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ASYMPTOTIC STABILITY AND THE DERIVATIVES OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. We consider asymptotic stability of the zero solution of the functional differential equation $X'(t)=F(t,X_t)$ by Liapunov's second method with a basic condition

$$V'_{(1)}(t,X_t) \le -\eta_1(t)W_1(m(X_t)) - \eta_2(t)W_2\left(\int_{t-h}^{t} |X(s)||X'(s)|\,ds\right),$$

or a similar condition. Some examples are given. As a consequence, the condition that $F(t,\phi)$ is bounded if ϕ is bounded is weakened in a classical result of stability of Krasovskii.

1. Introduction. The objective of this paper is to investigate asymptotic stability of the zero solution of the functional differential equation

$$X'(t) = F(t, X_t), \tag{1}$$

where $X_t(\theta) = X(t+\theta)$ for $-h \le \theta \le 0$ and h is a positive constant. Before proceeding we shall set forth some notation and terminology that will be used throughout this paper. Denote by C the space of continuous functions $\phi : [-h, 0] \to R^n$. For $\phi \in C$ we will use the norm $||\phi|| := \max_{-h\le s\le 0} |\phi(s)|$, where $|\cdot|$ is any convenient norm in R^n. Given $H > 0$, C_H denotes the set of $\phi \in C$ with $||\phi|| < H$. $X'(t)$ denotes the right-hand derivative at t if it exists and is finite. It is supposed that $F : R_+ \times C_H \to R^n$, that F is continuous, and that it

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takes bounded sets into bounded sets. Here $R_+ = [0,\infty)$. Then it is known [**2, 5, 6, 9**] that for each $t_0 \in R_+$ and each $\phi \in C_H$ there is at least one solution $X(t_0,\phi)$ defined on an interval $[t_0, t_0+\alpha)$ and if there is an $H_1 < H$ with $|X(t,t_0,\phi)| \le H_1$, then $\alpha = \infty$. We also suppose $F(t,0) \equiv 0$ so that $X = 0$ is a solution of (1), and is called the zero solution.

By means of Liapunov's second method, throughout this paper we work with continuous functionals $V : R_+ \times C_H \to R_+$ (called Liapunov functionals) with $V(t,0) = 0$, whose derivative V' with respect to (1) is defined by

$$V'_{(1)}(t,\phi) = \limsup_{\delta\to 0^+}[V(t+\delta, X_{t+\delta}(t,\phi)) - V(t,\phi)]/\delta.$$

We also work with wedges, denoted by $W_i : R_+ \to R_+$, which are continuous and strictly increasing, which also satisfy $W_i(0) = 0$.

Definition 0. The zero solution of (1) is stable if for each $\varepsilon > 0$ and $t_0 \ge 0$ there exists a $\delta > 0$ such that $[\phi \in C_\delta, t \ge t_0]$ imply that $|X(t,t_0,\phi)| < \varepsilon$. The zero solution of (1) is uniformly stable (U.S.) if it is stable and if the δ is independent of t_0. The zero solution of (1) is asymptotically stable (A.S.) if it is stable and for each $t_0 \ge 0$, there is an $r > 0$ such that for each $\phi \in C_r$, $|X(t,t_0,\phi)| \to 0$, as $t \to \infty$. The zero solution of (1) is uniformly asymptotically stable (U.A.S.) if it is U.S. and if there is an $r > 0$ and for each $\mu > 0$ there is a $T > 0$ such that $[t_0 \ge 0, \phi \in C_r, t \ge t_0 + T]$ imply that $|X(t,t_0,\phi)| < \mu$.

In the study on asymptotic stability of the zero solution of (1), Krasovskii [**7**] (cf. Driver [**5**]) required the negative definiteness of $V'_{(1)}(t,X_t)$ in the form

$$V'_{(1)}(t,X_t) \le -W_1(||X_t||). \tag{2}$$

Although his theorem has been mainly of theoretical importance, it has not proved to be useful. In applications, investigators have proposed different conditions. For instance, Burton, Casal and Somolinos [**3**] proved that if $X(t)$ is a solution of (1) on $[t_0,\infty)$ with $|X(t)| < H$, and

$$V'_{(1)}(t,X_t) \le -W_1(|X(t)|) - W_2\bigg(\int_{t-h}^{t} |X'(s)|\,ds\bigg), \tag{3}$$

then there is a convex downward wedge W_1 such that (2) holds.

If we have a Liapunov functional $V : R_+ \times C_H \to R_+$ satisfying

$$V'_{(1)}(t, X_t) \leq -W_1(|X(t)|) - W_2(|X'(t)|),$$

and if W_2 is convex downward, then we can find another Liapunov functional $U : R_+ \times C_H \to R_+$ satisfying

$$U'_{(1)}(t, X_t) \leq -W_1(|X(t)|) - W_3\bigg(\int_{t-h}^{t} |X'(s)|\, ds\bigg)$$

(see [**8**]). This implies that U also satisfies (2) for some W_1.

In applications, investigators would often have some other similar but more general conditions, such as

$$(4) \qquad V'_{(1)}(t, X_t) \leq -\eta_1(t)W_1(|X(t)|) - \eta_2(t)W_2\bigg(\int_{t-h}^{t} |X'(s)|\, ds\bigg),$$

or

$$(5) \qquad V'_{(1)}(t, X_t) \leq -\eta_1(t)W_1(|X(t)|) - \eta_2(t)W_2(|X'(t)|).$$

Numerous results have been obtained with conditions similar to (4) and (5), see [**1, 3, 4**]. Papers [**1, 3**] discussed asymptotic stability with conditions similar to (4) or (5). Paper [**4**] discussed uniform asymptotic stability with (5). For reference, we reorganize and summarize the theorems in Paper [**1, 3**] as following. Definitions of some of the terms used here will be given later.

Theorem 1.1. *Let $V : R_+ \times C_H \to R_+$ be continuous and $\eta_1 \geq 0$, $\eta_2 \geq 0$ with*

(i) $X = 0$ *stable,*

(ii) $V'_{(1)}(t, X_t) \leq -\eta_1(t)W_1(|X(t)|) - \eta_2(t)W_2(|X(t)|)W_3(|X'(t)|)$. *Then $X = 0$ is* A.S. *if one of the following conditions holds*:

(a) $\eta_1 \notin L^1[0, \infty)$, *$\eta_2 > 0$ a constant, and $W_3(r) = r$.*

(b) *$\eta_1 \in IP(\delta)$ for some $\delta > 0$, $\eta_2 > 0$ a constant, and W_3 convex downward.*

(c) $\eta_1 \in IP(\delta)$ *for some* $\delta > 0$, $\eta_2 > 0$ *and* $\int_{-\delta}^{0} [\eta_2(t+s)]^q \, ds < B$ *for* $0 < B < \infty$ *and* $t \geq 0$, $[W_3(r)]^p$ *convex downward, where* $0 < p < 1$, $q = p/(p-1)$.

(d) $X = 0$ U.S., $\eta_1 > 0$ *a constant,* $\eta_2 \notin L^1[0, \infty)$ *and decreasing,* W_3 *convex downward.*

(e) $X = 0$ U.S., $\eta_1(t) = \eta_2(t) \notin L^1[0, \infty)$ *and decreasing, and* W_3 *convex downward.*

In these conditions, (a), (b) and (c) are given in [**3**]. (d) is the restated condition given in [**3**]. (e) is a generalized condition of [**1**] which can be shown by the discussions of [**8**] and this paper. It can also be easily seen that (c) is often weaker than (b), and (e) is weaker than (d).

In this paper we are going to consider asymptotic stability with conditions similar to but more general than (4) or (5) and give some examples to show the applications.

2. Preliminaries.

Definition 2.1. A measurable function $\eta : R_+ \to R_+$ is said to be *integrally positive* with parameter $\delta > 0$ $(IP(\delta))$ if whenever $I = \cup_{m=1}^{\infty} [\alpha_m, \beta_m]$ with $\alpha_m < \beta_m < \alpha_{m+1}$ and $\beta_m - \alpha_m \geq \delta$, $m = 1, 2, 3, \ldots$, then $\int_I \eta(s) \, ds = \infty$.

It is then well known that η is $IP(\delta)$ for some $\delta > 0$ if and only if $\lim_{t\to\infty} \inf \int_t^{t+\delta} \eta(s) \, ds > 0$.

It is also clear that as a set of functions $IP(\delta_1) \subseteq IP(\delta_2)$ if $\delta_1 \leq \delta_2$.

Definition 2.2. A continuous function $\eta : R_+ \to R_+$ is said to be *weakly divergent* in series with parameter $\delta > 0$ $(WDIS(\delta))$ if there are a constant $c > 0$ and a sequence $t_n \to \infty$ with $t_{n+1} - t_n \geq c$ such that $\sum_{n=1}^{\infty} \bar{\eta}(t_n) = \infty$, where $\bar{\eta}(t_n) = \min_{t_n - \delta \leq s \leq t_n} \eta(s)$.

In the above definitions, we often use the abbreviation of the definition to denote the class of the functions. For instance, we will often use $WDIS(\delta)$ to denote the class of functions which are weakly divergent in series with parameter $\delta > 0$.

Lemma 2.1. *If* $\eta : R_+ \to R_+$ *is* $WDIS(\delta)$, *then for any constant* $\lambda > 0$, *there is a sequence* $s_n \to \infty$ *such that* $s_{n+1} - s_n \geq \lambda$ *and* $\sum_{n=1}^{\infty} \bar{\eta}(s_n) = \infty$, *where* $\bar{\eta}(s_n) = \min_{s_n - \delta \leq s \leq s_n} \eta(s)$.

Proof. Suppose that a constant c and a sequence $\{t_n\}$ are given by the definition of $WDIS(\delta)$. Given a $\lambda > 0$, if $\lambda \leq c$, then we are done. Therefore, we may assume $\lambda > c$.

Let $N = [\lambda/c] + 1$, where $[x]$ is the largest integer function. Clearly, in every interval $I_i = [i\lambda, (i+1)\lambda]$, $i = 0, 1, 2, \dots$, there are at most N t_n's. Consider the intervals I_{2i+1} with the odd indexes and the intervals, I_{2i}, with the even indexes, $i = 0, 1, 2, \dots$. In the sequence of intervals, $\{I_{2i+1}\}$, we construct the first subsequence of $\{t_n\}$ by picking the first t_n in each I_{2i+1}. We can also pick the second t_n in each I_{2i+1} to get the second subsequence of $\{t_n\}$. In this way, we can find the k-th subsequence of $\{t_n\}$ by taking the k-th t_n in each I_{2i+1}. If the number of t_n's in some interval is less than k, then we assume the k-th t_n in that interval is zero. Thus, with the sequence of the intervals I_{2i+1}, $i = 0, 1, 2, \dots$, we can find at most N subsequences of $\{t_n\}$. Similarly, we can also find at most N subsequences of $\{t_n\}$ from the sequence of the intervals, $\{I_{2i}\}$. Therefore, we can have at most $2N$ subsequences of $\{t_n\}$. The absolute value of the difference of any two elements in each of such subsequences is not less than λ by the construction of the subsequences. Since $\sum_{n=1}^{\infty} \bar{\eta}(t_n) = \infty$, there is at least one subsequence, say $\{s_n\}$, of $\{t_n\}$ such that $\sum_{n=1}^{\infty} \eta(s_n) = \infty$.

This completes the proof. □

Definition 2.1 was introduced in [**4**]. Note that a function $\eta \in IP(\delta)$ does not imply $\eta \in WDIS(\delta)$, and the converse is not true either. These can be illustrated by the following example.

Example 2.1. (a) $\eta(t) = |\sin t| \notin WDIS(\pi/2)$ since

$$\bar{\eta}(t) = \min_{t-\pi/2 \leq s \leq t} |\sin s| \equiv 0 \qquad \text{for all } t \in R.$$

But, clearly, $\eta \in IP(\pi/2)$. Although $\eta \notin WDIS(\pi/2)$, $\eta \in WDIS(\pi/4)$.

(b) $\eta(t) = 1/t \notin IP(\delta)$ for any $\delta > 0$. But $\eta \in WDIS(\delta)$ for any $\delta > 0$.

In fact, we can prove the next more general result.

Lemma 2.2. *Suppose that $\eta : R_+ \to R_+$ is continuous and nonincreasing with $\eta \notin L^1[0,\infty)$. Then $\eta \in WDIS(\delta)$ for any $\delta > 0$.*

Proof. Given $\delta > 0$, for $\eta \notin L^1[0,\infty)$, one of the following

$$\sum_{i=1}^{\infty} \int_{3i\delta}^{(3i+1)\delta} \eta(s)\, ds, \qquad \sum_{i=1}^{\infty} \int_{(3i+1)\delta}^{(3i+2)\delta} \eta(s)\, ds, \qquad \sum_{i=1}^{\infty} \int_{(3i+2)\delta}^{(3i+3)\delta} \eta(s)\, ds,$$

must be divergent. Suppose $\sum_{i=1}^{\infty} \int_{3i\delta}^{(3i+1)\delta} \eta(s)\, ds = \infty$. Since η is nonincreasing $\int_{3i\delta}^{(3i+1)\delta} \eta(s)\, ds \le \delta\eta(3i\delta)$.

For $\bar{\eta}(t) = \min_{t-\delta \le s \le t} \eta(s) = \eta(t-\delta)$, let $t_i = 3i\delta + \delta$. Then

$$\infty = \sum_{i=1}^{\infty} \int_{3i\delta}^{(3i+1)\delta} \eta(s)\, ds \le \delta \sum_{i=1}^{\infty} \eta(3i\delta) = \delta \sum_{i=1}^{\infty} \eta(t_i).$$

The proof is complete. □

In our research, Jensen's inequality related to convex functions and Sobolev's inequality are very useful. A simple form of Sobolev's inequality and the proof were given in [**3**]. For reference, we note that if $W : [a,b] \to (-\infty,\infty)$ with $W([t_1+t_2]/2) \le [W(t_1)+W(t_2)]/2$ for any $t_1,\, t_2 \in [a,b]$, then W is convex downward. About a convex function, we have the next lemma.

Lemma 2.3 (Jensen's inequality). *Let $W : R_+ \to R_+$ be convex downward, and let $f, p : [a,b] \to R_+$ be continuous with $\int_a^b p(t)\, dt > 0$. Then*

$$\int_a^b p(t) W(f(t))\, dt \ge \int_a^b p(t)\, dt W\bigg(\int_a^b f(t)p(t)\, dt \Big/ \int_a^b p(t)\, dt \bigg).$$

Lemma 2.4 (Sobolev's inequality). *Let $\phi : [a,b] \to R^n$ have a continuous derivative. Then*

(i) $\min_{a\le t\le b}|\phi(t)| + \int_a^b |\phi'(s)|\,ds \ge ||\phi||$.

(ii) $|\phi(a)| + \int_a^b |\phi'(t)|\,dt \ge ||\phi||$.

(iii) $\int_a^b [|\phi(t)| + |\phi'(t)|]\,dt \ge k||\phi||$,

where $k = \min\{1, b-a\}$ *and* $||\phi|| := \max_{a\le s\le b}|\phi(s)|$.

Lemma 2.5. *If* W_1, $W_2 : R_+ \to R_+$ *are two wedges, then there are wedges* W_3 *and* W_4 *defined on* R_+ *such that for any* $s, t \in R_+$,

$$W_3(s+t) \le W_1(s) + W_2(t) \le W_4(s+t).$$

Proof. Define $u(r) := \inf\{W_1(s)+W_2(t) | s+t \ge r, s \ge 0, t \ge 0\}$. Then u is well defined and $u(r) \ne 0$ if $r \ne 0$ since W_1 and W_2 are wedges. It is also clear that $u(r)$ is continuous, nondecreasing and $u(0) = 0$.

Since $f(r) = r/(r+1)$ is strictly increasing, $W_3(r) = f(r)u(r)$ is a wedge and also

$$W_3(s+t) \le u(s+t) \le W_1(s) + W_2(t).$$

To show the other part, define $W_4(r) = W_1(r) + W_2(r)$. Then

$$W_1(s) + W_2(t) \le W_1(s+t) + W_2(s+t) \le W_4(s+t).$$

This completes the proof. □

Lemma 2.6. *Let* $f : R_+ \to R_+$ *be continuous and* $G(t) = \int_{t-h}^t f(s)\,ds$. *Given* $\varepsilon > 0$ *and* $h_1 \in (0, h]$, *if* $G(t_1) \ge \varepsilon$ *for some* $t_1 \ge 2h$, *then there is a closed interval* $[a, b]$ *of length* h_1 *containing* t_1 *in which* $G(t) \ge \varepsilon/2$, *that is,* $b - a = h_1$, $t_1 \in [a, b]$, $G(t) \ge \varepsilon/2$ *for all* $[a, b]$.

This lemma is similar to that of [**4**]. But [**4**] only gave that $h_1 \in (0, h)$ and $G(t) \ge \delta := \varepsilon(h - h_1)/(2h - h_1)$ in which δ is clearly less than $\varepsilon/2$. The following proof was originally given by T. Krisztin and not published.

Proof. By the continuity of $k(x) = \int_{t_1-h}^{x} f(s)\,ds$ on $[t_1 - h, t_1]$, there is an $a \in (t_1 - h, t_1)$ such that $\int_{t_1-h}^{a} f(s)\,ds = \varepsilon/2$ and hence $\int_a^{t_1} f(s)\,ds \geq \varepsilon/2$. Then for $t \in [a, t_1]$ we have $t - h \leq t_1 - h < a \leq t$, that is, $[t_1 - h, a] \subset [t - h, t]$. Therefore

$$G(t) = \int_{t-h}^{t} f(s)\,ds \geq \int_{t_1-h}^{a} f(s)\,ds = \varepsilon/2 \qquad \text{for } t \in [a, t_1].$$

For $t \in [t_1, a+h]$, we have $t - h \leq a < t_1 \leq t$, that is, $[a, t_1] \subset [t - h, t]$. Therefore,

$$G(t) = \int_{t-h}^{t} f(s)\,ds \geq \int_{a}^{t_1} f(s)\,ds \geq \varepsilon/2 \qquad \text{for } t \in [t_1, a+h].$$

Thus $G(t) \geq \varepsilon/2$ for all $t \in [a, b]$, where $b = a + h$. This, of course, is true for any $b = a + h_1$ with $h_1 \in (0, h]$. □

3. Asymptotic stability. For convenience, we denote

$$m(\phi) = \min_{-h \leq s \leq 0} |\phi(s)|, \qquad \phi \in C.$$

Theorem 3.1. *Let* $V : R_+ \times C_H \to R_+$ *be continuous with*

(i) $X = 0$ *stable*;

(ii) $V'_{(1)}(t, X_t) \leq -\eta_1(t) W_1(m(X_t)) - \eta_2(t) W_2(\int_{t-h}^{t} |X(s)||X'(s)|\,ds)$.

Then $X = 0$ *is* A.S. *if one of the following holds*:

(a) $\eta_1 \in IP(\delta)$ *for some* $\delta > 0$ *and* $\eta_2 \in IP(h)$.

(b) $\eta_1 \notin L^1[0, \infty)$, $\eta_2 > 0$ *constant, and* $W_2(r) = r$.

(c) $X = 0$ *uniformly stable,* $\eta_1 \notin L^1[0, \infty)$, *and* $\eta_2 \in IP(h)$.

(d) $X = 0$ *uniformly stable,* $\eta(t) := \min\{\eta_1(t), \eta_2(t)\} \notin L^1[0, \infty)$.

Proof. Suppose that (i) and (ii) hold. Let $X(t)$ be a solution of (1) on $[t_0, \infty)$ with $|X(t)| < H$. It suffices to show $X(t) \to 0$ as $t \to \infty$.

Note that condition (ii) implies that

$$V'_{(1)}(t, X_t) \le -\eta_1(t)W_1(m(X_t)) - \eta_2(t)\overline{W}_2\left(\int_{t-h}^{t} |(X^2(s))'|\, ds\right), \tag{3.1}$$

where $X^2(s) = X(s) \cdot X(s)$, the inner product in R^n and $\overline{W}_2 = W_2(r/2)$.

First we consider the case with condition (a). For $\eta \in IP(\delta)$, there are constants $\xi > 0$ and $T > 0$ such that for all $t \ge T$, $\int_{t-\delta}^{t} \eta_1(s)\, ds \ge \xi$. Now we are going to show $\min_{t-\delta-h\le s\le t} |X^2(s)| \to 0$ as $t \to \infty$ and $\int_{t-h}^{t} |(X^2(s))'|\, ds \to 0$ as $t \to \infty$.

If $\min_{t-\delta-h\le s\le t} |X^2(s)|$ does not tend to zero as $t \to \infty$, then there are a sequence $t_n \to \infty$ and an $\varepsilon > 0$ such that $\min_{t_n-\delta-h\le s\le t_n} |X^2(s)| \ge \varepsilon$ for all n. We may assume that $t_{n+1} - t_n > 2\delta + h$ and $t_n > 2\delta$. Then, by condition (ii),

$$\begin{aligned} 0 &\le V(t_n, X_{t_n}) \le V(t_0, X_{t_0}) - \sum_{i=1}^{n} \int_{t_i-\delta-h}^{t_i} \eta_1(s)W_1(m(X_s))\, ds \\ &\le V(t_0, X_t) - W_1(\varepsilon)\sum_{i=1}^{n} \int_{t_i-\delta}^{t_i} \eta_1(s)\, ds \to -\infty \end{aligned}$$

as $t \to \infty$, a contradiction.

Therefore $\min_{t-\delta-h\le s\le t}|X^2(s)| \to 0$ as $t \to \infty$. Now if $\int_{t-h}^{t}|(X^2(s))'|\, ds$ does not tend to zero as $t \to \infty$, then there are a sequence $u_n \to \infty$ and an $\varepsilon > 0$ such that $\int_{u_n-h}^{u_n} |(X^2(s))'|\, ds \ge \varepsilon$. We may assume $u_{n+1} - u_n > 2h$, $u_n > 2h$. By Lemma 2.6, there are a_n and b_n such that $b_n - a_n = h$, $u_n \in (a_n, b_n)$ and, for every $t \in [a_n, b_n]$, $\int_{t-h}^{t} |(X^2(s))'|\, ds \ge \varepsilon/2$. By condition (3.1), for $t > b_n$,

$$\begin{aligned} 0 &\le V(t, X_t) \\ &\le V(t_0, X_{t_0}) - \sum_{i=1}^{n} \int_{a_i}^{b_i} \eta_2(s)\overline{W}_2\left(\int_{s-h}^{s} .|(X^2(r))'|\, dr\right) ds \\ &\le V(t_0, X_t) - \overline{W}_2(\varepsilon/2)\sum_{i=1}^{n} \int_{a_i}^{b_i} \eta_2(s)\, ds \to -\infty \end{aligned}$$

as $t \to \infty$, a contradiction.

Therefore $\int_{t-h}^{t} |(X^2(s))'|\, ds \to 0$ as $t \to \infty$.

Find an integer $N > 0$ such that $Nh \geq \delta$. Then, by Sobolev's inequality,

$$\begin{aligned}\max_{t-\delta-h\leq s\leq t} |X^2(s)| &\leq \min_{t-\delta-h\leq s\leq t} |X^2(s)| \\ &\quad + \int_{t-\delta-h}^{t} |(X^2(s))'|\, ds \\ &\leq \min_{t-\delta-h\leq s\leq t} |X^2(s)| \\ &\quad + \sum_{i=1}^{N} \int_{t-(i+1)h}^{t-ih} |(X^2(s))'|\, ds \to 0\end{aligned}$$

as $t \to \infty$. Therefore, $|X(t)| \to 0$ as $t \to \infty$. This proves the first case.

To prove the second case with condition (b), we may assume, for the sake of contradiction, that $|X(t)|$ does not tend to zero as $t \to \infty$. Then there are a sequence t_n and an $\varepsilon > 0$ such that $|X(t_n)| \geq \varepsilon$. But if there is a constant $S \geq 0$ such that $|X(t)| \geq \varepsilon/2$ for all $t \geq S$, then for $t \geq S + h$, by (ii),

$$\begin{aligned}0 \leq V(t, X_t) &\leq V(t_0, X_{t_0}) - \int_{S+h}^{t} \eta_1(s) W_1(m(X_s))\, ds \\ &\leq V(t_0, X_{t_0}) - W_1(\varepsilon/2) \int_{S+h}^{t} \eta_1(s)\, ds \to \infty\end{aligned}$$

as $t \to \infty$, a contradiction. Therefore, we may assume that there is a sequence $s_n \to \infty$ such that $|X(s_n)| < \varepsilon/2$. We may further assume, by considering its subsequence if necessary, $t_0 + 2h < t_n < s_n < t_{n+1} < s_{n+1}$. Then by (3.1), for $t \geq s_n + h$,

$$0 \leq V(t, X_t) \leq V(t_0, X_{t_0}) - (\eta_2/2) \int_{t_0+2h}^{t} \int_{s-h}^{s} |(X^2(r))'|\, dr\, ds$$

(note that η_2 is constant and $W_2(r) = r$ in this case)

$$\leq V(t_0, X_{t_0}) - (\eta_2/2) \int_{t_0+2h}^{t-h} \int_{r}^{r+h} |(X^2(r))'|\, ds\, dr$$

(by changing the order of the integration)

$$
\begin{aligned}
&\le V(t_0, X_{t_0}) - (\eta_2 h/2) \sum_{i=1}^{n} \int_{t_i}^{s_i} |(X^2(r))'| \, dr \\
&\le V(t_0, X_{t_0}) - (\eta_2 h/2) \sum_{i=1}^{n} [|X^2(t_i)| - |X^2(s_i)|] \\
&\le V(t_0, X_{t_0}) - (\eta_2 h/2)(3\varepsilon^2/4)n \to -\infty
\end{aligned}
$$

as $n \to \infty$, a contradiction. Therefore $X(t) \to 0$ as $t \to \infty$. This proves the second case.

To show the case with condition (c), we may assume that $|X(t)|$ does not tend to zero as $t \to \infty$ and consider the sequence $\{t_n\}$ and $\varepsilon > 0$ found in the second case. For this constant $\varepsilon > 0$, find the $\delta_0 > 0$ of U.S. Then, for each $t \ge t_0$, there is at least one $t_* \in [t-h, t]$ such that $|X(t_*)| \ge \delta_0$. We can also find a sequence $q_n \to \infty$ such that $|X(q_n)| \le \delta_0/2$ as we found the sequence $\{s_n\}$ in the second case. Now for each q_n, there is an $r_n \in [q_n - h, q_n]$ such that $|X(r_n)| \ge \delta_0$. Clearly, $0 < q_n - r_n < h$, and we may also assume $q_{n+1} - q_n > 2h$, $q_n > 2h$ for all n. Note that

$$
\begin{aligned}
\int_{q_n - h}^{q_n} |(X^2(s))'| \, ds &\ge \int_{r_n}^{q_n} |(X^2(s))| \, ds \\
&\ge |X^2(r_n)| - |X^2(q_n)| \ge 3\delta_0/4.
\end{aligned}
$$

By Lemma 2.6, there are a_n, b_n with $b_n - a_n = h$, $q_n \in (a_n, b_n)$ and $\int_{t-h}^{t} |(X^2(s))'| \, ds \ge 3\delta_0/8$ for all $t \in [a_n, b_n]$. Then by (3.1), for all $t \ge b_n$,

$$
\begin{aligned}
0 \le V(t, X_t) \le V(t_0, X_{t_0}) &- \sum_{i=1}^{n} \int_{a_i}^{b_i} \eta_2(s) \overline{W}_2 \left(\int_{s+h}^{s} |(X^2(r))'| \, dr \right) ds \\
&\le V(t_0, X_{t_0}) - \overline{W}_2(3\delta_0/8) \sum_{i=1}^{n} \int_{a_i}^{b_i} \eta_2(s) \, ds \to -\infty
\end{aligned}
$$

as $t \to \infty$, a contradiction. Therefore $X(t) \to 0$ as $t \to \infty$. This proves the third case.

Finally, we are going to show the case with condition (d). Note that $\eta(t) := \min\{\eta_1(t), \eta_2(t)\} \notin L^1[0, \infty)$. Therefore (3.1) implies

$$\begin{aligned} V'_{(1)}(t, X_t) &\le -\eta(t)\left[W_1(m(X_t)) + \overline{W}_2\left(\int_{t-h}^t |(X^2(s))'|\, ds\right)\right] \\ &\le -\eta(t)W_3\left(m^2(X_t) + \int_{t-h}^t |(X^2(s))'|\, ds\right) \\ &\quad (W_3 \text{ is some wedge by Lemma 2.5}) \\ &= -\eta(t)W_4(\|X_t\|), \quad \text{where } W_4(r) = W_3(r^2) \\ &\quad (\text{by Sobolev's inequality}). \end{aligned}$$

Thus we have

$$V'_{(1)}(t, X_t) \le -\eta(t)W_4(\|X_t\|). \tag{3.2}$$

Suppose that $X = 0$ is U.S. and $X(t)$ does not tend to zero as $t \to \infty$. Then, as we discussed previously, there is a constant δ_0 such that for each $t \ge t_0$, there is at least one $t_* \in [t-h, t]$ satisfying $|X(t_*)| \ge \delta_0$. This implies that, for all $t \ge t_0$, $\|X_t\| \ge \delta_0$. Then (3.2) implies $V'_{(1)}(t, X_t) \le -\eta(t)W_4(\delta_0)$. Integrating this inequality, we obtain

$$0 \le V(t, X_t) \le V(t_0, X_{t_0}) - W_4(\delta_0)\int_{t_0}^t \eta(s)\, ds \to -\infty$$

as $t \to \infty$, a contradiction. Therefore $X(t) \to 0$ as $t \to \infty$ and $X = 0$ is A.S. □

Remark. It is clear that Theorem 3.1 remains true if condition (ii) of the theorem is changed to

(ii)′ $$V'_{(1)}(t, X_t) \le -\eta_1(t)W_1(m(X_t)) - \eta_2(t)W_2\left(\int_{t-h}^t |X'(s)|\, ds\right),$$

which is more often seen.

Theorem 3.2. *Let $V : R_+ \times C_H \to R_+$ be continuous and suppose that*

(i) $X = 0$ *is* U.S.

and

(ii) *either* (a)

$$V'_{(1)}(t, X_t) \leq -\eta_1(t) W_1\left(\int_{t-h}^{t} |X(s)|\, ds\right) - \eta_2(t) W_3(|X(t)|) W_4(|X'(t)|),$$

or (b)

$$V'_{(1)}(t, X_t) \leq -\eta_1(t) W_1(|X(t)|) - \eta_2(t) W_3(|X(t)|) W_4(|X'(t)|),$$

where $\eta_1 \in IP(\delta_1)$ *for some* $\delta_1 > 0$, $\eta_2 \in WDIS(\delta_2)$ *for some* $\delta_2 \geq \max(\delta_1, h)$, *and* W_4 *convex downward.*

Then $X = 0$ *is* A.S.

Proof. For $\eta_2 \in WDIS(\delta_2)$, find the sequence $t_n \to \infty$ with $\sum_{i=1}^{\infty} \bar{\eta}_2(t_n) = \infty$, $\bar{\eta}_2(t_n) = \min_{t_n - \delta_2 \leq s \leq t_n} \eta_2(s)$. We may assume $t_n \geq \delta_1$ and $t_{n+1} - t_n \geq \max(\delta_1, \delta_2, h)$ by Lemma 2.1.

Let $X(t) = X(t, t_0, \phi)$ be a solution of (1) with $|X(t)| \leq H$ and suppose that $|X(t)|$ does not tend to zero as $t \to \infty$. Then there are an $\varepsilon_0 > 0$ and a sequence $s_n \to \infty$ with $|X(s_n)| \geq \varepsilon_0$. For this $\varepsilon_0 > 0$, find the δ_0 of U.S. Then in any interval $[t-h, t]$ there is a $\bar{t} \in [t-h, t]$ with $|X(\bar{t})| \geq \delta_0$ since $|X(s_n)| \geq \varepsilon_0$.

In particular, consider $I_n = [t_n - \delta_1, t_n]$. We may assume $h \leq \delta_1 \leq \delta_2$ since $IP(\delta) \subseteq IP(h)$ for any $\delta \leq h$. Then there is a $b_n \in I_n$ with $|X(b_n)| \geq \delta_0$. If there are infinite many I_n's, say I_{n_j}, such that $|X(t)| \geq \delta_0/2$ for all $t \in I_{n_j}$, then either by (a)

$$\begin{aligned} 0 &\leq V(t_{n_k} + h/2, X_{t_{n_k}+h/2}) \\ &\leq V(t_0, X_{t_0}) - \sum_{j=1}^{k} \int_{t_{n_j}-\delta_1+h/2}^{t_{n_j}+h/2} \eta_1(s) W_1\left(\int_{s-h}^{s} |X(r)|\, dr\right) ds \\ &\leq V(t_0, X_{t_0}) - W_1(\delta_0 h/4) \sum_{j=1}^{k} \int_{t_{n_j}-\delta_1+h/2}^{t_{n_j}+h/2} \eta_1(s)\, ds \to -\infty, \end{aligned}$$

(since, in any case, $[s-h, s]$ always contains a subinterval of length $h/2$ of $[t_{n_j} - \delta_1, t_{n_j}]$ if $s \in [t_{n_j} - \delta_1 + h/2, t_{n_j} + h/2]$), as $t \to \infty$, or by (b)

$$0 \le V(t_{n_k}, X_{t_{n_k}}) \le V(t_0, X_{t_0}) - \sum_{j=1}^{k} \int_{t_{n_j}-\delta_1}^{t_{n_j}} \eta_1(s) W_1(|X(s)|)\, ds$$
$$\le V(t_0, X_{t_0}) - W_1(\delta_0/2) \sum_{j=1}^{k} \int_{t_{n_j}-\delta_1}^{t_{n_j}} \eta_1(s)\, ds \to -\infty$$

as $t \to \infty$, a contradiction.

Therefore, we may assume that for each I_n there is an $a_n \in I_n$ such that $a_n < b_n$ (by renaming if necessary), $|X(a_n)| = \delta_0/2$, and $|X(t)| \ge \delta_0/2$ for $t \in [a_n, b_n]$. Now, by condition (ii),

$$\begin{aligned}
0 &\le V(t_n, X_{t_n}) \\
&\le V(t_0, X_{t_0}) - \sum_{i=1}^{n} \int_{t_i-\delta_2}^{t_i} \eta_2(s) W_3(|X(s)|) W_4(|X'(s)|)\, ds \\
&\le V(t_0, X_{t_0}) - \sum_{i=1}^{n} \bar{\eta}_2(t_i) \int_{t_i-\delta_2}^{t_i} W_3(|X(s)|) W_4(|X'(s)|)\, ds \\
&\le V(t_0, X_{t_0}) - \sum_{i=1}^{n} \bar{\eta}_2(t_i) \int_{t_i-\delta_2}^{t_i} W_3(1) W_4\left(\frac{W_3(|X(s)|)}{W_3(1)} |X'(s)|\right) ds
\end{aligned}$$

(assume $|X(t)| < 1$, and for W_4 a convex downward wedge,

$$\begin{aligned}
&W_4(at) \le aW_4(t) \text{ if } 0 \le a \le 1) \\
&\le V(t_0, X_{t_0}) \\
&\quad - W_3(1) \sum_{i=1}^{n} \bar{\eta}_2(t_i) \delta_2 W_4\left(\frac{1}{W_3(1)\delta_2} \int_{t_i-\delta_2}^{t_i} W_3(|X(s)|) |X'(s)|\, ds\right)
\end{aligned}$$

(by Jensen's inequality)

$$\le V(t_0, X_{t_0}) - \delta_2 W_3(1) \sum_{i=1}^{n} \bar{\eta}_2(t_i) W_4\left(\frac{W_3(\delta_0/2)}{W_3(1)\delta_2} \int_{a_i}^{b_i} |X'(s)|\, ds\right)$$

$$\leq V(t_0, X_{t_0}) - \delta_2 W_3(1) W_4\left(\frac{W_3(\delta_0/2)}{W_3(1)\delta_2} \cdot (\delta_0/2)\right) \sum_{i=1}^{n} \bar{\eta}_2(t_i) \to -\infty$$

as $t \to \infty$, a contradiction.

Therefore, $X(t) \to 0$ as $t \to \infty$ and $X = 0$ is A.S.

The proof is complete. □

As consequences of the theorems, we have the following interesting corollaries which generalize a well-known result, Theorem 4.2.12(iii) of [**2**, p. 277] in some sense.

Corollary 3.1. *Let $V : R_+ \times C_H \to R_+$ be continuous, and suppose that*

(i) *$X = 0$ is stable,*

(ii) *$F(t, \phi)$ is bounded for ϕ bounded, and*

(iii) *$V'_{(1)}(t, X_t) \leq -\eta(t) W_1(\int_{t-h}^{t} |X(s)|\, ds)$.*

Then $X = 0$ is A.S. *if one of the following holds*:

(a) *$\eta \in IP(h)$, or*

(b) *$X = 0$ is* U.S. *and $\eta \notin L^1[0, \infty)$.*

Proof. By condition (ii), there is a constant $k > 0$ such that $|F(t, \phi)| \leq k$ for all $t \in R_+$ and $||\phi|| \leq H$. Then, for all solutions $X(t)$ with $|X(t)| \leq H$, condition (iii) implies that

$$V'_{(1)}(t, X_t) \leq -\frac{\eta(t)}{2} W_1\left(\int_{t-h}^{t} |X(s)|\, ds\right) - \frac{\eta(t)}{2} W_1\left(\frac{1}{k}\int_{t-h}^{t} |X(s)|\, |X'(s)|\, ds\right).$$

Therefore conditions (i), (ii), and either (a) or (d) of Theorem 3.1 are satisfied so $X = 0$ is A.S. □

Corollary 3.2. *Let $V : R_+ \times C_H \to R_+$ be continuous and suppose that*

(i) $X = 0$ *is* U.S.

(ii) $F(t, \phi)$ *is bounded for* ϕ *bounded, and*

(iii) $V'_{(1)}(t, X_t) \le -\eta(t) W_1(|X(t)|)$, *where* η *is nonincreasing and* $\eta \notin L^1[0, \infty)$.

Then $X = 0$ *is* A.S.

Proof. By the discussion of [**8**], there is another Liapunov functional $U : R_+ \times C_{\overline{H}} \to R_+$ satisfying

$$U'_{(1)}(t, X_t) \le -\eta(t) W_1\bigg(\int_{t-h}^{t} |X(s)|\, ds \bigg).$$

This is just condition (iii) of Corollary 3.1. Hence $X = 0$ is A.S. ◻

Corollary 3.3. *Let* $V : R_+ \times C_H \to R_+$ *be continuous and suppose that*

(i) $X = 0$ *is* U.S.,

(ii) $V'_{(1)}(t, X_t) \le -W_1(|X(t)|)$,

(iii) *there is an* $R > 0$ *such that, for*

$$f_R(t) := \sup_{t-h \le s \le t} \sup_{||\phi|| \le R} |F(t, \phi)|,$$

and

(iv) *there is a sequence* $t_n \to \infty$ *with* $t_{n+1} - t_n \ge \delta$ *for some constant* $\delta > 0$ *and* $\sum_{i=1}^{\infty} 1/(f_R(t_n) + 1) = \infty$.

Then $X = 0$ *is* A.S.

Proof. Define

$$\eta(t) = \frac{1}{1 + \sup_{||\phi|| \le R} |F(t, \phi)|}.$$

Then

$$\begin{aligned}\bar{\eta}(t) = \min_{t-h \le s \le t} \eta(s) &= \frac{1}{\sup_{t-h \le s \le t}(1 + \sup_{||\phi|| \le R} |F(s, \phi)|)} \\ &\ge \frac{1}{1 + \sup_{t-h \le s \le t} \sup_{||\phi|| \le R} |F(s, \phi)|} \ge \frac{1}{1 + f_R(t)}.\end{aligned}$$

Then $\sum_{n=1}^{\infty} \eta(t) \geq \sum_{n=1}^{\infty} 1/(1+f_R(t_n)) = \infty$. Therefore, $\eta \in WDIS(h)$.

Note that for any solution $X(t)$ with $|X(t)| \leq \min(H, R)$,

$$|X'(t)| = |F(t, X_t)| \leq \sup_{||\phi|| \leq R} |F(t, \phi)| + 1 = 1/\eta(t).$$

By condition (ii),

$$V'_{(1)}(t, X_t) \leq -(1/2)W_1(|X(t)|) - (1/2)\eta(t)W_1(|X(t)|)|X'(t)|.$$

So conditions (i) and (ii)(b) of Theorem 3.2 are satisfied and then $X=0$ is A.S. ◻

4. Applications.

Example A. Consider the scalar equation

$$\text{(A)} \qquad x'(t) = a(t)x(t) + b(t)x(t-h)$$

where $b(t) = t\ln t + (\sin t + |\sin t|)e^t$, $a(t) = -[b(t+h)+1]$ and $h = \pi/2$.

Then $X = 0$ of (A) is A.S.

Proof. Define $V(t, x_t) = |x(t)| + \int_{t-h}^{t} |b(s+h)||x(s)|\, ds$. Then

$$V'(t, x_t) \leq (a(t) + |b(t+h)|)|x(t)| = -|x(t)|.$$

Let $t_n = (2n-1)\pi + \pi/2$, $n = 1, 2, 3, \dots$. Then

$$\begin{aligned} f_1(t_n) &= \sup_{t_n - \pi/2 \leq s \leq t_n} \sup_{||\phi|| \leq 1} |F(s, \phi)| \\ &\leq \sup_{t_n - \pi/2 \leq s \leq t_n} [b(s+h) + 1 + b(s)] \\ &\leq 2b(t_n - \pi/2) + 1 = 2(2n-1)\pi \ln(2n-1)\pi + 1. \end{aligned}$$

Clearly $t_{n+1} - t_n \geq \pi/2$ and $\sum_{n=1}^{\infty} 1/(f_1(t_n)+1) = \infty$.

It is not hard to prove that $x = 0$ is U.S. by Razumikhin argument with x^2.

Thus all conditions of Corollary 3.3 are satisfied and $x = 0$ of (A) is A.S. The proof is complete □

The elementary function $b(t)$ is given by T.A. Burton.

Remark. (a) In this example $b(t)$ can be unbounded of order e^t and even higher order. We can still have asymptotic stability.

(b) Theorem 1.1 cannot be applied to this example.

Example B. Burton and Hatvani [**4**] considered the scalar equation

$$x'(t) = b(t)x(t-h) \tag{B1}$$

with $b : [-h, \infty) \to [-1, 0]$ continuous. Under the conditions:

$$-2 + \int_{t-h}^{t} |b(u)|\, du + h \le 0, \tag{B)(i}$$

and

$$b(t+h) - b(t), \qquad \int_{t-h}^{t} [1 - |b(s)|]\, ds > 0, \tag{B)(ii}$$

they proved that $x = 0$ of (B1) is A.S.

They also considered the generalization of (B1), the nonlinear scalar equation

$$x'(t) = b(t)f(x(t-h)) \tag{B2}$$

with continuous functions $b : R_+ \to R$ and $f : R \to R$. If there are constants $c > 0$ and $\alpha > 0$ such that

(B)(iii)

$$xf(x) > 0 \qquad \text{for } x \ne 0 \quad \text{and} \quad |f(x)| \le c|x| \qquad \text{for all } x \in R,$$

and

$$|b(t)| \le \alpha \le 2/(\alpha h) - (1/h) \int_{t}^{t+h} |b(u)|\, du \qquad \text{for } t \in R_+, \tag{B)(iv}$$

and

$$\text{(B)(v)} \qquad \begin{gathered}\alpha - |b(t)| \in IP(h_1)\\ \text{for some } h_1 \in (0,h) \quad \text{and} \quad b \notin L^1[0,\infty),\end{gathered}$$

then they proved that $x = 0$ is A.S.

To prove the result, they considered the Liapunov functional

$$\begin{aligned} V(t,x_t) = {} & \left(x(t) + \int_{t-h}^{t} b(s+h) f(x(s))\,ds\right)^2 \\ & + \alpha \int_{-h}^{0} \int_{t+u}^{t} |b(s+h)| f^2(x(s))\,ds\,du, \end{aligned} \tag{4.1}$$

and showed that (also see [**8**])

$$\begin{aligned} |x(t+h/2)|^4/16 & \le V(t,x_t) \\ & \le 2x^2(t) + 3\alpha^2 c^2 h \int_{t-h}^{t} x^2(s)\,ds, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} V'(t,x_t) \le {} & -\gamma(t) x(t) g(x(t)) \\ & - \eta(t) \int_{t-h}^{t} |b(s+h)| f^2(x(s))\,ds, \end{aligned} \tag{4.3}$$

where $\gamma(t) = |b(t+h)|[2 - c(\alpha h + \int_t^{t+h} |b(s)|\,ds)]$, and $\eta(t) = \alpha - |b(t+h)|$.

Note that (4.3) implies $V'(t,x_t) \le 0$. Therefore $x = 0$ of (B2) is A.S. Then by our Theorem 3.1 with conditions (i), (ii) and (d), we can prove the next theorem.

Theorem 4.1. *If both* $b : [-h,\infty) \to R$ *and* f *of* (B2) *satisfy conditions* (B)(iii), (B)(iv) *and*
(B)(vi)

$$m(t) := \min\left\{\eta(t)\int_t^{t+h} |b(s)|\,ds, \eta(t-h)\int_t^{t+h} |b(s)|\,ds\right\} \notin L^1[0,\infty),$$

where $\eta(t) = \alpha - |b(t)| \ge 0$*, then* $x = 0$ *of* (B2) *is* A.S.

Proof. We also consider the Liapunov functional $V(t,x_t)$ defined by (4.1). Then (4.2) and (4.3) hold, too. Therefore $x=0$ is U.S.

Consider a new Liapunov functional $U(t,x_t)$ defined by

$$U(t,x_t) = V(t,x_t) + V(t-h, x_{t-h}).$$

Then

$$\begin{aligned} U'(t,x_t) &= V'(t,x_t) + V'(t-h,x_{t-h}) \\ &\le -\eta(t)\int_{t-h}^{t} |b(s+h)| f^2(x(s))\,ds \\ &\quad - \eta(t-h)\int_{t-2h}^{t-h} |b(s+h)| f^2(x(s))\,ds. \end{aligned}$$

Since we only consider the local stability, there is a wedge W such that

$$\begin{aligned} U'(t,x_t) &\le -\eta(t)\int_{t-h}^{t} |b(s+h)|\,ds W(m(x_t)) \\ &\quad - \eta(t-h)\int_{t-h}^{t} |b(s)| f^2(x(s))\,ds \\ &\le -\eta(t)\int_{t-h}^{t} |b(s+h)|\,ds W(m(x_t)) \\ &\quad - \frac{\eta(t-h)}{\alpha h}\left[\int_{t-h}^{t} |b(s)|\,|f(x(s-h)|\,ds\right]^2 \end{aligned}$$

(by Hölder's inequality and $|b(t)| \le \alpha$. $\frac{1}{\alpha}\int_{t-h}^{t} |b(s)|\,ds \le 1$)

$$\begin{aligned} &\le -\eta(t)\int_{t-h}^{t} |b(s+h)|\,ds W(m(x_t)) \\ &\quad - [\eta(t-h)/(\alpha h)^2]\int_{t-h}^{t} |b(s+h)|\,ds \left[\int_{t-h}^{t} |x'(s)|\,ds\right]^2. \end{aligned}$$

Therefore, conditions (i), (ii) and (d) of Theorem 3.1 are satisfied. Hence, $x=0$ of (B2) is A.S. □

It is clear that if $b(t)$ is either an h-periodic function with $\eta \notin L^1[0,\infty)$ or an increasing function such that $\eta(t) \to 0$ as $t \to \infty$ and $\eta \notin$

$L^1[0,\infty)$, then condition (B)(vi) can be satisfied. Therefore, (B)(vi) is different from (B)(v) and also generalizes (B)(ii).

Example C. Consider the nonlinear scalar equation

$$x'(t) = -a(t)x^3(t) + b(t)x^3(t-h) \tag{C}$$

with $a, b : [-h,\infty) \to R$ continuous.

Theorem 4.2. *Let* $\lambda(t) = \max\{|a(t)|, |b(t+h)|\}$ *satisfy*
(C1)

$$\lambda \notin L^1[0,\infty) \quad \textit{and} \quad \int_{t-h}^{t} \lambda(s)\,ds \le K \quad \textit{for some constant } K > 0.$$

If there are constants $\alpha > 0$, $\beta > 0$ *and* $\gamma > 0$ *such that*

(C2)
$$2[-a(t) + b(t+h)] + \beta|a(t) - b(t+h)| \int_{t-h}^{t} |b(u+h)|\,du + \alpha h\lambda(t) \le 0,$$

and

$$\gamma|a(t) - b(t+h)| - \alpha \le 0, \tag{C3}$$

then $x = 0$ *of* (C) *is* A.S.

Proof. Define

$$V(t, x_t) = \left[x(t) + \int_{t-h}^{t} b(u+h)x^3(u)\,du\right]^2 + \alpha \int_{-h}^{0}\int_{t+s}^{t} \lambda(u)x^4(u)\,du\,ds.$$

Then

$$V'(t, x_t) = 2\left(x(t) + \int_{t-h}^{t} b(u+h)x^3(u)\,du\right)[-a(t) + b(t+h)]x^3(t)$$

$$
\begin{aligned}
&+\alpha\int_{-h}^{0}\lambda(t)x^4(t)\,ds-\alpha\int_{-h}^{0}\lambda(t+s)x^4(t+s)\,ds\\
&=\{2[-a(t)+b(t+h)]+\alpha h\lambda(t)\}x^4(t)\\
&-\alpha\int_{t-h}^{t}\lambda(u)x^4(u)\,du\\
&+|a(t)-b(t+h)|\int_{t-h}^{t}|b(u+h)|(x^6(t)+x^6(u))\,du\\
&\le\Big\{2[-a(t)+b(t+h)]\\
&+\beta|a(t)-b(t+h)|\int_{t-h}^{t}|b(u{+}h)|\,du+\alpha h\lambda(t)\Big\}x^4(t)\\
&-(\alpha/2)\int_{t-h}^{t}\lambda(u)x^4(u)\,du\\
&+(1/2)[\gamma|a(t)-b(t+h)|-\alpha]\int_{t-h}^{t}|b(u+h)|x^4(u)\,du
\end{aligned}
$$

(assume $|x(t)|\le\min(\sqrt{\beta},\sqrt{\gamma/2},1)$)

$$
\begin{aligned}
&\le-(\alpha/4)\int_{t-h}^{t}\lambda(u)\,du m^4(x_t)-(\alpha/4)\int_{t-h}^{t}\lambda(u)x^4(u)\,du\\
&\le-(\alpha/4)\int_{t-h}^{t}\lambda(u)\,du m^4(x_t)-\frac{\alpha}{4k^3}\bigg(\int_{t-h}^{t}\lambda(u)|x(u)|\,du\bigg)^4
\end{aligned}
$$

since $(\int_{t-h}^{t}\lambda(u)|x(u)|\,du)^4 \le (\int_{t-h}^{t}\lambda(u)\,du)^3\int_{t-h}^{t}\lambda(u)x^4(u)\,du$ by Hölder's inequality.

Define $U(t,x_t) = V(t,x_t) + V(t-h, x_{t-h})$. Then

$$\begin{aligned}U'(t,x_t) &\le -(\alpha/4)\int_{t-h}^{t} \lambda(u)\,du m^4(x_t) - \frac{\alpha}{4k^3}\bigg(\int_{t-h}^{t} \lambda(u)|x(u)|\,du\bigg)^4 \\ &\quad - \frac{\alpha}{4k^3}\bigg(\int_{t-2h}^{t-h} \lambda(u)|x(u)|\,du\bigg)^4 \\ &\le -(\alpha/4)\int_{t-h}^{t} \lambda(u)\,du m^4(x_t) \\ &\quad - \frac{\alpha}{32k^3}\bigg(\int_{t-h}^{t} \lambda(u)|x(u)|\,du + \int_{t-h}^{t} \lambda(u-h)|x(u-h)|\,du\bigg)^4\end{aligned}$$

($(a+b)^4/8 \le a^4 + b^4$ for any real numbers a and b)

$$\le -(\alpha/4)\int_{t-h}^{t} \lambda(u)\,du m^4(x_t) - \frac{\alpha}{32k^3}\bigg(\int_{t-h}^{t} |x'(u)|\,du\bigg)^4.$$

If we apply Theorem 3.1 with (a), then we get asymptotic stability of the zero solution easily if $\int_{t-h}^{t} \lambda(u)\,du \in IP(\delta)$ for some $\delta > 0$. To prove Theorem 4.2, we need more work. We have to prove that $x = 0$ is U.S. to apply Theorem 3.1 with (c).

First, we have

$$\begin{aligned}V(t,x_t) &\le 2x^2(t) + 2\bigg(\int_{t-h}^{t} b(u+h)x^3(u)\,du\bigg)^2 \\ &\quad + \alpha h\int_{t-h}^{t} \lambda(u)x^4(u)\,du \\ &\le 2x^2(t) + 2K^2||x_t||^6 + \alpha hK||x_t||^4.\end{aligned} \tag{4.4}$$

Note that

$$V(t,x_t) \ge \alpha\int_{-h}^{0}\int_{t+s}^{t} \lambda(u)x^4(u)\,du = \alpha\int_{t-h}^{t} \lambda(u)x^4(u)(u-t+h)\,du$$

(by changing the order of integration)

$$\geq (\alpha h/2) \int_{t-h/2}^{t} \lambda(u) x^4(u)\, du.$$

Therefore,

$$V(t, x_t) + V(t - h/2, X_{t-h/2}) \geq (\alpha h/2) \int_{t-h}^{t} \lambda(u) x^4(u)\, du.$$

Define $I = |\int_{t-h}^{t} b(u+h) x^3(u)\, du|$.

If $I \leq |x(t)|/2$, then $V(t, x_t) \geq [|x(t)| - I]^2 \geq |x(t)|^2/4$.

If $I \geq |x(t)|/2$, then by Hölder's inequality

$$\begin{aligned} |x(t)|^2/4 \leq I^2 &\leq \int_{t-h}^{t} |b(u+h)|\, du \int_{t-h}^{t} |b(u+h)| x^6(u)\, du \\ &\leq K \int_{t-h}^{t} \lambda(u) x^4(u)\, du \end{aligned}$$

(assume $|x(t)| \leq \min(\sqrt{\beta}, \sqrt{\gamma/2}, 1)$)

$$\begin{aligned} &\leq \frac{2K}{\alpha H}(V(t, x_t) + V(t - h/2, x_{t-h/2})) \\ &\leq \frac{4K}{\alpha h} V(t - h/2, x_{t-h/2}), \qquad V' \leq 0. \end{aligned}$$

Therefore,

$$|x(t)|^2/4 \leq \left(1 + \frac{4K}{\alpha h}\right) V(t - h/2, x_{t-h/2}). \tag{4.5}$$

Now (4.4), (4.5) and $V' \leq 0$ imply that $x = 0$ of (C) is U.S. Then by Theorem 3.1 with (c), $x = 0$ of (C) is A.S.

The proof is complete. □

This example was also discussed in [**3**]. Our conditions are different from those there. Since we only discuss the local stability, our result is also interesting.

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