forall\, i \le k, d(x_i, F) < d(x_i, A) + \varepsilon\},$$

where $\{x_1, x_2, \dots, x_k\} \subset X$, $\varepsilon > 0$, and B is a bounded subset of X.

The definition of the Attouch-Wets topology presented in Section 1 says that a local base for $\tau_{AW} = \tau_{AW_d}$ at $A \in \mathrm{CL}(X)$ consists of all sets of the form

$$\{F \in \mathrm{CL}(X) : \sup_{x \in B} |d(x, F) - d(x, A)| < \varepsilon\},$$

where B is a bounded subset of X and $\varepsilon > 0$. In view of the local presentation (3) for σ_d above, we have $\sigma_d \subset \tau_{AW}$ on $\mathrm{CL}(X)$. Also, (3) shows again that $\tau_{W_d} \subset \sigma_d$. Another local base for τ_{AW} at $A \in \mathrm{CL}\,(X)$ consists of all sets of the form

$$\Sigma_n[A] \equiv \{F \in \mathrm{CL}\,(X) : F \cap S_n[x_0] \subset S_{1/n}[A], \\ \text{and } A \cap S_n[x_0] \subset S_{1/n}[F]\},$$

where $n \in Z^+$ (see, e.g., [**5, 8, 3, 12**]). Comparison with the local presentation (1) for σ_d confirms the aforementioned inclusion $\sigma_d \subset \tau_{AW}$.

3. Topological properties of σ_d. In this section we nail down the basic properties of σ_d: (i) what properties of the underlying metric determine the topology; (ii) when it is metrizable; and (iii) when it coincides with nearby topologies. As shown in [**12**], compatible metrics d and ρ define the same Attouch-Wets topologies if and only if they define the same bounded sets and admit the same class of functions that are uniformly continuous on bounded sets. Similar arguments, in which the Efremovic lemma [**32**, p. 77] plays a key role, yield the same result for the weaker topology σ_d.

Theorem 3.1. *Let d and ρ be metrics for a set X. The following are equivalent*:

(i) $\sigma_d = \sigma_\rho$ *on* $\mathrm{CL}\,(X)$;

(ii) $\mathrm{CLB}_{\,d}(X) = \mathrm{CLB}_{\,\rho}(X)$, *and for each metric space $\langle Y, d'\rangle$ and, for each $f : X \to Y$, $f : \langle X, d\rangle \to \langle Y, d'\rangle$ is uniformly continuous on bounded subsets of X if and only if $f : \langle X, \rho\rangle \to \langle Y, d'\rangle$ is uniformly continuous on bounded subsets of X.*

We note that there is no need to assume in the hypotheses of Theorem 3.1 that ρ and d define the same topologies. Equivalence of the metrics is guaranteed by condition (i), for $x \to \{x\}$ is easily verified to be an embedding of X into $\mathrm{CL}\,(X)$ equipped with the bounded proximal topology (Michael [**30**] calls this property *admissibility* for the hyperspace). Equivalence is also guaranteed by condition (ii), for (ii) ensures continuity of the identity functions $id : \langle X, d\rangle \to \langle X, \rho\rangle$ and $id : \langle X, \rho\rangle \to \langle X, d\rangle$.

First countability, second countability, and metrizability, results for σ_d largely parallel similar results for the proximal topology τ_{δ_d}. Often, results here can be derived from results of Section 4 of [**13**] using the following relativization lemma.

Lemma 3.2. *Let* $\langle X, d\rangle$ *be a metric space, and let* $A_0 \in \mathrm{CL}(X)$. *Then* $\langle \mathrm{CL}(A_0), \sigma_d\rangle$ *coincides with* $\mathrm{CL}(A_0)$ *equipped with the relative topology it inherits from* $\langle \mathrm{CL}(X), \sigma_d\rangle$.

Proof. Fix $A \in \mathrm{CL}(A_0)$. Then, for each $n \in Z^+$ and $\{a_1, a_2, \dots, a_k\} \subset A$, $\mathrm{CL}(A_0) \cap \Phi_A[n; a_1, a_2, \dots, a_k] = \{F \in \mathrm{CL}(A_0) : F \cap S_n^{A_0}[x_0] \subset S_{1/n}^{A_0}[A]$, and, for all $i \le k$, $d(a_i, F) < 1/n\}$. □

Similar relativization results are valid for the Hausdorff metric topology, the Attouch-Wets topology, the Vietoris topology, the Fell topology, the proximal topology, and the Mosco topology (where A_0 is now closed and convex). Instead, the Wijsman topology is pathological in this regard. For example, if ρ is the metric on Z^+ defined by $\rho(1,2) = 2$ and $\rho(i,j) = 1$ for $j \ge 3$ and $j > i$, then relativization does not behave properly on $A_0 = \{2, 3, 4, \dots\}$. For example, if $A_n = \{2, n+2, n+3, \dots\}$, then $\langle A_n\rangle$ converges to $\{2\}$ in $\langle \mathrm{CL}(A_0), \tau_{W_\rho}\rangle$, but not in $\mathrm{CL}(A_0)$ as a subspace of $\langle \mathrm{CL}(Z^+), \tau_{W_\rho}\rangle$.

Theorem 3.3. *Let* $\langle X, d\rangle$ *be a metric space. Then* $\langle \mathrm{CL}(X), \sigma_d\rangle$ *is first countable if and only if* X *is second countable.*

Proof. Suppose X is second countable, with $\{x_i : i \in Z^+\}$ dense in X. Then a countable local base for σ_d at $A \in \mathrm{CL}(X)$ consists of all sets of the form

$$\Lambda_A[S_n[x_1]; 1/n, x_1, x_2, \dots, x_n]$$

where $n \in Z^+$. Conversely, if $\langle \mathrm{CL}(X), \sigma_d\rangle$is first countable, then by Lemma 3.2, for each closed ball B, $\langle \mathrm{CL}(B), \sigma_d\rangle = \langle \mathrm{CL}(B), \tau_{\delta_d}\rangle$ is first countable. By Theorem 4.2 of [**13**], B is second countable with the relative topology and is thus separable. Hence, X is separable and is thus second countable. □

Theorem 3.4. *Let* $\langle X, d\rangle$ *be a metric space. The following are equivalent*:

(1) *Each bounded subset of X is totally bounded*;

(2) $\tau_{AW_d} = \tau_{W_d}$ *on* $\mathrm{CL}(X)$;

(3) $\tau_{AW_d} = \sigma_d$ *on* $\mathrm{CL}(X)$;

(4) $\langle \mathrm{CL}(X), \sigma_d \rangle$ *is metrizable*;

(5) $\langle \mathrm{CL}(X), \sigma_d \rangle$ *is second countable.*

Proof. (1) $\Rightarrow$ (2). Fix $A \in \mathrm{CL}(X)$, B closed and bounded, and $\varepsilon > 0$. Choosing points $\{b_1, b_2, \ldots, b_n\}$ in B with $S_{\varepsilon/4}[\{b_1, b_2, \ldots, b_n\}] \supset B$, we have

$$\bigcap_{i=1}^{n} \{F \in \mathrm{CL}(X) : |d(b_i, F) - d(b_i, A)| < \varepsilon/4\} \subset \{F \in \mathrm{CL}(X) : \sup_{x \in B} |d(x, F) - d(x, A)| < \varepsilon\}.$$

This shows that $\tau_{AW_d} \subset \tau_{W_d}$, and the reverse inclusion is always valid.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (4). The topology τ_{AW_d} is always metrizable [**5, 8, 3**].

(4) $\Rightarrow$ (5). Since $\langle \mathrm{CL}(X), \sigma_d \rangle$ is metrizable, it is first countable. Thus, X is separable by Theorem 3.3. Now if E is a countable dense subset of X, it easily follows from presentation (2) of σ_d in Theorem 2.2 that the finite subsets of E are σ_d-dense in $\mathrm{CL}(X)$. Thus, $\langle \mathrm{CL}(X), \sigma_d \rangle$ is separable and metrizable and is thus second countable.

(5) $\Rightarrow$ (1). If (5) holds, then for each closed and bounded set B, $\langle \mathrm{CL}(B), \sigma_d \rangle = \langle \mathrm{CL}(B), \tau_{\delta_d} \rangle$ is second countable. By Theorem 4.3 of [**13**], B must be totally bounded. Condition (1) now follows, since total boundedness is a hereditary property. □

We now look at some other coincidences.

Theorem 3.5. *Let $\langle X, d \rangle$ be a metric space. Then*

(1) $\sigma_d = \tau_{\delta_d}$ *on* $\mathrm{CL}(X)$ *if and only if the metric d is bounded*;

(2) $\sigma_d = \tau_{W_d}$ *on* $\mathrm{CL}(X)$ *if and only if each bounded subset of X is totally bounded.*

Proof. In (1), boundedness of X is obviously sufficient. For necessity, if X is unbounded, let $\langle x_n \rangle$ be a sequence in X with $d(x_1, x_n) > n$ for $n > 1$. Then if $A_n = \{x_1, x_{n+1}\}$ we have $\{x_1\} = \sigma_{d^-} \lim_{n\to\infty} A_n$. But $D_d(A_n, \{x_n : n \geq 2\})$ does not converge to $D_d(\{x_1\}, \{x_n : n \geq 2\})$, so that $\{x_1\} \neq \tau_{\delta_{d^-}} \lim_{n\to\infty} A_n$.

Sufficiency in (2) follows from Theorem 3.4, since σ_d is trapped between the Wijsman and Attouch-Wets topologies. Proof of necessity, following the proofs of Lemmas 5.3 and 5.4 of [**13**] is complex, and will not be presented here. □

Theorem 3.6. *Let X be a normed linear space, and let d be the metric determined by the norm. Then on* CL(X)*, the Mosco topology equals σ_d if and only if X is finite dimensional.*

Proof. If X is finite dimensional, then the Wijsman topology τ_{W_d} equals the Attouch-Wets topology τ_{AW_d} on the closed sets, because pointwise convergence of distance functions implies their uniform convergence on compact sets (by equicontinuity), and thus their uniform convergence on bounded sets. Since the Mosco topology τ_M and σ_d are trapped between them, $\tau_M = \sigma_d$ on the closed convex sets.

We next show that if X is infinite dimensional, then $\sigma_d \neq \tau_M$. If X is infinite dimensional and not reflexive, either Theorem 2.2 of [**17**] or Theorem 4.2 of [**11**] shows that the Mosco topology does not contain the Wijsman topology even restricted to compact convex sets, so that $\sigma_d \not\subset \tau_M$ for closed convex sets. Now suppose that X is infinite dimensional and reflexive. Let W be a closed separable subspace. By reflexivity of W, its continuous dual W^* is separable when equipped with the norm topology. Let $\{y_n : n \in Z^+\}$ be norm dense in W^*, and, for each $n \in Z^+$, let $A_n = \{x \in W : \forall\, i \leq n\ (x, y_i) = 0\}$. Let x_n be a norm one element of A_n. Since $\langle A_n \rangle$ is a decreasing sequence, $\langle A_n \rangle$ is Mosco convergent to $\cap_{n=1}^{\infty} A_n = \{\theta\}$, where θ is the origin of X. Notice that, for each $n \in Z^+$, $x_n \in S_2[\theta] \cap A_n \not\subset S_{1/2}[\theta]$. Thus, $\langle A_n \rangle$ is not σ_d-convergent to $\{\theta\}$. □

We are not aware that the necessity of finite dimensionality for the equality of τ_{AW_d} and τ_M on CLC(X) has been noted in the literature. This of course follows from Theorem 3.6.

One of the nice properties enjoyed by the Attouch-Wets topology on convex sets but not by the Mosco topology (as the last proof plainly shows) is continuity of the diameter functional [**14**, Lemma 3.2]. This is also true for the weaker topology σ_d. Actually, one can establish a quite general purely metric result in this direction.

Definition. Let A be a nonempty set in a metric space $\langle X, d\rangle$, and let $\alpha > 0$. We call A *α-connected* provided for each a and b in A, there exists a finite set of points $a = a_0, a_1, \dots, a_n = b$ such that $d(a_{i-1}, a_i) < \alpha$ for $i = 1, 2, 3, \dots, n$.

Connected sets are α-connected for each $\alpha > 0$, and compact sets that are $\alpha > 0$ connected for each $\alpha > 0$ are connected [**23**, p. 442].

Lemma 3.7. *Let $\langle X, d\rangle$ be a metric space, let $A \in \mathrm{CLB}\,(X)$, and let $a \in A$. If F is α-connected and $F \in \Theta_A[S_{2\alpha}[A]; \alpha; a]$, then $F \subset S_{2\alpha}[A]$.*

Proof. We have $d(a, F) < \alpha$ so that $F \cap S_\alpha[A] \neq \varnothing$. Suppose $F \not\subset S_{2\alpha}[A]$. By α-connectedness, F must intersect $\{x \in X : \alpha \leq d(x, A) < 2\alpha\}$. Choosing x_0 in the intersection, since $x_0 \in S_{2\alpha}[A]$, there exists $a_0 \in A$ with $d(x_0, a_0) < \alpha$. Since $d(x_0, A) \geq \alpha$, this implies that $a_0 \notin A$, a contradiction. □

Lemma 3.8. *Let $\langle X, d\rangle$ be a metric space. Then*

(i) *$F \to \mathrm{diam}\, F$ is lower semicontinuous on $\langle \mathrm{CL}\,(X), \sigma_d\rangle$;*

(ii) *If A is bounded and $\langle A_\lambda\rangle$ is a net of closed α-connected sets σ_d-convergent to A, then* $\mathrm{diam}\, A = \lim \mathrm{diam}\, A_\lambda$.

Proof. (i). Fix $A \in \mathrm{CL}\,(X)$. If A is singleton, then the diameter functional is clearly lower semicontinuous at A. Otherwise, let $\varepsilon > 0$, and pick a_1 and a_2 in A with $d(a_1, a_2) > \mathrm{diam}\, A - \varepsilon/2$. With $B \in \mathrm{CLB}\,(X)$ arbitrary, we have $\mathrm{diam}\, F > \mathrm{diam}\, A - \varepsilon$ for each $F \in \Theta_A[B; \varepsilon/4, a_1, a_2]$.

(ii) We now assume that A is bounded. Fix $a \in A$ and $\varepsilon < \alpha$. For all λ sufficiently large, we have $A_\lambda \in \Theta_A[S_{2\alpha}[A]; \varepsilon/4; a]$. By Lemma 3.7, we have $A_\lambda \subset S_{2\alpha}[A]$, so that $\mathrm{diam}\, A_\lambda \leq \mathrm{diam}\, A + \varepsilon$. □

Theorem 3.9. *Let $\langle X, d\rangle$ be a metric space, and let $\alpha > 0$. Then $F \to \operatorname{diam} F$ is continuous on the α-connected subsets of X equipped with σ_d, and thus on the space of convex sets $\langle \mathrm{CLC}\,(X), \sigma_d\rangle$, when X is a normed linear space.*

Proof. If $A \in \mathrm{CL}\,(X)$ is unbounded, then the diameter functional is obviously upper semicontinuous at A. Apply Lemma 3.8. □

It is easy to see that the diameter functional need not be σ_d-continuous on $\mathrm{CL}\,(X)$, e.g., on the line with the usual metric, $\{0\} = \sigma_d\text{-}\lim_{n\to\infty}\{0, n\}$.

4. Applications to one-sided optimization. The present section deals with the use of the topology σ_d in minimization problems. We shall work with the space of the lower semicontinuous functions defined on a metric space $\langle X, d\rangle$ with values in $(-\infty, \infty]$. Lower semicontinuity of f is equivalent to saying that the *epigraph* of f, $\operatorname{epi} f = \{(x, \alpha) : x \in X,\ \alpha \in R, \text{ and } \alpha \geq f(x)\}$, is a closed subset of $X \times R$, while $\operatorname{epi} f$ is a convex set if and only if f is a convex function in the usual sense. The function f is called *proper* provided $\operatorname{epi} f$ is nonempty. We denote the proper lower semicontinuous functions on X by $\mathrm{LSC}\,(X)$, and we write $\Gamma(X)$ for the proper lower semicontinuous convex functions on a normed space X.

For $f \in \mathrm{LSC}\,(X)$, each *level set* of $f \in \mathrm{LSC}\,(X)$ at height α,

$$\operatorname{lev}(f, \alpha) \equiv \{x \in X : f(x) \leq \alpha\},$$

is closed (but possibly empty), and if f is convex, so is $\operatorname{lev}(f, \alpha)$ for each $\alpha \in R$. The basic parameters for the problem of minimizing $f \in \mathrm{LSC}\,(X)$ (over X) are its *solution value* $v(f) \equiv \inf\{f(x) : x \in X\}$, and its possibly empty *solution set* $\operatorname{Argmin} f \equiv \{x \in X : f(x) = v(f)\}$. As is now standard in one sided optimization [**1**], giving $X \times R$ the box metric ρ,

$$\rho[(x_1, \alpha_1), (x_2, \alpha_2)] = \max\{d(x_1, x_2), |\alpha_1 - \alpha_2|\},$$

we may equip $\mathrm{LSC}\,(X)$ with various hyperspace topologies, under the identification $f \leftrightarrow \operatorname{epi} f$.

We shall consider problems that are well-posed problems in the generalized sense, which play an important role in minimum problems, for at least two reasons. First, they are not difficult to solve(at least in principle!), because every minimizing sequence has cluster points that are solutions. Thus, any method giving points with function values close to $v(f)$ offers at the same time points close to some solution. Furthermore, such problems behave well with respect to perturbations of the data (see [**14, 29, 36**], for instance).

Definition. A function $f \in \mathrm{LSC}\,(X)$ is declared *well-posed in the generalized sense* (abbreviated by g.w.p.) [**22, 29**], provided both of the following conditions hold:

(a) $\operatorname{Argmin} f$ is nonempty and compact;

(b) whenever $\langle x_n \rangle$ is a minimizing sequence for f, i.e., $v(f) = \lim_{n\to\infty} f(x_n)$, then $\langle x_n \rangle$ must have a subsequence convergent to some point of $\operatorname{Argmin} f$.

If, in addition, $\operatorname{Argmin} f$ is a singleton, then f is called *Tyhonov* well-posed (T.w.p.) [**39**].

There are several characterizations of well-posedness (see [**20**] for a systematic treatise on the subject). Here we mention only that in a complete metric space, T.w.p. amounts to saying that $\inf\{\operatorname{diam}(\operatorname{lev}(f, \beta)) : \beta > v(f)\} = 0$, while g.w.p. is equivalent to the condition $\inf\{\alpha(\operatorname{lev}(f,\beta)) : \beta > v(f)\} = 0$ where, for a given set A, $\alpha(A)$ is its Kuratowski measure of noncompactness (see, e.g., [**6**]).

Observe that compactness alone of $\operatorname{Argmin} f$ is not sufficient to guarantee T.w.p., even for continuous convex functions, as is shown by the following well-known example: Let X be a separable Hilbert space, with $\{e_i : i \in Z^+\}$ an orthonormal base. Consider

$$f(x) = \sum_{i=1}^{\infty} (x, e_i)^2 / i^2.$$

Here the origin is the unique minimizer of f, but there are minimizing sequences that are not even bounded.

The previous notion of well-posedness applies well to unconstrained problems. Suppose now that we are given a closed subset A of X, representing a constraint set, and that we have to minimize f over A. Such a problem will be represented by the *variational pair* (f, A). Formally, the above concept of Tyhonov well-posedness can be used in this setting, too, simply by considering A as the underlying metric space (with the metric inherited from X) or, equivalently, by minimizing the function $f \mid A \equiv f + I_A$ over X, where I_A is the indicator function of the set A, taking the value 0 on the points of A and ∞ outside. Nevertheless, it is clear that this approach does not take into account the fact that some minimization procedures search for points that are allowed to violate the constraints, to some extent. For this reason, other, more stringent notions of well-posedness are also used for constrained problems [**20, 34, 35, 15**]. Here, to get the sharpest results, we shall make use of the notion of strongly well-posed problems.

Definition. A pair (f, A) such that $f \mid A$ is not identically ∞, is called *strongly well-posed in the generalized sense* provided:

(i) the set $\{a \in A : f(a) = v(f \mid A)\}$ is nonempty and compact;

(ii) whenever $\langle x_n \rangle$ is a sequence satisfying $\langle d(x_n, A) \rangle \to 0$ and $\limsup_{n\to\infty} f(x_n) \le v(f \mid A)$, then $\langle x_n \rangle$ has a subsequence convergent to a point of A.

In [**15**], a pair (f, A) is called *strongly well-posed* provided it satisfies the above definition, and Argmin $f \mid A$ is a singleton. The only fact we mention here about strongly well-posed problems is that a Furi-Vignoli type condition holds for these problems, too [**15, 34, 35**]. For more about connections between well-posedness notions, see [**20**].

To conclude this short introduction to the second part of the paper, let us observe that $f \to v(f)$ is upper semicontinuous on $(\mathrm{LSC}\,(X), \sigma_\rho)$ and that $\{(f, x) : f \in \mathrm{LSC}\,(X) \text{ and } x \in \mathrm{Argmin}\, f\}$ is closed in $(\mathrm{LSC}\,(X), \sigma_\rho) \times X$, i.e., the Argmin multifunction has closed graph. These facts are not difficult to show, and they hold for weaker topologies on LSC (X), too.

Our first theorem intends to show that if f is g.w.p. and $f = \sigma_\rho\text{-}\lim_{n\to\infty} f_n$, where again ρ is the box metric on $X \times R$, then Argmin f can be recovered as a σ_d-limit of certain level sets of the f_n. This is rather surprising, in that $v(f) = \lim_{n\to\infty} f_n$ may not hold;

e.g., on the line, let $f(x) = x^2$ and, for each n, define f_n by

$$f_n(x) = \begin{cases} x^2 & \text{if } |x| < n \\ -1 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *Let $\langle X, d\rangle$ be a metric space, and let ρ denote the box metric on $X \times R$. Suppose $f \in \mathrm{LSC}\,(X)$ is well-posed in the generalized sense. Let $\langle f_n\rangle$ be a sequence in $\mathrm{LSC}\,(X)$ that is σ_ρ-convergent to f. Then there is a positive sequence $\langle \varepsilon_n\rangle$ convergent to zero such that $\mathrm{Argmin}\, f = \lim_{n\to\infty} \mathrm{lev}\,(f_n, v(f) + \varepsilon_n)$, in the σ_d-sense.*

Proof. For each $m \in Z^+$, choose a finite subset K_m of $\mathrm{Argmin}\, f$ with $\mathrm{Argmin}\, f \subset S_{1/m}[K_m]$. Since $f = \sigma_{\rho^-} \lim_{n\to\infty} f_n$, there exists an index N_m and a finite subset E_{mn} of $\mathrm{epi}\, f_n$ such that for each $n \geq N_m$ we have $\{(x, v(f)) : x \in K_m\} \subset S_{1/m}[E_{mn}]$. Let D_{mn} be the projection of E_{mn} on X. By the definition of the box metric, we have $f_n(x) < v(f) + 1/m$ for each $x \in D_{mn}$, so that for $n \geq N_m$,

$$(*) \qquad \mathrm{Argmin}\, f \subset S_{2/m}[D_{mn}] \subset S_{2/m}[\mathrm{lev}\,(f_n, v(f) + 1/m)].$$

There is no loss of generality in assuming that $\langle N_m\rangle$ is a strictly increasing sequence. We can now define our sequence $\langle \varepsilon_n\rangle$: take $\varepsilon_n = 1$ for $n < N_1$, and for $N_m \leq n < N_{m+1}$, let $\varepsilon_n = 1/m$. Let us now verify that this choice works.

Write $A = \mathrm{Argmin}\, f$, and let $(B^c)^{++} \cap V_1^- \cap \cdots \cap V_k^-$ be a basic neighborhood of A in the bounded proximal topology. There exist $m_0 \in Z^+$ and points $\{a_1, a_2, \ldots, a_k\} \subset A$ such that, for each $i \leq k$, we have $S_{2/m_0}[a_i] \subset V_i$. Choose $\delta > 0$ such that $S_\delta[A] \cap S_{2\delta}[B] = \varnothing$. As f is well-posed in the generalized sense, there exists $m_1 > m_0$ with $\inf\{f(x) : x \in S_{2\delta}[B]\} > v(f) + 1/m_1$. This means that

$$\mathrm{epi}\, f \in ((\mathrm{cl}\, S_\delta[B] \times \{v(f) + 1/m_1\})^c)^{++},$$

so that there exists $m_2 > m_1$ such that, for each $n \geq N_{m_2}$, we have $\mathrm{epi}\, f_n \in ((\mathrm{cl}\, S_\delta[B] \times \{v(f) + 1/m_1\})^c)^{++}$ since epigraphs recede in the vertical direction $f_n(x) > v(f) + 1/m_1$ for each $x \in S_\delta[B]$ and each $n \geq N_{m_2}$. Put differently, we have

$$(**) \qquad \mathrm{lev}\,(f_n, v(f) + 1/m_1) \cap S_\delta[B] = \varnothing \qquad \text{whenever } n \geq N_{m_2}.$$

We claim that for each $n \geq N_{m_2}$, we have $\text{lev}\,(f_n, v(f)+\varepsilon_n) \in (B^c)^{++} \cap V_1^- \cap \cdots \cap V_k^-$. Fix such an index n. By construction, $\varepsilon_n = 1/m$ for some $m \geq m_2$. Also, $(*)$ gives $A \subset S_{2/m}[\text{lev}\,(f_n, v(f)+\varepsilon_n)]$. By the choice of m_0, which is less than m_2, this means that $\text{lev}\,(f_n, v(f)+\varepsilon_n) \cap V_i \neq \varnothing$ for $i = 1, 2, \ldots, k$. On the other hand, since $n \geq N_{m_2}$ and $\varepsilon_n \leq 1/m_2 < 1/m_1$, $(**)$ yields $\text{lev}\,(f_n, v(f)+\varepsilon_n) \cap S_\delta[B] = \varnothing$. This means that $\text{lev}\,(f_n, v(f)+\varepsilon_n) \in (B^c)^{++}$, completing the proof. □

Let us make some comments on the previous theorem. First, observe that compactness of $\text{Argmin}\, f$ implies a stronger convergence of sublevel sets, namely, convergence in the sense of the τ_{AW} topology [**16**, Corollary 3.3]. In the case that we are dealing with approximating functions having connected level sets, even more can be said: we get convergence in the Hausdorff metric sense. This follows immediately from Lemma 3.7. One of the most compelling consequences in the connected case is boundedness of the level sets below certain heights of the approximating functions.

The following examples show that the assumption of generalized well-posedness is crucial in Theorem 4.1 and that, in the convex case, σ_ρ cannot be replaced by the weaker τ_M topology.

Example. Let X be a separable Hilbert space, with $\{e_n : n \in Z^+\}$ an orthonormal base. As we have already mentioned,

$$f(x) = \sum_{i=1}^{\infty} (x, e_i)^2/i^2$$

has a unique minimizer but is not T.w.p. If we consider $f_n \equiv f$, then for no choice of positive ε_n can we have in the σ_d-sense $\text{Argmin}\, f = \lim_{n\to\infty} \text{lev}\,(f_n, v(f)+\varepsilon_n)$, because for each n, there is some e_i with large i belonging to $\text{lev}\,(f_n, v(f)+\varepsilon_n)$. As a result, the condition

$$\text{lev}\,(f_n, v(f)+\varepsilon_n) \cap U \subset S_{1/2}(\text{Argmin}\, f) = \{x : ||x|| < 1/2\},$$

as required by Theorem 2.2 to get σ_d-convergence, must fail. Observe that all the functions of this example are continuous and convex, and that $\langle f_n \rangle$ converges to f in every reasonable way, so that the assumption of g.w.p. for f cannot be dispensed with, without affecting the conclusion of the theorem.

Example. Here we show that, in the convex case, the Mosco topology cannot guarantee the same result offered by the stronger bounded proximal topology. Again, in a separable Hilbert space, consider $f(x) = ||x||^2$ and $f_n(x) = ||x||^2 - (x, e_n)^2$, for $n = 1, 2, \ldots$. It is not difficult to show that $f = \tau_M\text{-}\lim_{n\to\infty} f_n$, but $\operatorname{Argmin} f_n$ is again "too big" to permit σ_d-convergence of $\langle \operatorname{lev}(f_n, v(f) + \varepsilon_n)\rangle$ to $\operatorname{Argmin} f$, for any choice of the ε_n.

Finally, it is easy to show with functions of a real variable that we may not have convergence of $\langle \operatorname{lev}(f_n, v(f) + \varepsilon_n)\rangle$ to $\operatorname{Argmin} f$ in the lower half of the hyperspace topology σ_d, which is just the lower Vietoris topology by virtue of Lemma 2.1, unless the sequence $\langle \varepsilon_n \rangle$ converges to 0 slowly enough. That is, it is important to make a judicious choice of $\langle \varepsilon_n \rangle$.

Evidently, well-posedness of a particular problem cannot guarantee well-posedness of all nearby problems, even with respect to strong topologies. For example, in any normed linear space, if $f(x) = ||x||$ and $f_n(x) = \max\{1/n, f(x)\}$, then f is the limit of $\langle f_n \rangle$ in any reasonable sense. We now intend to see whether well-posedness of a particular constrained problem guarantees solvability of nearby problems. In other words, given a pair (f, A) which is well-posed, we would like to know if, for some topology, close problems have minimizers. The first examples we provide show that continuity of f is not enough to get positive results, even in finite dimensions, where, of course, the bounded proximal topology, the Attouch-Wets topology and the Wijsman topology all coincide.

Example. Let $X = R$. The Tyhonov well-posed function $f(x) = |x|$ is the limit of $\langle f_n \rangle$ where, for each n, $f_n(x) = \min\{|x|, n - |x|\}$. Clearly, each f_n is unbounded below.

We now keep the objective function fixed and vary constant sets.

Example. Let $X = R^2$, and let (f, A) be the following pair:

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } |y| \le 1 \\ x^2 + 2 - y^4 & \text{if } |y| > 1 \end{cases} \quad \text{and } A = R \times \{0\}.$$

Then $f \mid A$ is strongly well-posed, but (f, A_n) is not even lower-bounded, if $A_n = \{(x, (1/n)x) : x \in R\}$.

Thus, we shall restrict our attention to convex problems (see [**21, 25**] as general references for convex problems), where positive results are more likely to be true. Indeed, from genericity theorems in the convex setting, we know that in any reasonable topology the well-posed problems are usually dense, and, frequently, they form (or contain) a G_δ set of problems [**14, 15, 34**]. For constrained problems, following [**15**] and [**36**], we focus on problems (f, A) where $f \in \Gamma(X)$ and $A \in \mathrm{CLC}\,(X)$ satisfy the following condition: either there is a point in the interior of A where f is finite, or there is a point of A where f is (real-valued and) continuous. In symmetric terms, either $\mathrm{epi}\, f \cap \mathrm{int}\,\mathrm{epi}\, I_A \neq \varnothing$ or $\mathrm{int}\,\mathrm{epi}\, f \cap \mathrm{epi}\, I_A \neq \varnothing$. This is really a modest requirement, for without such a constraint qualification, properness of nearby problems may fail even for strong topologies.

We start by providing two simple examples where it is shown that the Mosco topology is not strong enough to get results.

Example. Let X again be any separable Hilbert space, and let $f(x) = ||x||^2$. Consider the following sequence of convex functions $\langle f_n \rangle$ Mosco convergent to $f : f_n(x) = ||x||^2 - (x, e_n)^2 + (1/n)(x, e_n)$. Then the functions f_n are not even lower bounded on X.

In the next example, we again keep the objective function fixed, and we move the constraint set.

Example. Once more with X any separable Hilbert space, let $f(x) = (x, y)$ where $y = \sum_{n=1}^{\infty} (1/n) e_n$. Consider the sequence of lines $\langle A_n \rangle$, where $A_n = \{\lambda e_n : \lambda \in R\}$, Mosco convergent to $A = \{\theta\}$. Then (f, A) is trivially strongly well-posed, but for each n, the problem (f, A_n) is unbounded below.

As a result of the two previous examples, we see that close (with respect to the Mosco topology) to a given well-posed problem, we can find problems without a solution. Actually this fact, at least in the unconstrained case, can be established in full generality, as shown by Theorem 6.9 of [**14**]. The situation changes if we use stronger topologies.

Theorem 4.2. *Let X be a Banach space, and let ρ be the box metric on $X \times R$. The following are equivalent*:

(i) *X is reflexive*;

(ii) *for each pair $(f, A) \in \Gamma(X) \times \mathrm{CLC}\,(X)$ which is well-posed in the generalized sense, for each sequence $\langle f_n \rangle$ in $\Gamma(X)$ converging to f in the σ_ρ (respectively, τ_{AW}) sense, and for each sequence $\langle A_n \rangle$ in $\mathrm{CLC}\,(X)$ converging to A in the τ_{AW} (respectively, σ_d) sense, if $\mathit{int}\, A \cap \mathit{dom}\, f \neq \varnothing$ (respectively, f is continuous at some point of A), then eventually (f_n, A_n) has a minimum point*;

(iii) *for each $f \in \Gamma(X)$ which is well-posed in the generalized sense and for each sequence $\langle f_n \rangle$ in $\Gamma(X)$ converging to f in the σ_ρ sense, then eventually f_n has a minimum point*;

(iv) *for each $f \in \Gamma(X)$ which is well-posed in the generalized sense and for each sequence $\langle f_n \rangle$ in $\Gamma(X)$ converging to f in the τ_{AW} sense, then eventually f_n has a minimum point*;

(v) *for some $f \in \Gamma(X)$ which is well-posed in the generalized sense and for each sequence $\langle f_n \rangle$ in $\Gamma(X)$ converging to f in the τ_{AW} sense, then eventually f_n has a minimum point.*

Proof. (i) $\Rightarrow$ (ii). The constraint qualification condition guarantees, in both cases, that the sequence $\langle f_n + I_{A_n} \rangle$ converges in the σ_ρ sense to $f + I_A$ [**36**, Theorem 5.2.5]. Now, use convexity, Lemma 3.7, and Theorem 4.1 to conclude that sublevel sets of $f_n + I_{A_n}$ are bounded for all large n, from which (ii) follows by reflexivity. The implications (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (v) are obvious. It remains to show that (v) $\Rightarrow$ (i). We will show that (v) and nonreflexivity of X are incompatible.

Suppose X is nonreflexive and y is a norm one element of X^* which is not norm achieving on the unit ball of X. For each $n \in Z^+$, let g_n be this convex function on X:

$$g_n(x) = \begin{cases} n \cdot |(x, y) - 1/n^2| & \text{if } ||x|| \leq 1/n^2 \\ \infty & \text{otherwise.} \end{cases}$$

Let us take $\langle w_k \rangle$ a sequence in the unit ball U of X such that $(w_k, y) \geq 1 - 1/k$ for each k. Finally, let $f \in \Gamma(X)$ be a g.w.p. function for which (v) holds. Without loss of generality, let us suppose that $f(0) = 0 = \inf f$. For each $n \in Z^+$, we define f_n by

$$f_n(x) = f \nabla g_n(x),$$

where ∇ indicates the inf-convolution operator (see, e.g., [**18**, p. 18]). We claim that:

(a) for each n, $f_n \in \Gamma(X)$;

(b) for each n, f_n does not have a minimum point;

(c) $\langle f_n \rangle$ converges in the τ_{AW} sense to f.

From the commutativity of ∇, we have

$$f_n(x) = \inf \{ f(x - w) + n|(w, y) - 1/n^2| : ||w|| \leq 1/n^2 \}, \tag{1}$$

$$= \inf \{ f(w) + n|(x - w, y) - 1/n^2| : ||w|| \leq 1/n^2 \}. \tag{2}$$

From (1), it follows that for each $x \in X$ and $n \in Z^+$ we have

$$f_n(x) \leq f(x) + 1/n, \tag{3}$$

$$f_n(x) \geq \inf \{ f(x - w) : ||w|| \leq 1/n^2 \}. \tag{4}$$

Moreover, from (2) we get

$$f_n(w_k/n^2) \leq 1/(nk). \tag{5}$$

Convexity of the inf-convolution operator holds in general. From the inequalities (2) and (4) we easily get that f_n is proper and continuous (being upper bounded on bounded sets), and (4) and (5) yield $\inf f_n = 0$, for each n. Now fix $n \in Z^+$ and suppose that $f_n(x) = 0$ for some x. Then, by (1), there must exist a sequence $\langle v_k \rangle$ of norm one elements in X such that $\lim_{k \to \infty} (v_k, y) = 1$ and $\lim_{k \to \infty} f(x - v_k/n^2) = 0$. By the generalized well-posedness of f, we conclude that, necessarily, for a subsequence, $\lim_{k \to \infty} x - v_k/n^2 = x_0$, where x_0 is a minimizer of f. It follows that $(n^2[x - x_0], y) = 1$, a contradiction, because $n^2[x - x_0]$ is a norm one element. The proof is completed by observing that condition (c) is an easy consequence of (3) and (4) above. □

Theorem 4.2 shows that reflexivity, which is not usually needed for obtaining results concerning the Attouch-Wets and bounded proximal topologies, here plays a fundamental role. This is due to the special feature of the existence issue. The two aforementioned topologies guarantee a regular behavior of the level sets (essentially boundedness) for functions close to a well-posed function, without appealing to reflexivity, which is on the other hand necessary (only) to guarantee the weak compactness of the bounded level sets.

In our initial example showing that functions nearby $x \to ||x||$ need not be well-posed in the generalized sense, uniqueness of the solution sets for the approximating problems failed. Suppose we do have uniqueness, as might be guaranteed by strict convexity, for example. In this case, can we make a further step? Unfortunately, our final example shows that, even here, we have little hope to get a general result.

Example. Let X be an infinite dimensional separable Hilbert space, and let H be any hyperplane not passing through the origin. Then there is a sequence $\langle f_n \rangle$ of strictly convex functions such that:

(i) $|| \cdot || = \tau_{AW} - \lim_{n\to\infty} f_n$;

(ii) for each n, (f_n, H) is not Tyhonov well-posed.

Proof. Let $\{y_i : i \in Z^+\}$ be a countable dense subset of U, and let $H = \{x \in X : (x, y_1) = c\}$ for some $c > 0$. Choose $x_1 \in cU$ such that $(x_1, y_1) = c$. Observe that $H = H_0 + x_1$ where $H_0 = \ker y_1$. Let $S_n = \{x \in cU : (x, y_1) \leq c - 1/n\}$. Let g_n be the following function: $g_n(x) = \lambda c$, if $x = \lambda z$ for some $\lambda \geq 0$, where z belongs to the boundary of S_n. Then g_n is a function which is positively homogeneous of degree 1 and such that lev $(g_n, c) = S_n$. Let h be the convex function defined by

$$h(x) = \sum_{i=1}^{\infty} (x - x_1, y_i)^2 / i^2,$$

and finally let $f_n = g_n + (1/n)h$. We claim that $\langle f_n \rangle$ has the required properties. That f_n is strictly convex follows from the strict convexity of h, which in turn follows from the fact that $\{y_i : i \in Z^+\}$ is dense in the unit ball.

(i) First, $f = \tau_{AW}\text{-} \lim_{n\to\infty} g_n$. This can be easily seen by observing that $\text{lev}(\|\cdot\|, c) = \tau_{AW}\text{-}\lim_{n\to\infty} \text{lev}(g_n, c)$, by using the homogeneous character of the functions under consideration and finally by appealing to Theorem 3.8 of [**14**], stating that convergence of the values and τ_{AW}-convergence of the level sets imply τ_{AW}-convergence of the functions.

(ii) Choose $x_2 \in H_0 \cap U$, and choose $x_3 \in H_0 \cap U \cap (\ker y_2)$. This is possible because H_0 is an infinite dimensional space, and hence, $\ker(y_2|H_0)$ is either a hyperplane in H_0 or is all of H_0. Continuing, for each j, choose $x_j \in U \cap H_0$ such that $(x_j, y_i) = 0$ for each $i < j$. Now observe that, for each n, $\text{Argmin} f_n = \{x_1\}$, because x_1 minimizes g_n on H and is the unique minimizer of h on all of the space. Moreover, $g_n(x_1) = c(1 + 1/n)$. For a sufficiently small constant a, $g_n(x_1 + ax_j) = c(1 + 1/n)$ for all j. As a result, $\langle x_1 + ax_j \rangle$ will be a minimizing sequence for f_n, because $\lim_{j\to\infty} h(x_1 + ax_j) = 0$. This implies that (f_n, H) is not well-posed and concludes the proof. □

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Department of Mathematics, California State University, Los Angeles, CA 90032

Dipartimento di Matematica, Universita degli studi di Milano, Via C. Saldini, 50, 20133, Milano, Italy