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POPULATIONS WITH AGE AND DIFFUSION: EFFECTS OF THE FERTILITY FUNCTION

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. When studying the existence of solutions of the nonlinear population problem with age dependence and diffusion,

$$u(x,t) = \int_0^\infty \rho(x,t,a)\,da$$
$$\rho_t + \rho_a = k(\rho u_x)_x - \mu(a,u)\rho$$
$$\rho(x,t,0) = \int_0^\infty \beta(a,u)\rho(x,t,a)\,da$$

some simplifying assumptions are necessary. Here we discuss the effects of a birth function of the form

$$\beta(a,u) = \beta(u)ae^{-\alpha a}$$

and a death function $\mu(a,u) = \mu_0(u)$, in terms of existence of solutions and localization of the population.

1. Introduction. We consider here a nonlinear one-dimensional population problem with age dependence and diffusion as proposed by Gurtin and MacCamy through several papers [**9, 12, 8**].

Let $\rho(x,t,a)$ denote the number of individuals per unit age and unit length who are of age a at time t and position x. The total population at x and t is

$$u(x,t) = \int_0^\infty \rho(x,t,a)\,da. \tag{1.1}$$

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Assuming that the population disperses to avoid crowding Gurtin and MacCamy [**9**] arrived at the following equations

$$\rho_t + \rho_a = k(\rho u_x)_x - \mu(a,u)\rho \tag{1.2}$$

$$\rho(x,t,0) = \int_0^\infty \beta(a,u)\rho(x,t,a)\,da \tag{1.3}$$

$$\rho(x,0,a) = \rho_0(x,a) \ge 0 \tag{1.4}$$

k is a constant that can be normalized to be 1.

The balance law (1.2) is of Malthusian type. If age and diffusion are ignored the population would tend to decay exponentially depending on the *death modulus* μ. On the other hand, if only diffusion is neglected and μ is assumed to depend on a the balance equation reduces to

$$\rho_t + \rho_a = -\mu(a)\rho \tag{1.5}$$

which can be readily integrated along characteristics.

Letting $t = a - t_0$ for $a \le t$ and $t = a + a_0$ for $t \le a$, we obtain a formal solution

$$\rho(t,a) = \begin{cases} \rho(0,a-t)e^{-\int_0^t \mu(a-t+s)\,ds}, & t \le a \\ \rho(t-a,0)e^{-\int_0^t \mu(s)\,ds}, & t \ge a \end{cases}. \tag{1.6}$$

Although $\rho(0,a-t) = \rho_0(a-t)$ is the prescribed initial age distribution, the term $\rho(t,0)$ needs to be determined. Assuming that the population sex ratio remains constant the birth rate $\beta(a,u)$ is defined such that $\beta(a,u)\,da$ represents the average number of offsprings produced per unit time by an individual aged between a and $a+da$. In this form equation (1.3) is the *birth law*, $\beta(a,u)$ is called the *birth module*.

The diffusion mechanism in (1.2) is such that the flux of individuals is proportional to the gradient of the total population. The first model considering diffusion was given by Skellam [**20**] in 1951. With ρ independent of age he assumed random diffusion of individuals, which gives a balance law of the form

$$\rho_t = k\rho_{xx} + \sigma(t) \tag{1.7}$$

where k is constant. It has been observed, however, that several species actually disperse to avoid crowding rather than dispersing randomly (see, for instance, [**4**]). This fact is modeled by considering

$$\rho_t = \varphi(\rho)_{xx} + \sigma(\rho) \tag{1.8}$$

where $\varphi'(0) = 0$, $\varphi'(\rho) > 0$ for $\rho > 0$. This was done by Gurtin and MacCamy in [**10**]. In [**7**] Gurney and Nisbet arrived at a similar equation with $\varphi(\rho) = \rho^2$ after considering a probabilistic walk in which individuals either stay at their present location or move in a direction of decreasing population.

The system (1.1)–(1.4) is just too general to be treated in that form, and some simplifying assumptions are necessary. In [**12**] Gurtin and MacCamy assumed that $\mu(a,u) = \mu_0$ and $\beta(a,u) = \beta_0 e^{-\alpha a}$, where μ_0, β_0 and α are constants, which reduces the system to a pair of partial differential equations. The first supposition models the case of a harsh environment in which age is not a significant cause of death (for instance, a population in the presence of predators that do not discriminate with age), the latter corresponds to a population with higher fertility at age 0. Of course, such an assumption is not to be taken literally but rather as an approximation to higher fertility at younger ages. In [**15**] and [**14**] the author studied the existence of solution and the spatial localization of the population when

$$\beta(a,u) = \beta_0(u) e^{-\alpha a} \tag{1.9}$$

and

$$\mu(a,u) = \mu_0(u) \tag{1.10}$$

β_0, μ_0 smooth positive functions. This problem was proposed in [**8**] by M. Gurtin.

A more realistic birth modulus was suggested in [**10**]

$$\beta(a,u) = \beta(u) a e^{-\alpha a} \tag{1.11}$$

with an expected number of zero births at age $a = 0$ an increasing fertility up to a maximum age $a = 1/\alpha$ and then a monotonical decrease to zero. For human-like populations the birth modulus looks like

$$\beta(a,u) = \begin{cases} 0, & 0 \le a \le a_0 \\ \beta(u)(a - a_0)e^{-\alpha a}, & a \ge a_0 \end{cases}. \tag{1.12}$$

In [**8**] M. Gurtin proposed a still more general birth function

$$\beta(a,u)=\beta_0(u)\sum_{k=1}^{n}\beta_k(u)a^k. \tag{1.13}$$

This would reduce (1.1)–(1.4) to a mixed system of $n+1$ nonlinear partial differential equations.

Here we shall compromise to a birth modulus of the form (1.11) and the death modulus as in (1.10). We shall assume for definiteness that $\beta(a,u)=\beta(u)b_0ae^{-\alpha a}$, where $b_0=\alpha e$ is introduced only as a normalizing factor. We believe this function carries most of the important aspects of the model and the one obtained with (1.13) will present similar characteristics. We shall study the existence of solutions, the localization of the population and its asymptotic behavior as $t\to\infty$.

Results for the existence of solutions and the localization of the populations in higher dimensions have been obtained in [**13**] when the birth modulus is given by (1.9) and the initial distribution is radially symmetric.

From another point, Busenberg and Ianelli [**3**] have proved the existence of solutions when both the birth and death modules are independent of the total population and depend only on age, $\mu(a,u)=\mu_0(a)$, $\beta(a,u)=\beta_0(a)$. The latter supposition renders the birth law (1.3) linear in u. Gurtin and MacCamy [**11**] have looked into product solutions of the form $\rho(x,t,a)=g(a)u(x,t)$, under the assumption that the death process is the form

$$\mu(a,u)=\mu_n(a)+\mu_e(u) \tag{1.14}$$

with $\mu_n(a)$ the probability of dying of natural causes during $(a,a+da)$ and $\mu_e(u)$ the probability of death due to environmental factors during the same interval.

For the random dispersal model we refer the reader to Garroni-Langlais [**6**], Langlais [**18**], DiBlassio [**5**] and the references contained therein.

2. Weak solutions. Introducing the auxiliary functions

$$\begin{aligned} C(x,t) &= \int_0^\infty e^{-\alpha a}\rho(x,t,a)\,da, \\ G(x,t) &= \int_0^\infty b_0 a e^{-\alpha a}\rho(x,t,a)\,da \end{aligned} \tag{2.1}$$

the birth law (1.3) can then be written

$$\rho(x,t,0) = \beta(u)G(x,t).$$

Integrating (1.2) with respect to a from 0 to ∞, assuming $\rho = 0$ at $a = \infty$ we obtain

$$u_t = (uu_x)_x - \mu(u)u + \beta(u)G(x,t) \tag{2.2}$$

and multiplying (1.2) by $e^{-\alpha a}$ and $ae^{-\alpha a}$ and integrating

$$C_t = (Cu_x)_x - (\mu(u)+\alpha)C + \beta(u)G \tag{2.3}$$

and

$$G_t = (Gu_x)_x - (\mu(u)+\alpha)G + b_0 C. \tag{2.4}$$

The initial conditions for (2.1)–(1.3) are obtained in the same manner.

Note that if $\rho(x,t,a) > 0$ in a set of positive measure then $C(x,t) > 0$, $G(x,t) > 0$ and $u(x,t) > 0$. On the other hand, if $\rho(x,t,a) = 0$ a.e. in a, then $C(x,t) = G(x,t) = u(x,t) = 0$.

Introduce the rates $p(x,t) = C(x,t)/G(x,t)$ and $q(x,t) = G(x,t)/u(x,t)$ for $u, G \neq 0$.

Substituting p and q into equations (2.2)–(2.4) and including the corresponding initial conditions, we arrive at

$$u_t = (uu_x)_x + (\beta(u)q - \mu(u))u \tag{2.5}$$

$$q_t - u_x q_x = (b_0 p - \alpha)q - \beta(u)q^2 \tag{2.6}$$

$$p_x - u_x p_x = \beta(u) - b_0 p^2 \tag{2.7}$$

$$u(x,0) = u_0(x) = \int_0^\infty \rho_0(x,a)\,da \geq 0 \tag{2.8}$$

$$q(x,0) = q_0(x), \quad p(x,0) = p_0(x) \tag{2.9}$$

where

$$p_0(x) = \frac{\int_0^\infty e^{-\alpha a}\rho(x,0,a)\,da}{\int_0^\infty b_0 a e^{-\alpha a}\rho(x,0,a)\,da} \tag{2.10}$$

and

$$q_0(x) = \frac{\int_0^\infty b_0 a e^{-\alpha a}\rho(x,0,a)\,da}{\int_0^\infty \rho(x,0,a)\,da} \tag{2.11}$$

if $\rho_0(x,a) > 0$ in a set of positive measure. Otherwise, $q_0(x) = p_0(x) = 0$.

This is a mixed system composed of a degenerate parabolic and first order nonlinear hyperbolic differential equations.

Let $h(x,t,u) = \beta(u)q - \mu(u)$. Equation (2.6) is

$$\begin{aligned} u_t &= (uu_x)_x + h(x,t,u)u \\ u(x,0) &= u_0(x) \geq 0. \end{aligned} \tag{2.12}$$

With $h \equiv 0$ it is the porous medium equation:

$$u_t = (uu_x)_x, \qquad u(x,0) = u_0(x) \geq 0. \tag{2.13}$$

It models the diffusion of a homogeneous gas flow through a homogeneous porous medium. There is a large literature for this equation. Complete references are given in Aronson [**1**], Peletier [**19**] and Vazquez [**21**].

The most striking difference between the solutions of (2.13) and those of the usual heat equation

$$u_t = u_{xx}, \qquad u(x,0) = u_0(x) \tag{2.14}$$

is their speed of propagation. Assume that a population $u(x,t)$ satisfies the random dispersal equation (2.13) and it is initially restricted to a bounded interval I_0, i.e., $u_0(x) > 0$ in I_0, $u_0(x) \equiv 0$ on $\mathbf{R}\backslash I_0$. These solutions are characterized by an infinite speed of propagation: $u(x,t) > 0$ for all $x \in \mathbf{R}$, $t > 0$. Thus, the population would spread immediately to all the space. On the other hand, if a population $u(x,t)$

satisfies (2.13) it will have a finite speed of propagation: If $I_0 = (a_0, b_0)$, there are two monotone interface curves $\gamma_1(t)$, $\gamma_2(t)$ with $a_0 = \gamma_1(0)$, $b_0 = \gamma_2(0)$, that separate the region of positivity of $u(x,t)$ from the region in which $u(x,t) \equiv 0$.

In this work we shall assume that the initial population for the anticrowding model is nonnegative and satisfies $u_0(x) > 0$ on I_0, $u_0(x) \equiv 0$ on $\mathbf{R}\backslash I_0$. It is therefore reasonable to expect that it will present a behavior similar to that of the solutions of the porous medium equation.

In [**14**] it is proved that when $\beta(a,u) = \beta(u)e^{-\alpha a}$, the support of $u(x,t)$ increases with t and it is always an interval. Further, if

$$\sup_{0\le u\le K_1} \frac{\mu(u)}{\beta(u)} < \inf_{0\le u\le K_1} \frac{\beta(u)-\alpha}{\beta(u)} \tag{2.15}$$

then the support of $u(x,t)$ grows to $(-\infty,\infty)$ as $t \to \infty$. In this case all the real line will be ultimately populated.

On the other hand, if

$$\sup_{0\le u\le K_1} \frac{\beta(u)-\alpha}{\beta(u)} < \inf_{0\le u\le K_1} \frac{\mu(u)}{\beta(u)}, \tag{2.16}$$

then the population remains localized in an interval $[-L, L]$ for all times. In this case the interaction between age dependence and diffusion produces that the population persist in a limited region.

Similar results are given in [**13**] for the problem with radial symmetry in several variables.

It is well known that equation (2.13) does not have classical solutions unless the initial datum is strictly positive. (See Aronson [**2**], Kalashnikov [**17**]). This is because if u_0 has compact support the solutions will not have a continuous first derivative when crossing the interfaces. A *fortiori* equations (2.5) should not have a classical solution either.

We shall use the following definition of weak solutions (see [**15**]):

Let

$$K = \{\varphi(x,t) \in C^\infty(\Omega_T) : u \equiv 0 \text{ for large } |x| \text{ and } t \text{ near } T\}.$$

Given $\varphi \in K$, multiply (2.6) by φ and integrate on $\mathbf{R} \times (0,T)$ to obtain

$$\int_0^T \int_{\mathbf{R}} \left(\frac{1}{2}(u^2)x\varphi_x - u\varphi_t\right) dx\, dt = \int_0^T \int_{\mathbf{R}} (\beta(u)q - \mu(u))u\varphi\, dx\, dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\, dx. \tag{2.17}$$

For q we multiply (2.6) by q and (2.7) by u, add and integrate

$$\int_0^T \int_{\mathbf{R}} \left(\frac{1}{2}(u^2)_x q\varphi_x - uq\varphi_t\right) dx\, dt = \int_0^T \int_{\mathbf{R}} (b_0 p - \alpha - \mu(u))qu\varphi\, dx\, dt + \int_{\mathbf{R}} u_0(x)q_0(x)\varphi(x,0)\, dx \tag{2.18}$$

and for p

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}} \left(\frac{1}{2}(u^2)_x pq\varphi_x - upq\varphi_t\right) dx\, dt \\ &\quad = \int_0^T \int_{\mathbf{R}} (\beta(u) - (\beta(u) + \alpha)p))qu\varphi\, dx\, dt \\ &\qquad + \int_{\mathbf{R}} u_0(x)p_0(x)q_0(x)\varphi(x,0)\, dx \end{aligned} \tag{2.19}$$

We define a weak solution of (2.6)–(2.8) as a triple (u,q,p) such that $(u^2)_x \in L^2_{\text{loc}}(\Omega_T)$ in the sense of distributions, $q(x,t), p(x,t) \in L^2_{\text{loc}}(\Omega_T)$, and (2.17)–(2.19) are satisfied for any $\varphi \in K$.

The following notation will be used:

$C^{2,1}(\Omega_T)$ is the Banach space consisting of functions $u(x,t)$ defined in Ω_T with continuous second derivatives in x and continuous first derivative in t.

$C^\alpha(\Omega_T)$ is the Banach space of functions $u(x,t)$ defined in Ω_T for which the α-norm

$$||u||_\alpha = \sup_{\Omega_T} |u| + \sup_{\Omega} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\alpha/2}}$$

is bounded.

$C^{1+\alpha}(\Omega_T)$ is the Banach space for which the $1+\alpha$-norm $||u||_{1+\alpha} = ||u_x||_\alpha + ||u_t||_\alpha + ||u||_\alpha$ is bounded.

Similarly, $C^{2+\alpha}(\Omega_T)$ is the Banach space for which the $2+\alpha$-norm $||u||_{2+\alpha} = ||u_{xx}||_\alpha + ||u||_{1+\alpha}$ is bounded.

3. Existence results. Concerning the functions $\mu(u), \beta(u)$ and $\rho_0(x,a)$ the following assumptions will be adopted throughout.

(A1) $\beta(r), \mu(r)$ are bounded continuously differentiable functions on $[0,\infty)$.

(A2) $\rho_0(x,a)$ is sufficiently smooth so that u_0, q_0 and p_0 are bounded continuous functions in $\mathbf{R}$.

The continuous differentiability assumption on β and μ can be easily relaxed to require only continuity. We have preferred to avoid this apparently more general case in order to keep the notations simple. With respect to $\rho_0(x,a)$ we are basically assuming that it is integrable with respect to a for every x and that it is not overly concentrated at $a=0$ for any x. For definiteness we assume that there exist constants M_0 and M_1 such that u_0, $|u_0'| \le M_0$ and β, $|\beta'|, \mu, |\mu'| \le M_1$.

Theorem 3.1. *Under the previous assumptions there exists a* (*weak*) *solution* (u,q,p) *of* (2.5)–(2.11). *The population* u *is uniformly bounded in* $\mathbf{R}$ *and* u^2 *is differentiable with respect to* x.

Theorem 3.2. *Assume there exist* $x_1, x_2 \in \mathbf{R}$ *such that* $\rho_0(x,a) > 0$ *for all* $x \in (x_1,x_2)$ *and* $\rho_0(x,a) = 0$ *for* $x \in \mathbf{R}\backslash(x_1,x_2)$. *Then the support of* $u(x,t)$ *is a finite interval for every* t, *and there exist two interface curves* $\gamma_1(t), \gamma_2(t)$ *such that* $x_1 = \gamma_1(0)$ *and* $x_2 = \gamma_2(0)$, *and* $\operatorname{supp} u(\cdot,t) = [\gamma_1(t), \gamma_2(t)]$ *for every* t.

In Section 4 we study the localization of u and its behavior as $t \to \infty$.

Proof of Theorem 1. Assume $u \in \mathcal{C}^{2+\alpha}(\Omega_T)$ is given. We shall first study equations (2.7) and (2.6).

Define characteristics $x(t;\bar{x},\bar{t})$ by

$$\partial x/\partial t = -u_x(x,t), \qquad x(\bar{t};\bar{x},\bar{t}) = \bar{x}. \tag{3.1}$$

Since u_x is Lipschitz continuous there is always a local solution of this equation. Since u_x is also bounded, this solution can be made global by extending it to the boundary of $[0,T]\times\mathbf{R}$. With $P(t) = p(x(t;\bar{x},\bar{t}),t)$ we get

$$dP/dt = \beta(u) - b_0P^2, \qquad P(0) = p(x(0;\bar{x},\bar{t}),0). \tag{3.2}$$

We shall first establish a comparison lemma for this equation.

Lemma 3.1. *If $R, S \in' (0,T)$, $R(0) < S(0)$, $f(t) \le g(t)$ and*

$$\frac{dR}{dt} + b_0R^2 - f(t) \le \frac{dS}{dt} + b_0S^2 - g(t)$$

for all $t\in(0,T)$, then $R(t) \le S(t)$ on $[0,T)$.

Proof. If $w(t) = R(t) - S(t)$, then

$$dw/dt + b_0(R+S)w \le f(t) - g(t) \le 0;$$

thus,

$$w(t)e^{\int_0^t b_0(R+S)(s)\,ds}$$

is a decreasing function which is nonpositive at $t = 0$. Therefore, $w(t) \le 0$ and $R(t) \le S(t)$ on $[0,T)$.

Now if κ is a positive constant and $\kappa \ne R_0$ the unique solution of

$$dR/dt = \kappa - b_0R^2, \qquad R(0) = R_0 \tag{3.3}$$

is

$$R(t) = \kappa_1\frac{e^{\sqrt{\kappa b_0}\,t} - Ce^{-\sqrt{\kappa b_0}\,t}}{e^{\sqrt{\kappa b_0}\,t} + Ce^{-\sqrt{\kappa b_0}\,t}}$$

where $\kappa_1 = \sqrt{\kappa/b_0}$ and $C = (\kappa_1 - R_0)/(\kappa_1 + R_0)$. In fact, if $R_0 < \kappa_1$, $R(t)$ increases from R_0 to κ_1, if $R_0 > \kappa_1$, $R(t)$ decreases from R_0 to κ_1.

When $R_0 = \kappa_1$, $R = R_0$ is the constant solution. From this observation and Lemma 3.1, it follows that any solution of (3.2) is bounded above and below by the corresponding solutions of (3.3) with$\kappa = \beta^*$ and $\kappa = \beta_*$, thus if $b_1 = \min\{\sqrt{\beta_*}/b_0, P(0)\}$ and $b_2 = \max\{\sqrt{\beta^*/b_0}, P(0)\}$ then

$$b_1 \le P(t) \le b_2 \tag{3.4}$$

for any solution $P(t) \in \mathcal{C}(0,T)$ of (3.2).

It follows now by standard existence theorem that there is a unique solution of equation (3.2) in the rectangle $[0,T) \times [b_1, b_2]$.

Along the same characteristics given by (3.1)–(3.3), for $Q(t) = q(x(t;\bar{x},\bar{t}),t)$ we have

$$\begin{aligned} dQ/dt &= (b_0 P(t) - \alpha)Q - \beta(u)Q^2 \\ Q(0) &= q(x(0;\bar{x},\bar{t}),0). \end{aligned} \tag{3.5}$$

This is a Bernoulli type equation that can be solved letting $R = Q^{-1}$. Integrating along characteristics we obtain

(3.6)
$$q(\bar{x},\bar{t}) = e^{\int_0^{\bar{t}}(P(s)-\alpha)\,ds}\left[q(x(0;\bar{x},\bar{t}),0)^{-1} + \int_0^{\bar{t}} \beta(u) e^{\int_0^{\tau}(P-\alpha)\,ds}\,dt\right]^{-1}.$$

This is just a formal solution. Direct differentiation shows that it is an actual solution.

Let $y = (x,t)$. In R^2 we consider a mollifier $J(y)$, a symmetric $\mathcal{C}^\infty$-function such that $J(y) \ge 0$ if $|y| \le 1$ and $\int_{R^2} J(y)\,dy = 1$.

(For instance

$$J(y) = \begin{cases} ke^{-1/(1-|y|^2)}, & |y| < 1 \\ 0, & |y| > 1 \end{cases} \quad \text{for appropriate constant } k).$$

Let $J_n(y) = (1/n^2)J(ny)$ and $(q * J_n)(y) = \int_{|y-y'|\le 1/n} J(n(y - y'))q(y')\,dy'$.

It is then clear that if $q \in \mathcal{L}^2$, $\{q * J_n\}_{n=1}^\infty$ is a C^∞-sequence that converges to q in $\mathcal{L}^2$, and if q is continuous $\{q * J_n\}_{n=1}^\infty$ converges to q uniformly on compact sets.

We will apply Schauder's fixed point theorem to the following ε-n-approximating systems:

$$u_t = (uu_x)_x + (\beta(u)(q * J_n) - \mu(u))(u - \varepsilon) \tag{3.7}$$

$$q_t - u_x q_x = (b_0 p - \alpha)q - \beta(u)q^2 \tag{3.8}$$

$$p_x - u_x p_x = \beta(u) - b_0 p^2 \tag{3.9}$$

$$u(x,0) = u_0(x) + \varepsilon, \qquad q(x,0) = q_0(x) + \varepsilon \tag{3.10}$$
$$p(x,0) = p_0(x) + \varepsilon$$

Let K_1 be a constant to be specified later, and for $\alpha \in (0,1)$, let $V = \{w \in \mathcal{C}^{2+\alpha}(\Omega_T) : ||w||_{2+\alpha} \leq K_1, w \geq \varepsilon\}$. V is a closed convex set. □

Define $T : V \mapsto \mathcal{C}^{2+\alpha}(\Omega_T)$ in the following way:

Given $w \in V$, by the previous discussion there exist unique solutions (depending on w), $p \in \mathcal{C}^{1+\alpha}(\Omega_T)$, $q \in \mathcal{C}^{1+\alpha}(\Omega_T)$ of (2.7), (2.6), respectively. Further, (3.5) implies that

$$\frac{d}{dQ}t - (b_0 P - \alpha)Q = -\beta(w)Q^2 < 0$$

so

$$e^{-\int_0^t (b_0 P - \alpha)\, ds}$$

is a decreasing function of t that at $t = 0$ takes the value $Q(0) = q(x(0, \bar{x}, \bar{t}), 0) < 1$. It follows that $q(\bar{x}, \bar{t}) \leq 1$ for all x, t.

Since $0 \leq q \leq 1$, $||q * J_n||_\sigma \leq 2n$, for any $\sigma \in (0,1)$. Since $u(x,0) \geq$ is strictly positive, from the standard theory of parabolic differential equations we get that (3.8) has a unique solution $u \in \mathcal{C}^{2+\sigma}(\Omega_T)$. Further, there exists a constant K_2 (depending on ε and n) such that $||u||_{2+\sigma} \leq K_3$. Choose $K_1 \geq K_2$ and $\sigma > \alpha$. Then T maps V into V. Since bounded sets in $\mathcal{C}^{2+\sigma}$ are precompact in $\mathcal{C}^{2+\alpha}$ for $0 < \alpha < \sigma < 1$, we also obtain that T is precompact. Standard estimates show that T is continuous. Thus, Schauder's fixed point theorem implies the existence of a fixed point u of T. This u with the corresponding p and q is a solution of (3.8)–(1.13). Further, since $0 \leq q \leq 1$, from the maximum principle it follows that $\varepsilon \leq u(x,t) \leq M_2$ where M_2 depends only on the initial bounds M_0, M_1. The solution found will be denoted by $u_{\varepsilon,n}, q_{\varepsilon,n}$ and $p_{\varepsilon,n}$. $\square$

After this, the proof of existence of solutions follows similar lines as in [**13**] and we will just sketch it here.

From [**16**] we quote the following

Theorem 3.3. *Let $u \in \mathcal{C}^{2+\alpha}(\Omega_T)$ be a (classical) solution of*

$$\begin{aligned} u_t &= (uu_x)_x + h(x,t,u)(u-\varepsilon) \\ u(x,0) &= u_0(x) + \varepsilon \end{aligned} \tag{3.11}$$

where $u_0(x) \geq 0$. Then the α-norm of u is uniformly bounded independently of ε and the modulus of continuity of h, i.e., there exists $K_3 > 0$ depending only on $||u_0||_\infty$ and $||h||_\infty$ such that $||u||_{2+\alpha} \leq K_3$ in $\overline{\Omega}_T$.

Using the Arzela-Ascoli theorem in the sequence $\{u_{\varepsilon,n}\}$ we can extract a subsequence $\{u_{\varepsilon,n_k}\}$ that converges to an α-Hölder continuous function $u_\varepsilon(x,t)$. Since $||q_{\varepsilon,n}||_\alpha$ and $||p_{\varepsilon,n}||_\alpha$ are bounded independently of n (but depending on ε) we can also extract subsequences $\{q_{\varepsilon,n_k}\}$ of $\{q_{\varepsilon,n}\}$ and $\{p_{\varepsilon,n_k}\}$ of $\{p_{\varepsilon,n}\}$ that converges uniformly in compact subsets of Ω_T to continuous functions q_ε and p_ε, respectively.

Since $u_{\varepsilon,n} \geq \varepsilon$, the equations (3.7) are uniformly parabolic in n. It follows then from the standard theory that $||u||_{2+\alpha} \leq K_6$ a constant independent of n, and for $\alpha' < \alpha$ there exists a subsequence $\{u_{\varepsilon,n_k}\}$ that converges to u_ε in $\mathcal{C}^{2+\alpha'}(\Omega_T)$.

We rename these sequences $\{u_{\varepsilon,n}\}$, $\{q_{\varepsilon,n}\}$ and $\{p_{\varepsilon,n}\}$, respectively.

Let $\varphi(x,t)$ be a test function as defined in this introduction. Multiply equation (3.7) by φ and integrate. Multiplying (3.8) by u, (3.7) by q and subtract, to obtain the equation for q

$$(uq)_t - \frac{1}{2}(q(u^2)_x)_x = (b_0 p - \alpha - \mu)qu. \tag{3.12}$$

Now multiply (3.12) by φ and integrate. Multiply (2.7) by qu and (3.12) by p and subtract to obtain the equation for p

$$(pqu)_t - \frac{1}{2}(pq(u^2)_x)_x = (\beta(u) - (\beta(u) + \alpha)p)qu. \tag{3.13}$$

Now multiply (3.13) by φ and integrate. Performing this calculation with $u_{\varepsilon,n}, q_{\varepsilon,n}$ and $p_{\varepsilon,n}$, we obtain the equivalent of (2.17)–(2.19). The uniform convergence discussed above implies that the limiting functions u_ε, q_ε and p_ε are weak solutions of the ε-limiting equations

$$u_t = (uu_x)_x + (\beta(u)q - \mu(u))(u - \varepsilon) \tag{3.14}$$

$$q_t - u_x q_x = (p - \alpha)q - \beta(u)q^2 \tag{3.15}$$

$$p_x - u_x p_x = \beta(u) - p^2 \tag{3.16}$$

$$u(x,0) = u_0(x) + \varepsilon, \qquad q(x,0) = q_0(x) + \varepsilon$$

$$p(x,0) = p_0(x) + \varepsilon \tag{3.17}$$

(Actually, it can be shown that u_ε is a classical solution of (3.15) and that q_ε is continuous).

Again, using that $||u_\varepsilon||_\alpha \leq K_3$ we can extract a subsequence $\{u_{\varepsilon'}\}$ that converges uniformly in compact sets to an α-Hölder continuous function $u(x,t)$. As in [**15**] it can be shown that u^2 is differentiable, $\{(u_{\varepsilon'}^2)_x\}$ is uniformly bounded and $\{(u_{\varepsilon'}^2)_x\}$ converges pointwise to u_x^2. The corresponding $\{q_{\varepsilon'}\}$ and $\{p_{\varepsilon'}\}$ converge weakly to a pair of integrable functions q and p, and equations (2.17)–(2.19) are satisfied. Thus, (u,q,p) is a weak solution of (2.1)–(2.2).

4. Qualitative behavior of solutions.

4.1. *Populated and unpopulated regions.* The existence of the interface that separates the populated region from the unpopulated region will be investigated first.

Proof of Theorem 2. We begin with the following

Lemma 4.1. *If $u(x_0,t_0) \geq \eta > 0$ then $u(x_0,t) > 0$ for all $t \geq t_0$. Thus, if a region becomes populated at time t_0 it remains populated for all later times. In particular, the initial region (x_1,x_2) remains populated for all times (but it might tend to 0 as $t \to \infty$).*

Proof. Assume first $t_0 = 0$ and $x_1 < x_0 < x_2$. Using the continuity of $u_0(x)$ choose $\delta > 0$ small, $\delta < x_0/2$ such that $\delta^2 \leq \eta/2$ and $u_0(x) \geq \eta/2$ on $I_\delta = (x_0 - \delta, x_0 + \delta)$.

For $x \in [x_0-\delta, x_0+\delta]\times(0,T]$, let $v(x,t) = e^{-kt}(\delta^2-(x-x_0)^2)+\varepsilon/2$, where $k = 4+M_1$, $|h| \le M_1$.

If

$$\mathcal{L}[z] = z_t - (zz_x)_x + (M_1+2)(z-\varepsilon)$$

then $\mathcal{L}[u_\varepsilon] = (h+M_1+2)(u_\varepsilon-\varepsilon) \ge 0$ and

$$\begin{aligned}\mathcal{L}[v] = {} & e^{-kt}(\delta^2-(r-r_0)^2)(-4+2e^{-kt}) \\ & - 4e^{-2kt}(x-x_0)^2 - \frac{\varepsilon}{2}(2e^{-kt}-M_1-2) \le 0.\end{aligned}$$

Also,

$$v(x,0) = \delta^2-(x-x_0)^2+\frac{\varepsilon}{2} \le \frac{\eta}{2}+\frac{\varepsilon}{2} \le u_0(x)+\varepsilon = u_\varepsilon(x,0)$$

and

$$v(x_0\pm\delta,t) = \frac{\varepsilon}{2} \le \varepsilon \le u_\varepsilon(x_0\pm\delta,t).$$

The maximum principle now implies that

$$v(x,t) \le u(x,t) \quad \text{on } [x_0-\delta,x_0+\delta]\times[0,T].$$

As $\varepsilon \to 0$, we obtain

$$u(x_0,t) \ge e^{-(4+M_1)t}\delta^2.$$

If $t_0 > 0$ we use the Hölder continuity of u and the uniform convergence of u_ε to u to find a δ such that

$$u_\varepsilon(x,t_0) \ge \eta/2+\varepsilon \quad \text{on } [x_0-\delta,x_0+\delta].$$

Then the function $w(x,t) = v(x,t+t_0)$ satisfies all the requirements of the previous argument. □

Lemma 4.2. *For every $\bar{t} > 0$ there exists $R_* > 0$ such that $u(x,\bar{t}) = 0$ for $|x| > R_*$. Thus, the support of $u(x,t)$ is finite for every t.*

Proof. Choose $R > |x_1|, |x_2|$ so $u_0(x) \equiv 0$ for $|x| > R$.

Let $\sigma < 1$, $\sigma < 1/K_1$ (without loss of generality we assume that $K_1 > 1$). Fix x_0 such that $x_0 - 1/\sigma \geq R$ and $\tau = 1/(14M_1)$. Consider $g(t) = 2\tau/(2\tau - t)$ and

$$v(x,t) = \sigma g(t)(x - x_0)^2 + \varepsilon g^\sigma(t)$$

on $B_1 = [x_0 - 1/\sigma, x_0 + 1\sigma] \times [0, \tau]$.

We will use v as an upper bound for u and obtain $u(x,t) \leq 2^\sigma \varepsilon$ on B_1.

Let $\mathcal{L}[z] = z_t - (zz_x)_x - h(x,t,z)(z - \varepsilon)$. Then $\mathcal{L}[u_\varepsilon] = 0$ and

$$\begin{aligned} \mathcal{L}[v] = \sigma &\frac{(x - x_0)^2 g^2(t)}{2\tau}(1 - 12\sigma\tau - h(2\tau - t)) \\ &+ \varepsilon\left(\left(\frac{\sigma}{2\tau} - 2\sigma\right) g^{\sigma+1}(t) - h(x,t,u_\varepsilon)(g^\sigma(t) - 1)\right). \end{aligned}$$

By the mean value theorem

$$g^\sigma(t) - 1 = \sigma g^{\sigma-1}(s) g'(s) t = \sigma g^{\sigma+1}(s) t$$

for some $s \in (0,t)$.

Since $|h| \leq M_1$, $0 \leq t \leq \tau$ and $g(s)/g(t) = (2\tau - t)/(2\tau - s)$ lies between $1/2$ and 1, we have

$$\begin{aligned} \mathcal{L}[v] \geq \sigma &\frac{(x - x_0)^2 g^2(t)}{2\tau}(1 - 12\sigma\tau - 2\tau M_1) \\ &+ \varepsilon\sigma \frac{g(t)^{\sigma+1}}{2\tau}(1 - 4\tau - 2M_1\tau) \geq 0 \end{aligned}$$

by the choice of τ and σ.

On the parabolic boundary of B_1,

$$v(x,0) = \sigma(x - x_0)^2 + \varepsilon \geq \varepsilon = u_\varepsilon(x,0)$$

for $x \in [x_0 - 1/\sigma, x_0 + 1/\sigma]$, and

$$v(x_0 \pm 1/\sigma, t) = g(t)/\sigma + \varepsilon g^\sigma(t) \geq 1\sigma$$

for $0 \leq t \leq \tau$.

Thus $v(x_0 \pm 1/\sigma, t) \geq u(x_0 \pm 1/\sigma, t)$ for $1/\sigma \geq K_1$.

It follows then by the maximum principle that $u \leq v$ on $[x_0 - 1/\sigma, x_0 + 1/\sigma] \times [0, \tau)$. In particular

$$u(x_0, t) \leq \varepsilon \left(\frac{2\tau}{2\tau - t} \right)^{\sigma} \leq 2^{\sigma} \varepsilon. \tag{4.1}$$

Since the only restriction for x_0 is to be larger than $R + 1/\sigma$ equation (4.1) is valid for any $x \geq x_0$. We repeat the argument with $x_3 \geq x_0 + 1/\sigma$ on $[\tau, 2\tau]$ with initial datum $u(x, \tau) \leq 2^{\sigma}\varepsilon$ and obtain that $u(x,t) \leq 2^{2\sigma}\varepsilon$ for $x \geq x_3 \geq R + 2(1/\sigma)$.

Now for a given $\bar{t}$, after $k = [\bar{t}/(\tau + 1)]$ steps we arrive at

$$u_{\varepsilon}(x, t) \leq 2^{k\sigma}\varepsilon$$

for

$$x \geq x_k \geq R + k(1/\sigma), \qquad 0 \leq t \leq \bar{t}.$$

As $\varepsilon \to 0$ we obtain that $u(x, t) \equiv 0$ for

$$x \geq R + k(1/\sigma), \qquad 0 \leq t \leq \bar{t}.$$

Hence the theorem. □

This last lemma implies that the support of $u(x, t)$ is finite for all t. On the other hand, by Lemma 3.1, we know that once u becomes positive it stays positive for all later times; hence, the support of $u(\cdot, t)$ increases with t. In particular, if $\operatorname{supp} u_0(x)$ is an interval, another application of the maximum principle shows that $\operatorname{supp} u(x, t)$ is also an interval. This proves Theorem 3. □

4.2. Localization. We now turn to the question of localization. The fact that $\operatorname{supp} u(\cdot, t)$ increases with t leads us naturally to the question of determining the limit of $\operatorname{supp} u(\cdot, t)$ as $t \to \infty$.

We say that the population is *localized* if there exists $L > 0$ such that $\operatorname{supp} u(\cdot, t) \subset [-L, L]$ for all $t > 0$.

We shall use as comparison the solutions $v(x, t)$ of

$$v_t = (v v_x)_x - \delta v, \qquad v(x, 0) = v_0(x) \geq 0. \tag{4.2}$$

Our previous discussion shows that the support of $v(\cdot,t)$ is always finite and increasing with t.

Introduce the change of independent and dependent variables $t(s) = (1/\delta)\log(1-\delta s)$ for $0 \le s \le 1/\delta$ and $w(x,s) = [1/(1-\delta s)]v(x,t(s))$. With this change (4.2) reduces to

$$w_s = (ww_x)_x, \qquad w(x,0) = v(x,0) = v_0(x). \tag{4.3}$$

This is the standard porous medium equation. Thus $w(x,s)$ has finite support for every $s > 0$. In particular, for $s = 1/\delta$ there exists $L > 0$ such that $\operatorname{supp} w(x,1/\delta) \subset [-L,L]$. It follows that $\operatorname{supp} v(\cdot,t) \subset [-L,L]$ for all $t > 0$.

Our first localization result is:

Theorem 4.1. *Assume that $\beta(r) < \mu(r)$ on $[0,M_1]$. Then $u(x,t)$ is localized.*

Proof. There exists $\delta > 0$ such that $\beta(r) - \mu(r) \le -\delta < 0$ for all $r \in [0,K_1]$.

Let $v_\varepsilon(x,t)$ be the classical solution of

$$\begin{aligned} v_t &= (vv_x)_x - \delta(v-\varepsilon) \\ v(x,0) &= u_0(x) + \varepsilon \ge 0. \end{aligned} \tag{4.4}$$

If $\mathcal{L}[z] = z_t - (zz_x)_x + \delta(z-\varepsilon)$ we have that $\mathcal{L}[v_\varepsilon] = 0$ and $\mathcal{L}[u_\varepsilon] = \beta(u_\varepsilon)q_\varepsilon(x,t) - \mu(u_\varepsilon) \le -\delta < 0$, since $0 \le q(x,t) \le 1$.

The maximum principle implies that $u_\varepsilon(x,t) \le v_\varepsilon(x,t)$ in Ω_T.

As $\varepsilon \to 0$ we obtain that $u(x,t) \le v(x,t)$ in Ω_T. Since $v(x,t)$ is localized, so is $u(x,t)$. This ends the proof. $\square$

Now let us suppose that $h(r,t,u) = \beta(u)q - \mu(u) \ge 0$. The comparison principle implies that $u(x,t) \ge w(x,t)$, the solution of the porous medium equation

$$w_t = (ww_x)_x, \qquad w(x,0) = v(x,0) = v_0(x). \tag{4.5}$$

It is known [**17**] that $\operatorname{supp} w(x,t) \to [0,\infty)$ as $t \to \infty$. Therefore $\operatorname{supp} u(x,t) \to [0,\infty)$ in this case.

Thus, if the birth module is definitely less than the death module, the population will not diffuse further than a fixed interval. On the other hand, if $h(x,t,u) \geq 0$, the population will eventually cover all $[0,\infty)$. Of course, the latter condition cannot be checked based on the data alone. Other conditions for localization and nonlocalization have been given in [**14**] for the birth function $\beta(a,u) = \beta(u)e^{-\alpha a}$. We present here similar results.

Let

$$\beta^* = \sup_{0\leq r\leq K_1} \beta(r), \qquad \mu_* = \inf_{0\leq r\leq K_1} \mu(r)$$

and β_*, μ^* defined similarly.

Also, let

$$s_1 = \sup_{0\leq u\leq K_1} \frac{\sqrt{b_0\beta^*} - \alpha}{\beta(u)}, \qquad i_1 = \inf_{0\leq u\leq K_1} \frac{\mu(u)}{\beta(u)}$$

and

$$s_2 = \sup_{0\leq u\leq K_1} \frac{\mu(u)}{\beta(u)}, \qquad i_2 = \inf_{0\leq u\leq K_1} \frac{\sqrt{b_0\beta_*} - \alpha}{\beta(u)}.$$

Theorem 4.2. *Assume that $s_1 < i_1$ and $q_0(x) \leq c_1$ for some $c_1 \in (s_1, i_1)$. If also $b_0 M_0 \leq \sqrt{b_0\beta^*}$, then $u(x,t)$ is localized.*

Proof. There exists $\delta > 0$ such that

$$\frac{\sqrt{b_0\beta^*} - \alpha}{\beta(u)} < c_1 < \frac{\mu(u) - \delta}{\beta(u)}$$

for every $u \in [0, K_1]$.

From the bounds for p given in (3.4) we obtain

$$b_0 p_\varepsilon \leq \max\{\sqrt{b_0\beta^*}, b_0 M_0\} \leq \sqrt{b_0\beta^*}.$$

Consider $\mathcal{L}[q] = q_t - u_x q_x - q(b_0 p - \alpha - \beta q)$. The usual comparison principle is valid for this q (see [**13**]).

We have $\mathcal{L}[q_\varepsilon] = 0$ and

$$-\mathcal{L}[c_1] = c_1(b_0 p - \alpha - \beta(u)c_1) \le c_1\beta(u)\left(\frac{\beta^* - \alpha}{\beta(u)} - c_1\right) \le 0.$$

Thus $\mathcal{L}[c_1] \ge 0 = \mathcal{L}[q_\varepsilon]$.

Since $q_0(x) \le c_1$ we get $q(x,t) \le c_1$ in Ω_T. Therefore $\beta(u)q(x,t) - \mu(u) < -\delta < 0$ and the argument of Lemma 4.1 applies. Thus $u(x,t)$ is localized.

Theorem 4.3. *Assume that $s_2 < i_2$ and $c_2 < q_0(x) \le 1$ for some $c_2 \in (s_2, i_2)$. If also $b_0 M_0 \ge \sqrt{b_0\beta_*}$, then $u(x,t)$ is nonlocalized, i.e.,* $\operatorname{supp} u(\cdot, t) \to \infty$ *as $t \to \infty$.*

Proof. This time from the bounds for p given in (3.4) we obtain

$$b_0 p_\varepsilon \ge \min\{\sqrt{b_0\beta_*}, b_0 M_0\} \ge \sqrt{b_0\beta_*}$$

and

$$-\mathcal{L}[c_1] = c_2(b_0 p - \alpha - \beta(u)c_2) \ge c_2\beta(u)\left(\frac{\sqrt{b_0\beta_*} - \alpha}{\beta(u)} - c_2\right) \ge 0.$$

Thus $\mathcal{L}[c_2] \le 0 = \mathcal{L}[q_\varepsilon]$.

Since $c_2 \le q_0(x)$, we get $q(x,t) \ge c_2$ in Ω_T. Therefore $\beta(u)q(x,t) - \mu(u) \ge 0$ and by the previous arguments $u(x,t)$ is nonlocalized. □

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