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ASYMPTOTICALLY AUTONOMOUS DIFFERENTIAL EQUATIONS IN THE PLANE

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. We present a couple of examples where the solutions of asymptotically autonomous differential equations behave quite differently from the solutions of the corresponding limit equations. Nevertheless a Poincaré Bendixson type limit set trichotomy can be shown in the plane.

1. Introduction. An ordinary differential equation in $\mathbf{R}^n$,

$$\dot{x} = f(t,x), \tag{1.1}$$

is called *asymptotically autonomous*—with *limit equation*

$$\dot{y} = g(y), \tag{1.2}$$

if

$$f(t,x) \to g(x), \qquad t \to \infty, \quad \text{locally uniformly in } x \in \mathbf{R}^n,$$

i.e. for x in any compact subset of $\mathbf{R}^n$. For simplicity we assume that $f(t,x), g(x)$ are continuous functions and locally Lipschitz in x.

In an often quoted (and sometimes misquoted) paper, L. Markus [**23**] presents the following theorems concerning the ω-limit sets, $\omega(t_0,x_0)$, of forward bounded solutions x to (1.1), subject to $x(t_0) = x_0$,

$$\omega(t_0,x_0) = \bigcap_{s>t_0} \overline{\{x(t); t \geq s\}}.$$

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Theorem 1.1 (*Markus*). *The ω-limit-set ω of a forward bounded solution x to* (1.1) *is nonempty, compact, and connected. Moreover ω attracts x, i.e.,*

$$dist(x(t), \omega) \to 0, \qquad t \to \infty.$$

Finally ω is invariant under (1.2). *In particular any point in ω lies on a full orbit of* (1.2) *that is contained in ω.*

Theorem 1.1 [**23**, Theorem 1 and preceding remarks] has sometimes been misquoted in the form that the ω-limit sets of (1.1) are unions of ω-limit sets of (1.2). Planar counter-examples will be presented in Section 3 (Examples 3.1, 3.2, 3.4, 3.5).

Theorem 1.2 (*Markus*). *Let e be a locally asymptotically stable equilibrium of* (1.2) *and ω the ω-limit set of a forward bounded solution x of* (1.1). *If ω contains a point y_0 such that the solution of* (1.2) *through $(0, y_0)$ converges to e for $t \to \infty$, then $\omega = \{e\}$, i.e. $x(t) \to e$, $t \to \infty$.*

Actually Markus proves more in his Theorem 2, but most applications use the formulation in Theorem 1.2 which is a consequence of [**23**, Theorems 1 and 2]. [**23**, Theorem 2], in its original formulation, has been applied in [**8**], e.g., in the proof of their Theorem 5.3. A generalization of [**23**], Theorem 2 can be found in [**14**, III.2, Exercise 2.4].

Markus's [**23**, Theorem 7] generalizes the Poincaré & Bendixson theorem to asymptotically autonomous planar systems.

Theorem 1.3 (*Markus*). *Let $n = 2$ and ω the ω-limit set of a forward bounded solution x of* (1.1). *Then ω either contains equilibria of* (1.2) *or is the union of periodic orbits of* (1.2).

Theorem 1.1 has heavily stimulated the development of the qualitative theory of nonautonomous differential equations and dynamical systems (see [**24, 26, 27, 10**], as a small sample of references). It has been generalized to Volterra integral equations by Miller & Sell [**25**].

Theorems 1.2 and 1.3 are often applied to show that the solutions of population dynamic (notably chemostat) models converge to an equilibrium (Theorem 1.2: [**9, 18, 19, 20**]; Theorem 1.3: [**4**]). Theorem 1.3 has also triggered some research on almost periodic solutions of asymptotically autonomous ODEs in the plane [**34, 12**].

Somehow Markus's paper has generated the feeling that the so-called *Inverse Limit Problem* has been fairly completely solved for asymptotically autonomous systems [**27**, Section 7], and not much work on this topic has seemingly been done in the last ten or even twenty years [**1, 2**] other than applying the known results. In Sections 2 and 3 we present a couple of one- and two-dimensional examples for which the solutions to (1.1) display a large-time behavior that differs dramatically from that of the solutions to the limit system (1.2). So we raise the *Inverse Limit Problem* again phrasing it as follows:

Question 1.4. Assume that the equilibria of (1.2) are isolated and that any solution of (1.2) converges to one of them. Does any solution of (1.1.) converge to an equilibrium of (1.2) as well?

An answer to this question is helpful in analyzing certain chemostat/gradostat and epidemic models where one can show for a multidimensional system that some components converge to a limit independently of what the other components do [**21, 29, 30, 35, 3**]. It is also useful for determining the large-time behavior of solutions to nonlinear reaction-diffusion systems with Neumann boundary conditions [**8, 31**, Chapter 14, Section D].

When considering convergence in an epidemic model [**3**], (Castillo-Chavez, personal communication, triggering this investigation) noticed that Theorems 1.1 to 1.3 were not sufficient to answer Question 1.4 for their special case. The result by Ball [**2**] who assumes the existence of a Lyapunov function does not seem to help in this case either.

The examples in Section 3 illustrate that, even in the plane, Question 1.4 cannot be positively answered without further conditions (see also example (1) in Section 4 of [**2**]). One of the most powerful tools for checking that forward bounded solutions of planar autonomous systems converge towards equilibria is the Dulac criterion (see [**13**], e.g., and, for a generalization, [**5**]). Theorems 1.1 to 1.3 do not allow the application of Dulac's criterion except in the special case of a unique equilibrium that is locally stable. We remedy this situation by extending Theorem 1.3 to a Poincaré & Bendixson type limit set trichotomy.

Theorem 1.5. *Let $n = 2$ and ω the ω-limit set of a forward bounded solution x of* (1.1). *Assume there are at most finitely many equilibria of* (1.2) *in a sufficiently small neighborhood of ω. Then the following trichotomy holds*:

(i) *ω consists of an equilibrium of* (1.2).

(ii) *ω is the union of periodic orbits of* (1.2) *and possibly of centers of* (1.2) *that are surrounded by periodic orbits of* (1.2) *lying in ω.*

(iii) *ω contains equilibria of* (1.2) *that are cyclically chained to each other in ω by orbits of* (1.2).

In the third possibility the ω-limit set contains homoclinic orbits (phase unigons) of (1.2) connecting one equilibrium to itself and/or phase polygons of (1.2) with finitely many sides (connecting equilibria) all of which are traversed in the same direction.

Example 3 in Section 3 shows that a continuum of periodic orbits can coexist with a homoclinic or heteroclinic orbit in the ω-limit set of an asymptotically autonomous planar system. Such a phenomenon does not occur for autonomous planar systems.

Theorem 1.5 is proved in Section 4, where we also explain how convergence of solutions of (1.1) to equilibria of (1.2) follows from the Dulac (divergence) criterion.

The limit-set trichotomy in Theorem 1.5 gives an answer to Question 1.4, namely to rule out cyclic chains of equilibria. In the plane this is also necessary in general as illustrated in Examples 3.1, 3.2, and 3.5 and by the following result which states that any homoclinic cycle or any minimal heteroclinic cycle connecting saddles of a planar autonomous system is the ω-limit set of a nonautonomous exponentially decreasing smooth perturbation of (1.2).

Theorem 1.6. *Let $n = 2$ and assume that g in* (1.2) *is continuously differentiable. Let $e_1, \dots, e_m$ be m saddles of* (1.2) *that are cyclically chained to each other by orbits of* (1.2): $e_1 \mapsto e_2 \mapsto \cdots \mapsto e_m \mapsto e_1$.

Assume that this chain is minimal insofar as $e_k \mapsto e_j$ for $k \geq j$ only if $k = m$, $j = 1$. Consider an initial point q on one of the orbit connections of the saddles. Then there exists a smooth exponentially decreasing function $\phi : [0, \infty) \to \mathbf{R}^2$ —which depends on q—such that the whole chain is the ω-limit set of the orbit through $(0, q)$ of

$$\dot{x} = g(x) + \phi(t). \tag{1.3}$$

More precisely, one can choose $\phi \in C^\infty(\mathbf{R}, \mathbf{R}^2)$,

$$|\phi(t)| \leq ce^{-\varepsilon t}, \qquad t \geq 0, \tag{1.4}$$

with $0 < \varepsilon < -\lambda_j^-$, $j = 1, \ldots, m$, where λ_j^- are the negative eigenvalues of the variational matrix $g'(e_j)$.

Theorem 1.6 is proved in Section 5. Examples 3.4 and 3.5 present smooth autonomous planar systems displaying cyclic chains of equilibria that are not ω-limit sets. The discussion of these findings and of the examples in Sections 2 and 3 is continued in Section 6. Applications of Theorem 1.5 to epidemic models will be presented in a joint publication with Carlos Castillo-Chavez [**7**].

2. Examples: Autonomous planar systems that are asymptotically autonomous one-dimensional systems. In this section we list a couple of examples for asymptotically autonomous one-dimensional systems which come from autonomous planar systems. We emphasize how the behavior of the one-dimensional limit-system is changed by adding a second component that vanishes for large times.

2.1. Repellers can become asymptotically stable.

$$\dot{x} = x(|x| - |y|), \qquad \dot{y} = -2y^3. \tag{2.1}$$

Integration of the y equation yields

$$y(t) = \frac{1}{2}(t + c)^{-1/2}, \qquad c = (2y_0)^{-2} \tag{2.2}$$

or

$$y(t) = \frac{y_0}{\sqrt{1 + 4y_0^2 t}}. \tag{2.3}$$

Thus system (2.1) is equivalent to an asymptotically autonomous equation for x which has the limit system

$$\dot{x} = x|x|. \tag{2.4}$$

All solutions of (2.4) not starting at $x_0 = 0$ blow up in finite positive time.

Observe that the system (2.1) is symmetric about the x and y axes and that all quadrants are invariant, hence our analysis will be restricted to positive initial data and positive solutions.

For given y, the x equation is of Bernoulli type, hence we make the transformation $x = w^{-1}$. w satisfies the differential equation

$$\dot{w} = y(t)w - 1, \qquad w_0 = 1/x_0 > 0.$$

Thus

$$w(t) = \exp\bigg(\int_0^t y(s)ds\bigg)\bigg(w_0 - \int_0^t \exp\bigg(-\int_0^s y(r)dr\bigg)ds\bigg), \qquad t \geq 0,$$

with

$$\int_0^t y(s)ds = \sqrt{t+c} - \sqrt{c},$$

i.e.,

$$\begin{aligned} w(t) &= e^{\sqrt{t+c}}\bigg(w_0 e^{-\sqrt{c}} - \int_c^{t+c} e^{-\sqrt{s}}ds\bigg) \\ &= e^{\sqrt{t+c}}\bigg(w_0 e^{-\sqrt{c}} - \int_{\sqrt{c}}^{\sqrt{t+c}} 2ue^{-u}du\bigg) \\ &= e^{\sqrt{t+c}}\bigg(w_0 e^{-\sqrt{c}} + 2(\sqrt{t+c}+1)e^{-\sqrt{t+c}} - 2(\sqrt{c}+1)e^{-\sqrt{c}}\bigg). \end{aligned}$$

Hence

$$x(t) = \frac{e^{-\sqrt{t+c}}}{\left(\frac{1}{x_0} - 2\sqrt{c} - 2\right)e^{-\sqrt{c}} + (2\sqrt{t+c}+2)e^{-\sqrt{t+c}}}. \tag{2.5}$$

Recall that $\sqrt{c} = 1/(2y_0)$. If $1/x_0 - 1/y_0 < 2$, the solution x more or less behaves like the solution of the autonomous problem (2.4), it blows up in finite time. If

$$\frac{1}{x_0} - \frac{1}{y_0} \geq 2, \tag{2.6}$$

we have

$$x(t) \leq \frac{y_0}{\sqrt{1 + 4y_0^2 t} + 2y_0} \leq \frac{y_0}{1 + 2y_0} \tag{2.7}$$

and $x(t) \to 0$, $t \to \infty$.

Equation (2.1) is a useful counter-example for several purposes. Viewed as a one-dimensional nonautonomous equation for x it displays an equilibrium, $x \equiv 0$, that is locally asymptotically stable, but not uniformly stable.

In view of persistence theory, the equilibrium $x \equiv 0$ is a uniform strong repeller for the limit system (2.4), though it is locally asymptotically stable for the asymptotically autonomous x equation in (2.1).

Seen as a system, (2.1) has two unbounded regions in every quadrant, both reaching down to the origin, such that solutions starting in the first (open) region blow up in finite positive time, while solutions starting in the second (closed, but with nonempty interior) region converge to the origin as time goes to infinity.

2.2. Locally asymptotically stable equilibria become local repellers.

$$\dot{x} = \Big(-x(1-x) + y\Big)(2+x), \qquad \dot{y} = -y. \tag{2.8}$$

As $y(t) = y_0 e^{-t}$, the x equation is an asymptotically autonomous one-dimensional equation

$$\dot{x} = \Big(-x(1-x) + y_0 e^{-t}\Big)(2+x), \tag{2.9}$$

with limit system

$$\dot{x} = -x(1-x)(2+x). \tag{2.10}$$

Equation (2.10) has $x \equiv 0$ as a locally stable equilibrium whose basin of attraction ends at the equilibria $x \equiv 1$ and $x \equiv -2$. For (2.9), however, it is not difficult to see that the basin of attraction of $x \equiv 0$ becomes arbitrarily small, if y_0 is chosen large enough. In particular all solutions of (2.9) starting at $x_0 < -2, y_0 \geq 0$ converge to $-\infty$, whereas, for any $\varepsilon > 0$ one can find $y_\varepsilon > 0$ such that a solution x to (2.9) tends to $+\infty$ whenever it starts at $x_0 \geq \varepsilon - 2, y_0 \geq y_\varepsilon$.

2.3. A semi-stable equilibrium becomes a global repeller.

$$\dot{x} = |x| + |y|, \qquad \dot{y} = -\alpha y \tag{2.11}$$

with $0 < \alpha < 1$. As $y(t) = y_0 e^{-\alpha t}$, the x equation is asymptotically autonomous with limit equation

$$\dot{x} = |x|$$

for which $x \equiv 0$ is a semi-stable equilibrium that attracts all solutions starting from negative values and repels all solutions starting from strictly positive values. As we will show, if $y_0 \neq 0$, the line $x \equiv 0$ will be a repeller for the full system (2.11) in spite of the exponential decrease of y. Obviously $x(t) \to \infty, t \to \infty$, once x becomes nonnegative. To show that this happens in finite time we assume that $x(t) < 0,\ t \geq 0$. Then

$$\dot{x} = -x + |y_0|e^{-\alpha t}, \qquad t \geq 0.$$

Integrating this equation easily yields a contradiction, as we have chosen $0 < \alpha < 1$.

3. Examples: Asymptotically autonomous planar systems. In this section we collect some examples of asymptotically autonomous planar systems in which the solutions behave differently from the solutions of the limiting system. Preferably we consider systems that come from three or higher dimensional autonomous systems in which some components converge to 0. The examples are often constructed along the same ideas as in Section 2, but where in one dimension, by the transition from the limit system to its asymptotically autonomous counterpart, we have destroyed an ω-limit set, we will create new ω-limit sets in the plane. The headings of the subsequent subsections emphasize which new ω-limit sets occur for the asymptotically autonomous system compared to those of the autonomous limit system.

3.1. A heteroclinic orbit. Consider the following three-dimensional autonomous system of ODEs which is an asymptotically autonomous planar system:

$$\begin{aligned} \dot{x}_1 &= \alpha(1-r)x_1 - \left(\beta|x_2| + x_3^2\right)x_2 \\ \dot{x}_2 &= \alpha(1-r)x_2 + \left(\beta|x_2| + x_3^2\right)x_1 \\ \dot{x}_3 &= -\gamma x_3 \\ r &= \sqrt{x_1^2 + x_2^2}. \end{aligned} \tag{3.1}$$

α, β, and γ are positive constant parameters. Equation (3.2) is equivalent to the asymptotically autonomous planar system (in polar coordinates r, θ such that $x_1 = r\cos\theta, x_2 = r\sin\theta$)

$$\begin{aligned} \dot{r} &= \alpha r(1-r) \\ \dot{\theta} &= \beta r|\sin\theta| + x_3^2(0)e^{-2\gamma t} \end{aligned} \tag{3.2}$$

with limit system

$$\begin{aligned} \dot{r} &= \alpha r(1-r) \\ \dot{\theta} &= \beta r|\sin\theta|. \end{aligned} \tag{3.3}$$

Notice that the nonautonomous perturbation in (3.2) is exponentially decreasing.

The planar autonomous system (3.3) has three equilibria (in rectangular coordinates): $(0,0), (\pm 1, 0)$. Every solution of (3.3) that starts in the upper half plane is attracted (for $t \to \infty$) to the equilibrium $(-1,0)$, whereas every solution starting in the lower half plane is attracted to $(1,0)$. In particular the upper half plane, the lower half plane and the disk with radius 1 are invariant under (3.3). Further there are no nonwandering points of (3.3) except the three equilibria. The circle with radius 1 is a heteroclinic cycle, but not a heteroclinic limit cycle of (3.3).

We claim that, for $\beta > 2\gamma, x_3(0) \neq 0$, the ω-limit set of any solution of (3.2) is the whole circle with radius 1.

This follows if we can show that θ in (3.2) is unbounded. Otherwise, as θ is strictly increasing, $\theta(t)$ converges to some θ_∞ for $t \to \infty$, $\theta_\infty = 2k\pi$ or $\theta_\infty = (2k+1)\pi$ for some $k \in \mathbf{N}_0$. From (3.2) we derive

$$\frac{d}{dt}(\theta_\infty - \theta) = -\beta r|\sin(\theta - \theta_\infty)| - x_3^2(0)e^{-2\gamma t}.$$

This implies that for any $\varepsilon > 0$ we have

$$\frac{d}{dt}(\theta_\infty - \theta) \leq -(\beta - \varepsilon)(\theta_\infty - \theta), \qquad t \geq t_\varepsilon,$$

for sufficiently large t_ε. Hence the convergence of θ to θ_∞ is exponential, more precisely, for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\theta_\infty - \theta(t) \leq M e^{-(\beta-\varepsilon)t}, \qquad t \geq 0. \tag{3.4}$$

As we assume that $\beta > 2\gamma$, we can choose $\varepsilon > 0$ such that $\beta > 2\gamma + \varepsilon$. From the θ equation in (3.2), we see that θ also satisfies the differential inequality

$$\dot{\theta}(t) \geq x_3^2(0) e^{-2\gamma t}.$$

Integrating this inequality yields

$$\theta(u) \geq \theta(t) + \int_t^u x_3^2(0) e^{-2\gamma s} ds, \qquad u > t.$$

We take the limit for $u \to \infty$ and obtain

$$\theta_\infty - \theta(t) \geq x_3^2(0) \frac{1}{2\gamma} e^{-2\gamma t},$$

hence, by (3.4),

$$M e^{-(\beta-\varepsilon)t} \geq x_3^2(0) \frac{1}{2\gamma} e^{-2\gamma t}.$$

This holds for all $t > 0$. Choosing t large enough, we obtain a contradiction, because $2\gamma < \beta - \varepsilon$.

In terms of the autonomous three-dimensional system (3.1) this result, for $\beta > 2\gamma$, can be rephrased as the heteroclinic orbit $r = 1, x_3 = 0$ to be the ω-limit set of all solutions to (3.1) that do not start in the plane $x_3 = 0$.

3.2. Possibly many heteroclinic orbits. The following example has less predictable dynamics than the example in 3.1. Consider the planar system of asymptotically autonomous ODEs

$$\begin{aligned} \dot{x}_1 &= \alpha(r-1)(5-r)x_1^2 - \left(\beta|x_2| + x_3^2\right) x_2 \\ \dot{x}_2 &= \alpha(r-1)(5-r)x_1 x_2 + \left(\beta|x_2| + x_3^2\right) x_1 \\ \dot{x}_3 &= -\gamma x_3 \\ r &= \sqrt{x_1^2 + x_2^2}. \end{aligned} \tag{3.5}$$

Notice that any solution to (3.5) with $x_3 \equiv 0$, $x_1(0) = 0$ satisfies $x_1(-t) = -x_1(t)$ and $x_2(-t) = x_2(t)$ such that the orbits (considered in the x_1, x_2 plane) are symmetric about the x_2 axis. We consider initial data with

$$1 < r(0) < 5. \tag{3.6}$$

α, β, and γ are positive constant parameters. In cylindrical coordinates r, θ, x_3, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, this system takes the form

$$\begin{aligned} \dot{r} &= \alpha r^2(r-1)(5-r)\cos\theta \\ \dot{\theta} &= \beta r|\sin\theta| + x_3^2 \\ \dot{x}_3 &= -\gamma x_3 \\ r &= \sqrt{x_1^2 + x_2^2}. \end{aligned} \tag{3.7}$$

This system has four equilibria $(\pm 1, 0, 0)$ and $(\pm 5, 0, 0)$.

Let us consider the flow in the plane $x_3 \equiv 0$: The upper and the lower half-ring in $\{1 < r < 5\}$ are invariant. Any nonequilibrium solution starting in the upper half ring, $\{1 < r < 5, x_2 > 0\}$, converges to $(1, 0, 0)$ for $t \to -\infty$ and to $(-1, 0, 0)$ for $t \to \infty$, whereas any nonequilibrium solution starting in the lower half ring, $\{1 < r < 5, x_2 < 0\}$, converges to $(-5, 0, 0)$ for $t \to -\infty$ and to $(5, 0, 0)$ for $t \to \infty$. The upper circles $\{r = a, x_2 > 0\}$, $a = 1, 5$, are orbits from $(a, 0, 0)$ to $(-a, 0, 0)$, the lower circles $\{r = a, x_2 < 0\}$, $a = 1, 5$, are orbits from $(-a, 0, 0)$ to $(a, 0, 0)$. The line segment from $(-5, 0, 0)$ to $(-1, 0, 0)$ is an orbit as well as the line segment from $(1, 0, 0)$ to $(5, 0, 0)$.

Turning to (3.7) for $x_3(0) \neq 0$, we see from the x_3 equation that

$$x_3(t) = x_3(0)e^{-\gamma t}, \qquad t \geq 0.$$

In spite of the fact that $x_3(t)$ exponentially decreases to 0, we will find that, for $\beta > 2\gamma$, the ω-limit set of an arbitrary orbit starting at $1 < r(0) < 5$, $x_3(0) \neq 0$ does not consist of just one equilibrium. From the r equation in (3.7), we realize that the equilibria $(1, 0, 0)$ and $(-5, 0, 0)$ are repellers for the invariant set $1 < r < 5$ and so the ω-limit set cannot just consist of one of these. Let us consider a solution that converges towards $(5, 0, 0)$ or $(-1, 0, 0)$. Then θ converges in a strictly monotone way to $\theta_\infty = 0$ or $\theta_\infty = \pi$ modulo an integer multiple of 2π

and $r(t) \to r_\infty \in \{1, 5\}$. A contradiction is now derived in the same way as in Subsection 3.1.

It follows from Markus's Theorem 1.1 [**23**] and the consideration of the system for $x_3 \equiv 0$ that the ω-limit set consists of two to four equilibria and orbits connecting them. The Poincaré & Bendixson type limit set trichotomy (Theorem 1.5) for asymptotically autonomous planar systems implies that the ω-limit set contains a cyclic chain of equilibria. So the ω-limit set contains either two or four equilibria. The planar limit system has many cyclic chains in the closed ring $1 \le r \le 5$ with the following three standing out among the others: the first is the circle $r = 1$, the second the circle $r = 5$, and the third consists of the upper half circle $r = 5$, the line segment from $(-5, 0)$ to $(-1, 0)$, the lower semi-circle $r = 1$ and the line segment from $(1, 0)$ to $(5, 0)$. At this point it is not clear which and how many cyclic chains can be contained in an ω-limit set of the asymptotically autonomous system and whether there may be even a whole continuum. Numerical calculations, performed by J.A. Palacios, suggest that the ω-limit set often consists of the third cyclic chain described above.

3.3. Coexistence of periodic orbits and homoclinic cycles. The following five-dimensional autonomous system is an asymptotically autonomous planar system:

$$
\begin{aligned}
\dot{x}_1 &= x_1 x_3 x_5 - (r + x_1 + (1-r)^2) x_2 \\
\dot{x}_2 &= x_2 x_3 x_5 + (r + x_1 + (1-r)^2) x_1 \\
\dot{x}_3 &= -x_4 x_5 \\
\dot{x}_4 &= x_3 x_5 \\
\dot{x}_5 &= -x_5^2 \\
r &= \sqrt{x_1^2 + x_2^2}
\end{aligned}
\tag{3.8}
$$

subject to initial conditions

$$x_3(0) = 1, \qquad x_4(0) = 0, \qquad x_5(0) = 1, \qquad e^{-1} < r(0) < e.$$

We find

$$x_5 = \frac{1}{1+t}, \qquad x_3 = \cos(\ln(1+t)), \qquad x_4 = \sin(\ln(1+t)).$$

We obtain the following asymptotically autonomous planar system in polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$:

$$\begin{aligned} \dot{\theta} &= r(1+\cos\theta) + (1-r)^2 \\ \dot{r} &= r\frac{1}{t+1}\cos(\ln(1+t)). \end{aligned} \tag{3.9}$$

The limit-system is

$$\begin{aligned} \dot{\theta} &= r(1+\cos\theta) + (1-r)^2 \\ \dot{r} &= 0. \end{aligned}$$

The orbits of this equation are the center $(0,0)$ and the circles of all radii. The circle with radius 1 is a homoclinic orbit connecting the equilibrium $(-1,0)$ (in rectangular coordinates) to itself and the other circles are periodic orbits.

For the solutions of (3.9) we have

$$r(t) = r(0)\exp\sin(\ln(t+1)).$$

Thus the ω-limit sets of solutions to (3.9) consist of the circles with radii between $r(0)e^{-1}$ and $r(0)e$; hence, as we have chosen $e^{-1} < r(0) < e$, we have coexistence in the ω-limit set of a homoclinic orbit with a continuum of periodic orbits.

3.4. A homoclinic orbit for the undamped Duffing oscillator with smooth exponentially decreasing forcing. Markus's [**23**] results have sometimes been misquoted in the form that any ω-limit set of an asymptotically autonomous ODE system is contained in the union of ω-limit sets of the limit system. Example 3.1 is a counter-example for Lipschitz continuous vector fields and exponentially decreasing forcing. The example can be modified such that the vector field becomes smooth, but the forcing is no longer exponentially decreasing, but just converges to 0 as time tends to infinity (see [**2**, example (1) in Section 4].

We now construct a counter-example with smooth vector fields and exponentially decreasing forcing. The example will be drastically different from 3.1 insofar as almost any point in the plane lies on a periodic orbit of the autonomous limit system. As limit system we consider the unforced Duffing oscillator

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = y_1 - y_1^3. \tag{3.10}$$

The equilibrium $(0,0)$ is a saddle and the equilibria $(\pm 1, 0)$ are centers. Any other point is contained in a periodic orbit or in one of two homoclinic orbits connecting the saddle $(0,0)$ to itself. One homoclinic orbit lies in the right half-plane, the other in the left half-plane. The important point is that the homoclinic orbits are no ω-limit sets of (3.10). Actually the orbits of (3.10) are level sets of the functional

$$V(y_1, y_2) = \frac{y_2^2}{2} - \frac{y_1^2}{2} + \frac{y_1^4}{4}. \tag{3.11}$$

The equilibrium $(0,0)$ and the homoclinic orbits correspond to the level $V = 0$, the equilibria $(\pm 1, 0)$ to level $V = -1/4$, the periodic orbits inside the homoclinic orbits correspond to the levels between $-1/4$ and 0, whereas the periodic orbits outside the homoclinic orbits are associated with strictly positive levels. (See [**36**, Example 1.1.7], e.g.)

By Theorem 1.6 there exists an exponentially decreasing, smooth forcing function ϕ such that the homoclinic orbit of (3.10) in the right half-plane becomes an ω-limit set of the perturbed system

$$\begin{aligned} \dot{x}_1 &= x_2 + \phi_1(t), \\ \dot{x}_2 &= x_1 - x_1^3 + \phi_2(t) \end{aligned} \tag{3.12}$$

subject to

$$x_1(0) = \sqrt{2}, \qquad x_2(0) = 0.$$

Notice that the initial data lie in the level set $V = 0$, i.e. on the homoclinic orbit of (3.10) in the right half-plane. We mention (without proof) that for this special example one can choose $\phi_1 \equiv 0$.

3.5. A homoclinic orbit for a modified Duffing oscillator with smooth exponentially decreasing forcing. We now modify Example 3.4 such that all forward bounded solutions of the limit problem converge to one of the three equilibria $(0,0)$, $(\pm 1, 0)$, but the homoclinic orbits are preserved and are ω-limit sets of appropriate asymptotically autonomous systems which are obtained by exponentially decreasing smooth nonautonomous perturbations of the limit system. We recall the functional

$$V(y_1, y_2) = \frac{y_2^2}{2} - \frac{y_1^2}{2} + \frac{y_1^4}{4}. \tag{3.18}$$

and consider the autonomous planar system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_1 - y_1^3 + y_2 V. \tag{3.19}$$

The orbital derivative $\dot{V}$ along (3.19) is

$$\dot{V} = y_2^2 V. \tag{3.20}$$

It follows that the sets $\{V = 0\}, \{V < 0\}$, and $\{V > 0\}$ are invariant. On $\{V = 0\}$, (3.19) has the same dynamics as (3.10), in particular $\{V = 0\}$ consists of two homoclinic orbits of (3.19), one in the right half-plane, and another in the left half-plane. These two homoclinic orbits are not ω-limit sets of (3.19). V is a Lyapunov-function on $\{V < 0\}$ and $-V$ is a Lyapunov-function on $\{V > 0\}$. This implies that the ω-limit set of every forward bounded orbit in $\{V \neq 0\}$ must be contained in the maximal invariant (under (3.19)) subset of $\{y_2 = 0\}$. Hence any forward bounded solution of (3.19) starting at some point with $V \neq 0$ converges to one of the three equilibria $(0,0), (\pm 1, 0)$. Any solution y of (3.19) starting at some point with $V < 0$ is bounded because it remains in the bounded region surrounded by the homoclinic orbits. Moreover $V(y(t))$ is a decreasing function of t, hence $y(t)$ cannot converge to $(0,0)$ because $V(0,0) = 0$. Hence any solution y of (3.19) starting in $\{V < 0\}$ converges to one of the equilibria $(\pm 1, 0)$. On the other hand, a solution y of (3.19) starting in $\{V > 0\}$ remains there and cannot converge to one of the three equilibria which all are in $\{V \leq 0\}$ because $V(y(t))$ is an increasing function of t; thus y is unbounded in forward time. Hence the homoclinic orbits are separatrices, and any solution of (3.19) starting at a point with $V < 0$ converges to one of the two nontrivial equilibria, whereas solutions starting at a point with $V > 0$ are unbounded and, by (3.20), converge to infinity, as time tends to ∞. Solutions starting in $\{V = 0\}$ converge to $(0,0)$ for $t \to \infty$.

By Theorem 1.6 there exists a smooth exponentially decreasing nonautonomous perturbation ϕ such that the homoclinic orbit in the right half-plane becomes an ω-limit set of

$$\dot{x}_1 = x_2 + \phi_1(t), \qquad \dot{x}_2 = x_1 - x_1^3 + x_2 V + \phi_2(t), \tag{3.21}$$

which is not an ω-limit set of the autonomous limit system (3.19).

4. Limit set trichotomy and convergence in the plane. The key to extending Markus's theorem 1.3 to a limit-set trichotomy lies in proving a Butler & McGehee type result for asymptotically autonomous semiflows [**33**].

In order to shorten our notation, a set Y in $\mathbf{R}^n$ is called forward f-invariant (forward g-invariant) if all solutions of (1.1) (of (1.2)) that start in Y stay in Y for all forward times. Similarly the (forward, backward, full) orbits of solutions to (1.1) will be called f-orbits, and the orbits of solutions to (1.2) will be called g-orbits. The ω-limit set of a forward f-orbit is called ω-f-limit set, etc., an element e satisfying $g(e) = 0$, a g-equilibrium. Sometimes the reference to f or g will be omitted when it is clear from the context.

Let Y be a subset of $\mathbf{R}^n$. A g-invariant subset M of Y is called an *isolated compact g-invariant subset* of Y, if there is an open set U such that there is no compact g-invariant set $\tilde{M}$ with $M \subseteq \tilde{M} \subseteq U \cap Y$ except M. U is called a *g-isolating neighborhood* of M in Y. If $Y = \mathbf{R}^n$, an isolated compact g-invariant subset of Y is simply called an isolated compact g-invariant set.

Lemma 4.1. *Assume that the point (s, x), $s \geq t_0$, $x \in X$, has a pre-compact forward f-orbit and that $\omega = \omega_f(s, x)$ is its ω-f-limit set. Further let M be a g-invariant set such that $M \cap \omega \neq \varnothing$, but $\omega \not\subseteq M$. Finally assume that $M \cap \omega$ is an isolated compact g-invariant subset of ω. Then M has a nonempty stable and a nonempty unstable manifold in ω in the following sense*:

There exists an element $u \in \omega \setminus M$ with $\omega_g(u) \subset M$ and an element $w \in \omega \setminus M$ with a full g-orbit in ω whose α-g-limit set is contained in M.

u can be chosen such that its forward g-orbit is arbitrarily close to M. w can be chosen such that its backward g-orbit is arbitrarily close to M.

We recall that the *α-g-limit set* of a full g-orbit $\{\varphi(t)\}$ is defined by

$$\alpha_g(\varphi) = \bigcap_{t \geq 0} \overline{\varphi((-\infty, -t])}.$$

Lemma 4.1 follows from autonomous Butler & McGehee type results [**6; 15**, Lemma 4.3; **32**, Section 4] by embedding the asymptotically autonomous semiflow and its limit semiflow (induced by the solutions to (1.1) and (1.2)) into an autonomous semiflow on a larger state space [**33**].

4.1. The Poincaré & Bendixson type limit-set trichotomy. In order to prove Theorem 1.5 we start with the following lemmata. We assume $n = 2$.

Lemma 4.2. a) *Let γ_0 be a periodic g-orbit and M the closed region enclosed by γ. If M is not an isolated compact g-invariant set, then any sufficiently small neighborhood of M contains a periodic g-orbit outside M that surrounds M.*

b) *Let γ_1, γ_2 be two periodic g-orbits such that γ_1 lies in the interior of γ_2. Let M be the closed ring enclosed by γ_1 and γ_2. If M is not an isolated compact g-invariant set, then any sufficiently small neighborhood of M contains a periodic g-orbit that lies inside γ_1 or outside γ_2.*

Proof. a) Let $x \in U \setminus M$ with U being an open neighborhood of M and x being contained in a compact g-invariant subset N of U. Hence there is a full g-orbit γ through x contained in N. If U is chosen sufficiently small, $U \setminus M$ does not contain g-equilibria. Otherwise we would obtain a sequence of g-equilibria approaching γ_0. This would imply that γ_0 itself contains a g-equilibrium, a contradiction. Hence the ω-g-limit set ω and the α-g-limit set α of x are periodic g-orbits, contained in $N \subseteq U$, by the Poincaré & Bendixson theorem. If U is chosen small enough, the continuity of the flow induced by (1.2) forces these periodic g-orbits to surround M with the same orientation as γ_0. ω and α cannot both coincide with γ_0 because then γ would intersect itself. This proves statement a). Statement b) is proved similarly. □

Lemma 4.3. *Let ω be the ω-limit set of a bounded forward f-orbit. Then ω does not contain a g-orbit that approaches a g-limit cycle from outside.*

Proof. Assume that ω contains a g-orbit γ that is not periodic but approaches a periodic g-orbit γ_0 from outside in forward or backward time. Let M be the region surrounded by γ_0. Lemma 4.2 implies that M is an isolated compact g-invariant set. Otherwise there would be arbitrarily close periodic g-orbits surrounding M which are different from γ_0 that would not allow γ_0 to be approached from outside by γ. Lemma 4.1 now implies that there are two g-orbits γ_1 and γ_2 outside M such that $\omega(\gamma_1) \subseteq M$, $\alpha(\gamma_2) \subseteq M$. As the boundary of M is the periodic g-orbit γ_0, we have $\omega(\gamma_1) = \gamma_0 = \alpha(\gamma_2)$. But this cannot happen without γ_1 and γ_2 crossing each other. □

Lemma 4.4. *Let ω be the ω-limit set of a bounded forward f-orbit. There is no g-orbit in ω that connects two different periodic g-orbits.*

Proof. Assume that there is such a g-orbit. Then this g-orbit would have to approach one of the periodic g-orbits from outside in contradiction to Lemma 4.3. □

Lemma 4.5. *There is no chain $\alpha \mapsto e_1 \mapsto \cdots e_k \mapsto \beta$, $k \geq 1$ of g-orbits in ω connecting periodic g-orbits α, β and g-equilibria e_j, $j = 1, \dots, k$.*

Two not necessarily different sets M, N are chained—symbolically $M \mapsto N$—if they are connected by a g-orbit, i.e., if there exists some element $x \notin M \cup N$ such that $\alpha_g(x) \subseteq M, \omega_g(x) \subseteq N$. A chain $M_1 \mapsto \cdots M_k$ is called cyclic if $M_k = M_1$.

Proof of Lemma 4.5. Assume that such a chain exists. By Lemma 4.3, the periodic g-orbits α and β have to be approached from inside by the g-orbits connecting them to e_1 and e_k respectively. This can only happen if the whole chain is contained in the interiors of both α and β, in particular $\alpha = \beta$. But $\alpha = \beta$ cannot hold either, because a periodic g-orbit cannot simultaneously be the α-limit set and the ω-limit set of two g-orbits approaching from inside. □

Lemma 4.6. *Let Y be a subset of $\mathbf{R}^2$. Assume e is an isolated g-equilibrium (isolated in $\mathbf{R}^2$), but not an isolated compact g-invariant subset of Y. Then the following alternative holds*:

(i) *Every sufficiently small neighborhood of e contains a homoclinic g-orbit in Y connecting e to itself.*

(ii) *Every sufficiently small neighborhood of e contains a periodic g-orbit in Y surrounding e, in particular e is a center.*

Proof. First we observe that (i) and (ii) exclude each other. Indeed, if there is a homoclinic g-orbit connecting e to itself, there is a neighborhood V of e such that V does not contain periodic g-orbits surrounding e. Hence it is sufficient to show that any neighborhood of e contains a homoclinic g-orbit or a periodic g-orbit.

Now let M be an invariant subset of Y different from the singleton e, M contained in an open disk V containing e. As e is an isolated equilibrium, V contains no other equilibrium than e, provided V is close enough to e. By the Poincaré & Bendixson limit set trichotomy, M contains a periodic g-orbit in Y or a homoclinic g-orbit in Y connecting e to e. In the first case, the periodic g-orbit surrounds a critical point $\tilde{e}$ of g. See [**17**, Theorem 11.5.2]. As the periodic g-orbit lies in the disk V, $\tilde{e}$ is contained in V as well, hence $\tilde{e} = e$. So the periodic g-orbit has to surround e. □

4.7. *Proof of Theorem* 1.5. Let us assume that (i), (ii), and (iii) do not hold.

We first remark that then, by Lemma 4.6, any g-equilibrium in ω is an isolated compact g-invariant subset of ω.

Secondly we remark that then any ω-g-limit set and α-g-limit set of a g-orbit in ω is a periodic g-orbit or consists of a g-equilibrium.

Otherwise, by the classical Poincaré & Bendixson limit set trichotomy, this limit set consists of finitely many g-equilibria and connecting g-orbits. By applying Lemma 4.1 several times, for g rather than f, one finds that it contains a cyclic chain which is also contained in ω, in contradiction to our assumption.

As we have excluded (ii), ω contains a point x that is not contained in a periodic g-orbit. We can further assume that x is neither a center nor an equilibrium connected to itself by homoclinic g-orbits in ω. Hence we can assume that the ω-g-limit set of x contains an equilibrium e_1 that is neither a center nor an equilibrium connected to itself by homoclinic g-orbits in ω. By Lemma 4.6, $\{e_1\}$ is an isolated compact g-invariant set. By Lemma 4.1 there exist g-orbits $\gamma_1,\ \gamma_2$ in $\omega \setminus \{e_1\}$ with $\alpha(\gamma_2) = \{e_1\} = \omega(\gamma_1)$, i.e., we have a chain $\alpha(\gamma_1) \mapsto e_1 \mapsto \omega(\gamma_2)$ in ω. By our second remark at the beginning of this proof, $\omega(\gamma_2)$ and $\alpha(\gamma_1)$ may be equilibria or periodic g-orbits. Lemma 4.5 rules out that both $\omega(\gamma_2)$ and $\alpha(\gamma_1)$ are periodic g-orbits. By reversing the time if necessary, we can assume that $\omega(\gamma_2)$ consists of an equilibrium e_2. Two cases need to be considered separately:

Case 1: $\alpha(\gamma_1)$ is a periodic g-orbit. This means that we have a chain $\alpha(\gamma_1) \mapsto e_1 \mapsto e_2$. By Lemma 4.3, $\alpha(\gamma_1)$ has to be approached by γ_1 from inside. As $\omega(\gamma_1)$ consists of e_1, this implies that e_1 lies in the interior of the periodic g-orbit $\alpha(\gamma_1)$. Since $\alpha(\gamma_2)$ consists of e_1 as well, $\omega(\gamma_2) = \{e_2\}$ must lie in the closed interior of $\alpha(\gamma_1)$. As e_2 is not a center (otherwise the surrounding periodic g-orbits would intersect γ_2) and (iii) does not hold, Lemma 4.6 implies that $\{e_2\}$ is an isolated compact g-invariant set which is different from ω. By Lemma 4.1, there exists an g-orbit γ_3 in $\omega \setminus \{e_2\}$ such that $\alpha(\gamma_3)$ consists of e_2. So we have the chain $\alpha \mapsto e_1 \mapsto e_2 \mapsto \omega(\gamma_3)$. Lemma 4.5 implies that $\omega(\gamma_3)$ cannot be a periodic g-orbit. Hence $\omega(\gamma_3)$ has to consist of an equilibrium e_3. Continuing this argument we can construct arbitrarily large chains

$$\alpha(\gamma_1) \mapsto e_1 \mapsto e_2 \mapsto \cdots \mapsto e_k.$$

As there are only finitely many equilibria contained in the compact set ω this chain has to become a cyclical chain of equilibria in contradiction to our exclusion of (iii).

Case 2: $\alpha(\gamma_1)$ consists of an equilibrium e_0. This means we have a chain $e_0 \mapsto e_1 \mapsto e_2$. As we have excluded (iii) and e_0 cannot be a center, e_0 forms an isolated compact g-invariant set different from ω by Lemma 4.6. By Lemma 4.1 there exists an g-orbit γ_0 such that $\omega(\gamma_0) = \{e_0\}$. If $\alpha(\gamma_0)$ is a periodic g-orbit, we are in the situation of case 1 and obtain a contradiction as outlined there. Hence $\alpha(\gamma_0)$ must consist of an equilibrium e_{-1}. Continuing this type of argument we can construct chains of equilibria

$$e_{-k} \mapsto \cdots \mapsto e_0 \mapsto e_1 \mapsto e_2.$$

As there are only finitely many equilibria in ω, this chain of equilibria has to become cyclical, in contradiction to our exclusion of (iii). Hence at least one of (i), (ii), (iii) has to hold. □

4.2. Convergence in the plane. We say that a subset X of $\mathbf{R}^2$ has the *Dulac property*, if it contains no periodic g-orbits and no g-equilibria that are cyclically chained in X. First we notice from Lemma 4.6:

Lemma 4.8. *If X has the Dulac property, then every isolated equilibrium in X (isolated in $\mathbf{R}^2$) is an isolated compact g-invariant subset of X.*

Theorem 1.5 and the Dulac property immediately imply the following convergence result.

Theorem 4.9. *Assume that the g-equilibria in $X \subseteq \mathbf{R}^2$ are isolated in $\mathbf{R}^2$ and X has the Dulac property. Then every bounded forward g-orbit and every bounded forward f-orbit in X converges towards a g-equilibrium.*

A sufficient condition for the Dulac property to hold is the so-called *Dulac* (or *divergence*) *criterion* (see [**13**], e.g.).

Dulac Criterion. *Let X, Y, and D be subsets of $\mathbf{R}^2$, D open and simply connected, with the following properties*:

- *Every bounded forward orbit of* (1.1) *in X has its ω-limit set in Y.*
- *All possible periodic orbits of* (1.2) *in Y and the closures of all possible orbits of* (1.2) *that chain equilibria of* (1.2) *cyclically in Y are contained in D.*
- *g is continuously differentiable on D and there is a real-valued continuously differentiable function ρ on D such that the divergence of ρg,*

$$\nabla \cdot (\rho g)(x_1, x_2) = \frac{\partial}{\partial x_1}(\rho g_1)(x_1, x_2) + \frac{\partial}{\partial x_2}(\rho g_2)(x_1, x_2),$$

is either strictly positive almost everywhere on D or strictly negative almost everywhere on D.

Corollary 4.10. *Let the g-equilibria in X be isolated in $\mathbf{R}^2$ and assume that the Dulac criterion holds. Then every bounded forward g-orbit in X and every bounded forward f-orbit in X converges towards a g-equilibrium.*

5. Heteroclinic cycles as ω-limit sets of asymptotically autonomous systems. This Section is devoted to the proof of Theorem 1.6, namely that any homoclinic cycle or any minimal heteroclinic cycle connecting saddles of a planar autonomous system is the ω-limit set of a nonautonomous system obtained by an exponentially decreasing smooth perturbation. This result is of interest if the cycle is not an ω-limit set of the autonomous system itself.

The idea of the proof consists in letting the solution of (1.3) follow the cycle—i.e., $\phi(t) = 0$—for most of the time, but to make $\phi(t) \neq 0$ in shrinking neighborhoods of the saddles in order to by-pass the saddles and go from their stable to their unstable manifolds. We first investigate how one can bypass a single saddle this way.

Lemma 5.1. *Assume that g is differentiable and $e = e_j$ is one of the saddles of* (1.2) *in Theorem* 1.6. *Then there exist numbers $\eta_0, K > 0$ such that the following holds.*

If u° is an element in the stable manifold of e, connecting from e_{j-1}, and $|u^\circ - e| < \eta_0$, then there exists $\tau > 0$ and a C^∞ function $\psi : \mathbf{R} \to \mathbf{R}^2$ and some number $\xi \in (1/2, 3/2)$ (which all depend on u°) with the following properties:

(i) $\tau < K$,

(ii) *ψ has support in $(0, \tau)$, and $|\psi(t)| \leq 2$.*

(iii) *The solution u to*

$$\dot{u} = g(u) + |u^\circ - e|\xi\psi(t), \qquad u(0) = u^\circ \tag{5.1}$$

intersects the unstable manifold of $e = e_j$, connecting to e_{j+1}, at time τ, and

$$|u(t) - e| \leq K|u^\circ - e|, \qquad 0 \leq t \leq \tau. \tag{5.2}$$

Before we prove Lemma 5.1, we use it to show Theorem 1.6. Assume that we have managed to construct ϕ in (1.3) on an interval $[0, t_0]$ such that $\phi(t_0) = 0$ and $x(t_0)$ is a point on the orbit from e_{j-1} to e_j (where e_{m+1} is identified with e_1). We will apply Lemma 5.1 for $e = e_j$ to extend ϕ and x. We choose some δ, $0 < \delta < \eta_0$, $\delta < 1$, such that δ is smaller than the distances between any pair of saddles and smaller than the distances from the initial point q to any saddle. Let $0 < \varepsilon < -\lambda_j^-$, $j = 1, \dots, m$, as in Theorem 1.6. As long as $\phi(t) = 0$ for $t > t_0$, the solution to (1.3) obeys (1.2) and follows the stable manifold of e. Hence $|x(t) - e|$ decreases exponentially with a rate $\lambda < -\varepsilon$. So there is some time $t_1 > t_0$ such that $|x(t_1) - e| = \delta e^{-\varepsilon t_1}$. Recall that δ has been chosen to be smaller than the distance from e_j to e_{j-1} and smaller than the distance from e_j to the initial datum q. We apply Lemma 5.1 with $u^\circ = x(t_1)$ and pick τ, ψ, ξ provided by this Lemma. We modify ϕ to become

$$\phi(t) = \delta e^{-\varepsilon t_1} \xi \psi(t - t_1) \qquad \text{for } t_1 \le t \le t_1 + \tau.$$

On $[t_1, t_1 + \tau]$, the solution x to (1.3) then satisfies $x(t) = u(t - t_1)$ with u being the solution of (5.1) in Lemma 5.1. In particular $x(t_1 + \tau) = u(\tau)$ is a point on the unstable manifold of e, connecting to the next saddle e_{j+1}, and ϕ is still C^∞, and $\phi(t_1 + \tau) = 0$. Moreover, for $t_1 \le t \le t_1 + \tau$, we have

$$|\phi(t)| \le \delta e^{-\varepsilon t_1} (3/2) |\psi(t - t_1)| \le 3\delta e^{-\varepsilon t_1} \le 3 e^{\varepsilon \tau} \delta e^{-\varepsilon t} \le 3 e^{\varepsilon K} \delta e^{-\varepsilon t}$$

and, from (5.2),

$$|x(t) - e| \le K |x(t_1) - e| \le K \delta e^{-\varepsilon t_1} \le K e^{\varepsilon K} \delta e^{-\varepsilon t}.$$

Hence the number c in Theorem 1.6 can be chosen to be $3\delta e^{K\varepsilon}$.

This construction can be repeated infinitely often in the neighborhood of any saddle. Notice from the last estimate that, when by-passing a saddle, the distance from the saddle decreases exponentially. This completes the proof of Theorem 1.6. □

Proof of Lemma 5.1. In a neighborhood of $e = e_j$, after a linear invertible transformation, $\dot{y} = g(y)$ takes the form $y = e + w$, where w satisfies

$$\begin{aligned} \dot{w}_1 &= \lambda^- w_1 + F_1(w) \cdot w \\ \dot{w}_2 &= \lambda^+ w_2 + F_2(w) \cdot w \end{aligned} \tag{5.3}$$

with $\lambda^- < 0 < \lambda^+$ and continuous functions $F_1, F_2 : \mathbf{R}^2 \to \mathbf{R}^2, F_1(0) = 0 = F_2(0)$. Let $0 < \eta < \eta_0$ be a fixed number and consider the problem

$$\begin{aligned} \dot{u}_1 &= \lambda^- u_1 + F_1(u) \cdot u - \xi\eta\chi(t) \\ \dot{u}_2 &= \lambda^+ u_2 + F_2(u) \cdot u + \xi\eta\chi(t) \\ u(0) &= u^\circ, \qquad |u^\circ| = \eta \end{aligned} \tag{5.4}$$

with χ to be constructed and ξ being a tuning parameter, $\xi \approx 1$. The initial datum u° is supposed to be in the stable manifold of 0. It is sufficient to prove Lemma 5.1 for (5.4) instead of (5.1), $\psi(t) = \chi(t)(-1, 1)$. Reversing the linear transformation then yields Lemma 5.1.

We want to steer u to the unstable manifold of e by choosing $\chi \geq 0$ and ξ appropriately such that the unstable manifold is reached at some time $\tau > 0$.

We look for u in the form $u = \eta v$ and obtain the following system for v:

$$\begin{aligned} \dot{v}_1 &= \lambda^- v_1 + F_1(\eta v) \cdot v - \xi\chi(t) \\ \dot{v}_2 &= \lambda^+ v_2 + F_2(\eta v) \cdot v + \xi\chi(t) \\ v(0) &= v^\circ, \qquad |v^\circ| = 1. \end{aligned} \tag{5.5}$$

As ηv°, with small $\eta > 0$, is supposed to be in the stable manifold of 0 which is tangential to the vector $(1, 0)$, we consider initial data v° satisfying

$$v_1^\circ \approx 1, \qquad v_2^\circ \approx 0.$$

As solutions to (5.5) depend smoothly on η and ξ and we are interested in $\eta > 0$ being close to 0 and in ξ being close to 1, we study the system

$$\begin{aligned} \dot{z}_1 &= \lambda^- z_1 - \chi(t) \\ \dot{z}_2 &= \lambda^+ z_2 + \chi(t) \\ z(0) &= v^\circ. \end{aligned} \tag{5.6}$$

Integration yields

$$\begin{aligned} z_1(t) &= e^{\lambda^- t}\left(v_1^\circ - \int_0^t e^{-\lambda^- s}\chi(s)ds\right) \\ z_2(t) &= e^{\lambda^+ t}\left(v_2^\circ + \int_0^t e^{-\lambda^+ s}\chi(s)ds\right). \end{aligned} \tag{5.7}$$

We first try $\chi \equiv 1$. From the first equation in (5.7) we see the following: There exist $0 < \tau_1 < \tau_2 < \infty$ such that, for any v_1°, $1/2 < v_1^\circ \leq 1$, we find some τ, $\tau_1 < \tau < \tau_2$, such that

$$z_1(\tau) = 0.$$

From the second equation in (5.7) we see that there is some $\delta_1 > 0$, $\delta_1 = \int_0^{\tau_1} e^{-\lambda^+ s}\chi(s)ds$, e.g., such that

$$z_2(\tau) > 0 \quad \text{whenever } 1/2 < v_1^\circ \leq 1,\ |v_2^\circ| < \delta_1.$$

Though τ depends on v°, we can find some $K > 0$ such that $\tau \leq K$ and $|z_2(\tau)| \leq K$ whenever $1/2 < v_1^\circ \leq 1, |v_2^\circ| < \delta_1$.

We can replace $\chi \equiv 1$ by $\chi \in C^\infty(\mathbf{R})$ with support in $(0, \tau)$ and values between 0 and 2 such that the following statement still holds for the solutions z of (5.7):

There exists $\delta_0 > 0$ and $K > 0$ such that, for any v_0 with $1 - \delta_0 < v_1^\circ \leq 1, |v_2^\circ| < \delta_0$, we find some $\tau, 0 < \tau \leq K$, such that $z_1(\tau) = 0,\ 0 < z_2(\tau) \leq K$.

In the next step we see that the solutions z_ξ to

$$\begin{aligned} \dot{z}_{\xi,1} &= \lambda^- z_{\xi,1} - \xi\chi(t) \\ \dot{z}_{\xi,2} &= \lambda^+ z_{\xi,2} + \xi\chi(t) \\ z_\xi(0) &= v^\circ \end{aligned}$$

satisfy $z_{\xi,1}(\tau) > 0$ for $\xi < 1$, whereas $z_{\xi,1}(\tau) < 0$ for $\xi > 1$ and $z_{\xi,2}(\tau) > 0$ if ξ is sufficiently close to 1.

We now return to (5.5). The solutions depend continuously on ξ and on η. Hence there is some $\eta_0 > 0$ such that there exist some $1/2 < \xi_1$, $\xi_2 < 3/2$ with the following property: For any $0 < \eta < \eta_0$, the solutions $v = v_\xi$ to (5.5) satisfy

$$\begin{gathered} v_{\xi,1}(\tau) > 0, \quad \text{if } \xi = \xi_1; \qquad v_{\xi,1}(\tau) < 0, \quad \text{if } \xi = \xi_2; \\ v_{\xi,2}(\tau) > 0, \quad \text{if } \xi_1 \leq \xi \leq \xi_2. \end{gathered} \tag{5.8}$$

ξ_1, ξ_2 can be found independently of v° as long as $|v^\circ| = 1, v_1^\circ > 0$, $|v_2| < \delta_0$ with sufficiently small δ_0. Moreover it follows from our construction that there is some $K > 0$ that is independent of v° such that

$$|v_\xi(t)| \leq K, \qquad 0 \leq t \leq \tau, \tag{5.9}$$

whenever $0 < \eta < \eta_0$, $1/2 < \xi < 3/2$.

Finally we return to the solutions u of (5.4), $u = \eta v$ with v satisfying (5.5). If $0 < \eta < \eta_0$ and u° lies on the stable manifold of e, $|u^\circ| = \eta$, then $u^\circ = \eta v^\circ$ with $|v^\circ| = 1$, $v_1^\circ > 0$, $|v_2^\circ| < \delta_0$, provided that η_0 is small enough. For the stable manifold is tangential to the vector (1,0). Hence (5.8) holds for $v = (1/\eta)u$. As the unstable manifold of e is tangential to the vector $(0,1)$ we find that the solution $u_\xi = \eta v_\xi$ of (5.4) has the following property: $u_\xi(\tau)$ is on one side of the unstable manifold for some $1/2 < \xi < 3/2$ and on the other side of the unstable manifold for some other $1/2 < \xi < 3/2$. As $u_\xi(\tau)$ depends continuously on ξ, $u_\xi(\tau) = \eta v_\xi(\tau)$ has to intersect the unstable manifold for some $\xi \in (1/2, 3/2)$. (5.9) implies (5.2). This proves Lemma 5.1. □

6. Discussion. The examples in Sections 2 and 3 have illustrated that solutions to one- or two-dimensional asymptotically autonomous ordinary differential equations can behave quite differently in the long run from the solutions of the corresponding limit systems. By adding an asymptotically vanishing term repelling equilibria become stabilized, stable equilibria become locally repellent, and attracting equilibria are replaced by homoclinic or heteroclinic orbits. In particular the ω-limit set of a solution to the asymptotically autonomous system may be much larger than the union of the ω-limit sets of all solutions to the limit system. This can even happen if the vector fields are smooth and the nonautonomous parts are exponentially decreasing. The Examples 3.1, 3.2, 3.4 and 3.5 falsify what has seemingly become some kind of folklore theorem, namely that the large-time behavior of solution to asymptotically autonomous differential equations can a priori be reduced to the large-time behavior of solutions to the limit system. Though such a reduction is often possible, it has to be justified by a careful analysis of the limit system.

In the plane the limit set trichotomy in Theorem 1.5 limits the possible asymptotic behavior of solutions to asymptotically autonomous systems. It is not completely satisfactory, however, because there remains the following

Problem 6.1. Let $n = 2$ and ω the ω-limit set of a forward bounded solution x of (1.1). Assume that the equilibria of (1.2) in ω are isolated in $\mathbf{R}^2$. Is ω the union of periodic orbits, equilibria, and orbits connecting equilibria associated with (1.2)?

Notice that, though the examples in Section 3 have ω-limit sets that are quite different from those of the limit systems, they still are of the form conjectured in Problem 6.1.

The limit-set trichotomy in Theorem 1.5 gives an answer to Question 1.4, namely to rule out cyclic chains of equilibria. Example 3.1, 3.2, and 3.5 together with Theorem 1.6 illustrate that this is also necessary in general.

This answer to Question 1.4 is not restricted to planar systems: *Question* 1.4 *has a positive answer whenever the equilibria of* (1.2) *are isolated compact invariant* (*under* (1.2)) *sets and are not cyclically chained to each other.*

This answer actually holds for general asymptotically autonomous semiflows and so applies to asymptotically autonomous (parabolic and hyperbolic) partial differential equations, functional differential equations, and to Volterra integral and integro-differential equations [**33**, Theorem 4.2]. Another positive answer was given by Ball [**2**] who, instead of ruling out cyclic chains of equilibria, assumed the existence of a Lyapunov function.

The assumptions in answer to question 1.4 are satisfied, e.g., in Corollary 2 by Smith [**28**] who positively answers Question 1.4 for asymptotically autonomous tri-diagonal competitive and cooperative ODE systems modeling neural nets.

We wonder whether other Poincaré & Bendixson type results for autonomous semiflows (e.g. for competitive or cooperative three-dimensional ODE systems, [**16**]; monotone cyclic feed-back systems, [**22**]; or reaction-diffusion equations on the circle, [**11**]) have asymptotically autonomous extensions similar to Theorem 1.5.

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