ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 1, Winter 1994

SINGULAR PERTURBATIONS IN VISCOELASTICITY

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Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. We study the singular perturbation for a class of partial integro-differential equations in viscoelasticity of the form

$$\begin{aligned}\rho u^\rho_{tt}(t,x) = Eu^\rho_{xx}(t,x) + \int_{-\infty}^t a(t-s)u^\rho_{xx}(s,x)\,ds \\ + \rho g(t,x) + f(x),\end{aligned} \tag{a}$$

when the density ρ of the material goes to zero. We will prove that when $\rho \to 0$ the solutions of the dynamical systems (a) (with $\rho > 0$) approach the solution of the steady state obtained from equation (a) with $\rho = 0$. The technique of energy estimates is used. A similar result is also obtained for a nonlinear equation of the form

$$\rho u^\rho_{tt}(t,x) = \phi(u^\rho_x(t,x))_x + \int_{-\infty}^t a(t-s)\phi(u^\rho_x(s,x))_x\,ds + \rho g(t,x).$$

1. Introduction. Consider the following model in viscoelasticity in the one-dimensional case on the real line, (see [**4, 10**]),

$$\begin{aligned}\rho u^\rho_{tt}(t,x) &= Eu^\rho_{xx}(t,x) + \int_{-\infty}^t a(t-s)u^\rho_{xx}(s,x)\,ds \\ &\qquad + \rho g(t,x) + f(x), \qquad (t,x) \in \mathbf{R}^+ \times [0,1], \\ u^\rho(t,0) &= u^\rho(t,1) = 0, \qquad t \in \mathbf{R}^+, \\ u^\rho(t,x) &= v^\rho(t,x), \qquad (t,x) \in \mathbf{R}^- \times [0,1].\end{aligned}$$

Here u is the displacement, ρg is the body force, f is the external force, $\rho > 0$ is the density of the material and $\mathbf{R}^+ = [0,\infty)$, $\mathbf{R}^- = (-\infty, 0]$.

1980 Mathematics subject classification (1985 Revision): 35B25
Received by the editors on March 3, 1993.
The research of the first author was partially supported by NSF Grant DMS-8906840.

The singular perturbation in such a case amounts to examining the behavior of the solutions of (1.1) when the density $\rho \to 0$.

There are many studies concerning singular perturbations. For example, Fattorini [**5**] studied the abstract equation

$$\begin{aligned} \rho^2 u^\rho_{tt}(t) + u^\rho_t(t) = Au^\rho(t) + f^\rho(t), \qquad t \geq 0, \\ u^\rho(0) = u^\rho_0, \quad u^\rho_t(0) = u^\rho_1, \end{aligned} \tag{1.2}$$

and

$$w_t(t) = Aw(t) + f(t), \qquad t \geq 0, \qquad w(0) = w_0, \tag{1.2$'$}$$

in a Banach space $\mathbf{U}$ with A the generator of a strongly continuous cosine family and a C_0 semigroup. He proved that for any $T > 0$, if $f^\rho \to f$ in $L^1([0,T], \mathbf{U})$ and $u^\rho_0 \to w_0$, $\rho^2 u^\rho_1 \to 0$ as $\rho \to 0$, then $u^\rho(t) \to w(t)$ uniformly for t in $[0,T]$ as $\rho \to 0$, where u^ρ and w are solutions of (1.2) and (1.2)$'$, respectively.

This problem is also related to the quasi-static approximation in viscoelasticity. For example, MacCamy [**15**] studied

$$u_{tt}(t) = -A(0)g(u(t)) - \int_0^t A'(t-s)g(u(s))\,ds + F(t), \tag{1.3}$$

and

$$0 = -A(0)g(w(t)) - \int_0^t A'(t-s)g(w(s))\,ds + F(t), \tag{1.3$'$}$$

in a Hilbert space $\mathbf{H}$ with $A(t)$ a bounded linear operator, g a nonlinear and unbounded operator, and proved that if $F(t) \to$ constant vector $F(\infty)$ as $t \to \infty$, then, under appropriate conditions, one has

$$\begin{aligned} g(u(t)) \to A(\infty)^{-1}F(\infty) \quad \text{weakly in } \mathbf{H}, \qquad t \to \infty, \\ g(w(t)) \to A(\infty)^{-1}F(\infty) \quad \text{in } \mathbf{H}, \qquad t \to \infty, \end{aligned}$$

where u and w are solutions of (1.3) and (1.3)$'$, respectively. This result motivates the procedure of using the quasi-static approximation in viscoelasticity which drops the "acceleration" term u_{tt} when t is large.

Note that the existence and uniqueness of solutions of (1.1) has been obtained in [**1, 3, 6, 9, 11, 13, 14**], and we are only interested in singular perturbations in this paper, so we will assume that (1.1) has a unique solution u^ρ for each $\rho > 0$. For simplicity, we also assume that $E = 1$.

Next, assume that $f \in C[0,1]$ and $1 + \int_0^\infty a(s)\,ds \neq 0$ and define

$$w(x) \equiv -\left[1 + \int_0^\infty a(s)\,ds\right]^{-1} A^{-1}(f)(x), \qquad x \in [0,1], \tag{1.4}$$

where

$$A = \partial^2/\partial x^2, \qquad \mathbf{D}(A) = \{y \in C^2[0,1] : y(0) = y(1) = 0\}.$$

It will be proved that w is the unique solution on $\mathbf{R}$ of (1.1) with $\rho = 0$. By linearity we see that $u^\rho - w$ satisfies equation (1.1) with $f = 0$. Using an energy estimate and a differential inequality, we will show that if for $s \leq 0$, $\rho > 0$, $t \in \mathbf{R}$ and $L^2 \equiv L^2([0,1], \mathbf{R})$,

$$\begin{gathered} ||v^\rho(0,\cdot)||_{L^2}, ||v_t^\rho(s,\cdot)||_{L^2}, \frac{1}{\rho}||v_x^\rho(0,\cdot) - w_x(\cdot)||_{L^2}, \\ ||g(t,\cdot)||_{L^2} \leq \text{constant}, \end{gathered} \tag{1.5}$$

then, for every $T > 0$,

$$u^\rho(t,\cdot) \to w(\cdot) \quad \text{in } L^2([0,1], \mathbf{R}) \qquad \text{as } \rho \to 0 \tag{1.6}$$

uniformly for $t \in [0, T]$.

Next, we consider the same problem for a nonlinear equation

$$\begin{aligned} \rho u_{tt}^\rho(t,x) &= \phi(u_x^\rho(t,x))_x \\ &\quad + \int_{-\infty}^t a(t-s)\phi(u_x^\rho(s,x))_x\,ds + \rho g(t,x), \\ &\quad (t,x) \in \mathbf{R}^+ \times [0,1], \\ u^\rho(t,0) &= u^\rho(t,1) = 0, \qquad t \in \mathbf{R}^+, \\ u^\rho(t,x) &= v^\rho(t,x), \qquad (t,x) \in \mathbf{R}^- \times [0,1], \end{aligned} \tag{1.7}$$

where ϕ is nonlinear, $\phi(0) = 0$, there is $\varepsilon > 0$, $\phi' \geq \varepsilon$ on $\mathbf{R}$.

We will prove that zero is the unique solution on $\mathbf{R}$ of (1.7) with $\rho = 0$ and that if

$$||v^\rho(0,\cdot)||_{L^2}, \quad ||v_t^\rho(s,\cdot)||_{L^2}, \quad \frac{2}{\rho}\int_0^1 \int_0^{v_x^\rho(0,x)} \phi(\tau)\,d\tau\,dx,$$
$$||g(t,\cdot)||_{L^2} \leq \text{constant}, \qquad s \leq 0, \rho > 0, t \in \mathbf{R},$$

then for any $T > 0$, solutions u^ρ of (1.7) ($\rho > 0$) go to zero in L^2 uniformly for $t \in [0, T]$ as $\rho \to 0$.

This approach uses the assumption that the history v^ρ satisfies equation (1.1) (respectively (1.7)) on $\mathbf{R}^-$, see [**11, 13, 14**]. Thus (1.1) (respectively (1.7)) holds on $\mathbf{R}$ and hence we can solve u_{xx} from (1.1) (respectively (1.7)) in terms of $\rho(u_{tt} - g)$ as in [**13, 14**]. This step is very important in obtaining the energy estimate. If we only assume that v^ρ is a known function on $\mathbf{R}^-$, (which may not satisfy (1.1) (respectively (1.7)) on $\mathbf{R}^-$), then with additional conditions on v_{xx}^ρ, we are able to modify the proofs to get the same results.

2. Singular perturbations of equation (1.1) and (1.7). In this section we examine the behavior of the solutions of equations (1.1) and (1.7) when $\rho \to 0$. To this end, we introduce the following assumptions:

$$1 + \hat{a}(\lambda) \neq 0 \qquad \text{for } \operatorname{Re}\lambda \geq 0,$$
$$a \text{ and } a' \in L^1(\mathbf{R}^+, \mathbf{R}), \qquad f \in C[0,1], \tag{2.1}$$

where $\hat{a}$ is the Laplace transform of a.

$$a \in L^1(\mathbf{R}^+, \mathbf{R}) \quad \text{and} \quad \int_0^\infty a(s)\,ds \neq -1, \qquad f \in C[0,1]. \tag{2.1$'$}$$

Now we can state and prove

Theorem 2.1. *Assume that equation* (1.1) *has a unique solution* u^ρ *on* $\mathbf{R}$ *for each* $\rho > 0$ *with* v^ρ *satisfying* (1.1) *on* $\mathbf{R}^-$. *Also, let assumption* (2.1) *be satisfied and let* w *be defined by* (1.4). *In addition, assume that there is a constant* G *such that*

$$||v^\rho(0,\cdot)||_{L^2}, \quad ||v_t^\rho(s,\cdot)||_{L^2}, \quad \frac{1}{\rho}||v_x^\rho(0,\cdot) - w_x(\cdot)||_{L^2},$$
$$||g(t,\cdot)||_{L^2} \leq G, \qquad s \leq 0, \rho > 0, t \in \mathbf{R}. \tag{2.2}$$

Then w is the unique solution on $\mathbf{R}$ of (1.1) *with $\rho = 0$, and for every $T > 0$,*

$$u^\rho(t,\cdot) \to w(\cdot) \quad \textit{in } L^2([0,1],\mathbf{R}) \qquad \textit{as } \rho \to 0, \tag{2.3}$$

uniformly for $t \in [0,T]$.

Proof. Consider the resolvent kernel $r(t)$ defined by

$$\begin{gathered} r(t) = -a(t) - \int_0^t a(t-s)r(s)\,ds \in \mathbf{R}, \qquad t \geq 0; \\ r(t) = 0, \qquad t < 0. \end{gathered} \tag{2.4}$$

We can write this as

$$(\delta + r) * (\delta + a) = \delta,$$

where $*$ is the convolution on $\mathbf{R}$ and $\delta * f = f$. Now (2.1) implies that $r \in L^1(\mathbf{R}^+,\mathbf{R})$ [**16**]. Next, $a' \in L^1(\mathbf{R}^+,\mathbf{R})$, so by taking a derivative in (2.4) we obtain $r' \in L^1(\mathbf{R}^+,\mathbf{R})$.

Observe that w defined by (1.4) is a solution on $\mathbf{R}$ of (1.1) with $\rho = 0$. Next, assume that y satisfies (1.1) on $\mathbf{R}$ with $\rho = 0$. Then, using the above convolution notation, we have $0 = (\delta + a) * y_{xx} + f$. Thus, $y_{xx} = -(\delta + r) * f$ and hence $y = -A^{-1}(\delta + r) * f$. Thus $w = y$ and hence w is the unique solution on $\mathbf{R}$ of (1.1) with $\rho = 0$. Next, define

$$Q^\rho(t,x) \equiv u^\rho(t,x) - w(x), \qquad (t,x) \in \mathbf{R} \times [0,1],$$

then Q^ρ satisfies

$$\begin{aligned} &\rho Q^\rho_{tt}(t,x) = Q^\rho_{xx}(t,x) + \int_{-\infty}^t a(t-s)Q^\rho_{xx}(s,x)\,ds + \rho g(t,x), \\ &\qquad\qquad (t,x) \in \mathbf{R}^+ \times [0,1], \\ &Q^\rho(t,0) = Q^\rho(t,1) = 0, \qquad t \in \mathbf{R}^+, \\ &Q^\rho(t,x) = v^\rho(t,x) - w(x), \qquad (t,x) \in \mathbf{R}^- \times [0,1]. \end{aligned} \tag{2.5}$$

In the following we will use the energy method to show that

$$Q^\rho(t,\cdot) \to 0 \quad \text{in } L^2([0,1],\mathbf{R}) \qquad \text{as } \rho \to 0, \tag{2.6}$$

uniformly for $t \in [0, T]$. Define the energy of (2.5) as

$$E^\rho(t) \equiv \int_0^1 (Q^\rho(t,x))^2\,dx + \int_0^1 (Q_t^\rho(t,x))^2\,dx + \frac{1}{\rho}\int_0^1 (Q_x^\rho(t,x))^2\,dx.$$

Then

$$(||Q^\rho(t)||_{L^2})^2, (||Q_t^\rho(t)||_{L^2})^2 \le E^\rho(t), \qquad t \ge 0, 0 < \rho.$$

$$(||Q_x^\rho(t)||_{L^2})^2 \le \frac{1}{\rho}(||Q_x^\rho(t)||_{L^2})^2 \le E^\rho(t), \quad t \ge 0, 0 < \rho < 1.$$

Observe that (2.2) implies there is a constant $J > G$ such that

$$||Q^\rho(0)||_{L^2}, ||Q_t^\rho(s)||_{L^2}, \frac{1}{\rho}||Q_x^\rho(0)||_{L^2} \le J, \quad s \le 0, \rho > 0. \tag{2.7}$$

Then $E^\rho(0) \le J^2 + J^2 + J^2 = 3J^2$, $0 < \rho$. Next, we have

$$\begin{aligned} \frac{d}{dt}E^\rho(t) &= 2\int_0^1 Q^\rho(t,x)Q_t^\rho(t,x)\,dx \\ &\quad + 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\ &\quad + \frac{2}{\rho}\int_0^1 Q_x^\rho(t,x)Q_{xt}^\rho(t,x)\,dx \\ &= 2\int_0^1 Q^\rho(t,x)Q_t^\rho(t,x)\,dx \\ &\quad + 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\ &\quad - \frac{2}{\rho}\int_0^1 Q_t^\rho(t,x)Q_{xx}^\rho(t,x)\,dx \\ &\le E^\rho(t) + 2\int_0^1 Q_t^\rho(t,x)Q_{tt}^\rho(t,x)\,dx \\ &\quad - \frac{2}{\rho}\int_0^1 Q_t^\rho(t,x)Q_{xx}^\rho(t,x)\,dx. \end{aligned} \tag{2.8}$$

Now we solve Q^ρ_{xx} from (2.5) in terms of $\rho(Q^\rho_{tt} - g)$. Observe that a and $a' \in L^1(\mathbf{R}^+, \mathbf{R})$ imply that $a(+\infty) = 0$, hence [**16**] one has

$$r(+\infty) = -a(+\infty)\left(1 + \int_0^\infty a(s)\, ds\right)^{-1} = 0.$$

Since Q^ρ satisfies (2.5) on $\mathbf{R}$, we may write (2.5) as

$$\rho(Q^\rho_{tt} - g)(t) = (\delta + a) * Q^\rho_{xx}(t), \qquad t \in \mathbf{R}.$$

Then we obtain

$$\begin{aligned} Q^\rho_{xx}(t) &= (\delta + r) * (\rho(Q^\rho_{tt} - g))(t) \\ &= \rho\bigg(Q^\rho_{tt}(t) - g(t) + \int_{-\infty}^t r(t-s)(Q^\rho_{tt}(s) - g(s))\, ds\bigg) \\ &= \rho\bigg(Q^\rho_{tt}(t) - g(t) + r(0)Q^\rho_t(t) \\ &\qquad + \int_{-\infty}^t r'(t-s)Q^\rho_t(s)\, ds - \int_{-\infty}^t r(t-s)g(s)\, ds\bigg). \end{aligned} \tag{2.9}$$

Replacing this into (2.8), we have for $t \geq 0$, $\rho > 0$,

$$\begin{aligned} \frac{d}{dt}E^\rho(t) &\leq E^\rho(t) + 2\int_0^1 Q^\rho_t(t,x)Q^\rho_{tt}(t,x)\, dx \\ &\quad - 2\int_0^1 Q^\rho_t(t,x)\bigg\{Q^\rho_{tt}(t,x) - g(t,x) + r(0)Q^\rho_t(t,x) \\ &\qquad + \int_{-\infty}^t r'(t-s)Q^\rho_t(s,x)\, ds \\ &\qquad - \int_{-\infty}^t r(t-s)g(s,x)\, ds\bigg\}\, dx \\ &= E^\rho(t) + 2\int_0^1 Q^\rho_t(t,x)g(t,x)\, dx - 2r(0)\int_0^1 (Q^\rho_t(t,x))^2\, dx \\ &\quad - 2\int_0^1 Q^\rho_t(t,x)\int_{-\infty}^t r'(t-s)Q^\rho_t(s,x)\, ds\, dx \\ &\quad + 2\int_0^1 Q^\rho_t(t,x)\int_{-\infty}^t r(t-s)g(s,x)\, ds\, dx \end{aligned}$$

$$
\begin{aligned}
&\le E^\rho(t) + (1+2|r(0)|)(||Q_t^\rho(t)||_{L^2})^2 + (||g(t)||_{L^2})^2 \\
&\quad + \int_{-\infty}^t |r'(t-s)|[(||Q_t^\rho(t)||_{L^2})^2 + (||Q_t^\rho(s)||_{L^2})^2]\,ds \\
&\quad + \int_{-\infty}^t |r(t-s)|[(||Q_t^\rho(t)||_{L^2})^2 + (||g(s)||_{L^2})^2]\,ds \\
&= E^\rho(t) + (||g(t)||_{L^2})^2 + \int_{-\infty}^t |r(t-s)|(||g(s)||_{L^2})^2\,ds \\
&\quad + \left[1 + 2|r(0)| + \int_0^\infty (|r'(s)| + |r(s)|)\,ds\right](||Q_t^\rho(t)||_{L^2})^2 \\
&\quad + \int_{-\infty}^t |r'(t-s)|(||Q_t^\rho(s)||_{L^2})^2\,ds \\
&\le \left[2 + 2|r(0)| + \int_0^\infty (|r'(s)| + |r(s)|)\,ds\right]E^\rho(t) \\
&\quad + \int_0^t |r'(t-s)|E^\rho(s)\,ds + \int_{-\infty}^0 |r'(t-s)|(||Q_t^\rho(s)||_{L^2})^2\,ds \\
&\quad + \left(1 + \int_0^\infty |r(s)|\,ds\right)J^2 \\
&\le \left[2 + 2|r(0)| + \int_0^\infty (|r'(s)| + |r(s)|)\,ds\right]E^\rho(t) \\
&\quad + \int_0^t |r'(t-s)|E^\rho(s)\,ds + \left[1 + \int_0^\infty (|r(s)| + |r'(s)|)\,ds\right]J^2 \\
&\equiv HE^\rho(t) + \int_0^t |r'(t-s)|E^\rho(s)\,ds + K,
\end{aligned}
$$

where H and K are constants defined in an obvious way.

Now consider

$$
\begin{aligned}
y'(t) &= Hy(t) + \int_0^t |r'(t-s)|y(s)\,ds + K \in \mathbf{R}, \qquad t \ge 0, \\
y(0) &= 3J^2.
\end{aligned} \tag{2.10}
$$

And, for $n \ge 1$,

$$
\begin{aligned}
y'(t) &= Hy(t) + \int_0^t |r'(t-s)|y(s)\,ds + K + 1/n, \qquad t \ge 0, \\
y(0) &= 3J^2 + 1/n.
\end{aligned} \tag{2.10$_n$}
$$

Using standard methods of ordinary differential equations, [**12**], one can prove that (2.10) and $(2.10)_n$ have unique solutions y and y_n on $\mathbf{R}^+$, respectively, and

$$E^\rho(t) < y_n(t), \qquad t \geq 0,\ n \geq 1,\ \rho > 0. \tag{2.11}$$

$$y_n(t) \to y(t), \quad n \to \infty, \text{ uniformly on compact sets of } \mathbf{R}^+. \tag{2.12}$$

Therefore,

$$E^\rho(t) \leq y(t), \qquad t \geq 0,\ \rho > 0.$$

Since J, H, and K are independent of ρ, so is y. Thus, for any $T > 0$, there is a constant $C = C(T) > J$ such that

$$E^\rho(t) \leq C, \qquad t \in [0,T],\ \rho > 0.$$

Therefore, for $t \in [0,T]$ and $0 < \rho < 1$,

$$(||Q^\rho(t)||_{L^2})^2, (||Q_t^\rho(t)||_{L^2})^2, (||Q_x^\rho(t)||_{L^2})^2 \leq C. \tag{2.13}$$

This means that

$$\{Q^\rho(\cdot)|_{[0,T]}\}_{0<\rho<1} \subseteq C([0,T], L^2)$$

is equicontinuous and, for any $t_0 \in [0,T]$,

$$\{Q^\rho(t_0)\}_{0<\rho<1} \subseteq H_0^1([0,1], \mathbf{R})$$

is bounded. Next the embedding $H_0^1([0,1],\mathbf{R}) \to L^2([0,1],\mathbf{R})$ is compact so that

$$\{Q^\rho(t_0)\}_{0<\rho<1} \subseteq L^2([0,1], \mathbf{R})$$

is precompact.

Now we can apply the Arzela-Ascoli theorem [**2**] to conclude that there are $\rho_k \to 0$ as $k \to \infty$ and $v \in C([0,T], L^2)$ such that

$$Q^{\rho_k}(\cdot)|_{[0,T]} \to v \quad \text{in } C([0,T], L^2) \qquad \text{as } k \to \infty. \tag{2.14}$$

Next we prove that $v(t) = 0$ in $L^2([0,1], \mathbf{R})$, $t \in [0,T]$.

Note that the boundary condition in (2.5) implies that

$$||Q^\rho(t)||_{L^2} \le ||Q^\rho_x(t)||_{L^2}, \tag{2.15}$$

and

$$\int_0^1 (Q^\rho_x(t,x))^2\,dx = -\int_0^1 Q^\rho(t,x)Q^\rho_{xx}(t,x)\,dx.$$

So, for $\tau \in [0,T]$ fixed, we have from (2.15) and (2.9),

(2.16)

$$\begin{aligned}\int_0^\tau (||Q^\rho(t)||_{L^2})^2\,dt &\le \int_0^\tau (||Q^\rho_x(t)||_{L^2})^2\,dt \\ &= \left|\int_0^\tau\int_0^1 Q^\rho(t,x)Q^\rho_{xx}(t,x)\,dx\,dt\right| \\ &= \rho\left|\int_0^\tau\int_0^1 Q^\rho(t,x)\bigg[Q^\rho_{tt}(t,x) - g(t,x)\right. \\ &\qquad + r(0)Q^\rho_t(t,x) + \int_{-\infty}^t r'(t-s)Q^\rho_t(s,x)\,ds \\ &\qquad \left. - \int_{-\infty}^t r(t-s)g(s,x)\,ds\bigg]\,dx\,dt\right|.\end{aligned}$$

Next, from (2.13), one has

$$\begin{aligned}\left|\int_0^1\int_0^\tau Q^\rho(t,x)Q^\rho_{tt}(t,x)\,dt\,dx\right| &= \left|\int_0^1 \bigg[Q^\rho(\tau,x)Q^\rho_t(\tau,x)\right. \\ &\qquad - Q^\rho(0,x)Q^\rho_t(0,x) \\ &\qquad \left. - \int_0^\tau (Q^\rho_t(t,x))^2\,dt\bigg]\,dx\right| \\ &\le (||Q^\rho(\tau)||_{L^2})^2 + (||Q^\rho_t(\tau)||_{L^2})^2 \\ &\quad + (||Q^\rho(0)||_{L^2})^2 + (||Q^\rho_t(0)||_{L^2})^2 \\ &\quad + \int_0^\tau\int_0^1 (Q^\rho_t(t,x))^2\,dx\,dt \\ &\le (4+\tau)C,\end{aligned}$$

$$\begin{aligned}\left|\int_0^\tau\int_0^1 Q^\rho(t,x)g(t,x)\,dx\,dt\right| &\le \int_0^\tau \bigg[\Big((||Q^\rho(t)||_{L^2})^2 + (||g(t)||_{L^2})^2\bigg]\,dt \\ &\le 2\tau C,\end{aligned}$$

$$\begin{aligned}\Big|\int_0^\tau \int_0^1 r(0)Q^\rho(t,x)Q_t^\rho(t,x)\,dx\,dt\Big| \\ &\le |r(0)|\int_0^\tau \Big[\Big((\|Q^\rho(t)\|_{L^2})^2 + (\|Q_t^\rho(t)\|_{L^2})^2\Big]\,dt \\ &\le 2\tau|r(0)|C,\end{aligned}$$

$$\begin{aligned}\Big|\int_0^\tau \int_0^1 Q^\rho(t,x)\int_{-\infty}^t r'(t-s)Q_t^\rho(s,x)\,ds\,dx\,dt\Big| \\ &\le \int_0^\tau \int_{-\infty}^t |r'(t-s)|[(\|Q^\rho(t)\|_{L^2})^2 + (\|Q_t^\rho(s)\|_{L^2})^2]\,ds\,dt \\ &\le 2\tau C\int_0^\infty |r'(s)|\,ds,\end{aligned}$$

$$\begin{aligned}\Big|\int_0^\tau \int_0^1 Q^\rho(t,x)\int_{-\infty}^t r(t-s)g(s,x)\,dx\,dt\Big| \\ &\le \int_0^\tau \int_{-\infty}^t |r(t-s)|[(\|Q^\rho(t)\|_{L^2})^2 + (\|g(s)\|_{L^2})^2]\,ds\,dt \\ &\le 2\tau C\int_0^\infty |r(s)|\,ds.\end{aligned}$$

Now let $\rho = \rho_k$ in (2.16) and let $k \to \infty$; by (2.14) and the above estimates we have

$$\int_0^\tau (\|v(t)\|_{L^2})^2\,dt = 0.$$

Note that this is true for arbitrary $\tau \in [0,T]$; thus $\|v(t)\|_{L^2} = 0$, a.e., on $[0,T]$. But $\|v(t)\|_{L^2}$ is continuous in t, so $\|v(t)\|_{L^2} = 0$ for $t \in [0,T]$. Since (2.14) with $v \equiv 0$ is true for every sequence $\rho_k \to 0$, it is true for $\rho \to 0$. This completes the proof. □

Next let us consider the singular perturbation for

$$\begin{aligned}\rho u_{tt}^\rho(t,x) &= \phi(u_x^\rho(t,x))_x + \int_{-\infty}^t a(t-s)\phi(u_x^\rho(s,x))_x\,ds \\ &\quad + \rho g(t,x), \qquad (t,x)\in \mathbf{R}^+\times[0,1], \\ u^\rho(t,0) &= u^\rho(t,1) = 0, \qquad t\in\mathbf{R}^+, \\ u^\rho(t,x) &= v^\rho(t,x), \qquad (t,x)\in\mathbf{R}^-\times[0,1],\end{aligned} \tag{2.17}$$

where ϕ is nonlinear and satisfies

$$\phi(0) = 0, \quad \text{there is } \varepsilon > 0, \phi' > \varepsilon \quad \text{on } \mathbf{R}. \tag{2.18}$$

We recall that by assuming that v^ρ satisfies (2.17) on $\mathbf{R}^-$, [**11, 13, 14**], proved the existence and uniqueness of solutions for (2.17). So we may assume that (2.17) has a unique solution u^ρ on $\mathbf{R}$ for each $\rho > 0$.

In order to use the energy estimate, we define

$$\begin{aligned} E^\rho(t) \equiv \int_0^1 (u^\rho(t,x))^2\,dx + \int_0^1 (u_t^\rho(t,x))^2 \\ + \frac{2}{\rho}\int_0^1 \int_0^{u_x^\rho(t,x)} \phi(\tau)\,d\tau\,dx. \end{aligned}$$

Then, for $t \geq 0$ and $0 < \rho < 1$, one has from (2.18), (see [**13**]),

$$\begin{aligned} \int_0^1 (u_x^\rho(t,x))^2\,dx &\leq \frac{2}{\varepsilon}\int_0^1 \int_0^{u_x^\rho(t,x)} \phi(\tau)\,d\tau\,dx \\ &\leq \frac{1}{\varepsilon}\frac{2}{\rho}\int_0^1 \int_0^{u_x^\rho(t,x)} \phi(\tau)\,d\tau\,dx \\ &\leq \frac{1}{\varepsilon}E^\rho(t), \qquad t \geq 0, 0 < \rho < 1. \end{aligned}$$

Next, one has

$$\begin{aligned} \int_0^1 (u_x^\rho(t,x))^2\,dx &\leq \frac{1}{\varepsilon}\int_0^1 \phi(u_x^\rho(t,x))_x u_x^\rho(t,x)\,dx \\ &= -\frac{1}{\varepsilon}\int_0^1 u^\rho(t,x)\phi(u_x^\rho(t,x))_x\,dx. \end{aligned}$$

Similar to (2.9), one obtains

$$\begin{aligned} \phi(u_x^\rho(t))_x = \rho\Big[& u_{tt}^\rho(t) - g(t) + r(0)u_t^\rho(t) \\ & - \int_{-\infty}^t r(t-s)g(s)\,ds + \int_{-\infty}^t r'(t-s)u_t^\rho(s)\,ds\Big]. \end{aligned}$$

Therefore, with the same proof as in Theorem 2.1, we have

Theorem 2.2. *Assume that equation* (2.17) *has a unique solution* u^ρ *on* $\mathbf{R}$ *for each* $\rho > 0$ *with* v^ρ *satisfying* (2.17) *on* $\mathbf{R}^-$. *Also, let assumption* (2.1) *be satisfied. In addition, assume that there is a constant* G *such that for* $s \le 0$, $\rho > 0$, $t \in \mathbf{R}$,

$$||v^\rho(0,\cdot)||_{L^2}, ||v_t^\rho(s,\cdot)||_{L^2}, \frac{2}{\rho}\int_0^1 \int_0^{v_x^\rho(0,x)} \phi(\tau)\,d\tau\,dx, \\ ||g(t,\cdot)||_{L^2} \le G.$$

Then zero is the unique solution on $\mathbf{R}$ *of* (2.17) *with* $\rho = 0$, *and for every* $T > 0$,

$$u^\rho(t,\cdot) \to 0 \quad \text{in } L^2([0,1],\mathbf{R}) \qquad \text{as } \rho \to 0,$$

uniformly for $t \in [0,T]$.

Theorem 2.1 was proved under the assumption that v^ρ satisfies (1.1) on $\mathbf{R}^-$. If we only assume that v^ρ is known on $\mathbf{R}^-$, (which may not satisfy (1.1) on $\mathbf{R}^-$), then with additional conditions on the history v_{xx}^ρ we can write equation (1.1) as

$$\begin{aligned}\rho u_{tt}^\rho(t,x) &= u_{xx}^\rho(t,x) + \int_0^t a(t-s)u_{xx}^\rho(s,x)\,ds \\ &\qquad + \int_{-\infty}^0 a(t-s)v_{xx}^\rho(s,x)\,ds + \rho g(t,x) + f(x) \\ &= (\delta + a)\hat{*}u_{xx}^\rho(t,x) \\ &\qquad + \int_{-\infty}^0 a(t-s)v_{xx}^\rho(s,x)\,ds + \rho g(t,x) + f(x),\end{aligned}$$

and hence

$$u_{xx}^\rho(t) = (\delta + r)\hat{*}\bigg(\rho(u_{tt}^\rho - g) - f - \int_{-\infty}^0 a(t-s)v_{xx}^\rho(s)\,ds\bigg)(t),$$

where, in this case, the integral in convolution $\hat{*}$ is from 0 to t. With essentially the same proof as in Theorem 2.1, we have

Theorem 2.3. *Assume that equation* (1.1) *has a unique solution* u^ρ *on* $\mathbf{R}^+$ *for each* $\rho > 0$ *with* v^ρ *given on* $\mathbf{R}^-$. *Also, let assumption* $(2.1)'$

be satisfied and let w be defined by (1.4)*. In addition, assume that there is a constant G such that*

$$(2.2)' \qquad \begin{aligned} &||v^\rho(0,\cdot)||_{L^2}, ||v_t^\rho(0,\cdot)||_{L^2}, \frac{1}{\rho}||v_x^\rho(0,\cdot)-w_x(\cdot)||_{L^2}, \\ &\frac{1}{\rho}||v_{xx}^\rho(s,\cdot)-w_{xx}(\cdot)||_{L^2} \le G, \qquad s\le 0, \rho>0. \end{aligned}$$

Then w is the unique solution on $\mathbf{R}$ *of* (1.1) *with $\rho = 0$, and for every $T > 0$,*

$$u^\rho(t,\cdot)\to w(\cdot) \quad in\ L^2([0,1],\mathbf{R}) \qquad as\ \rho\to 0,$$

uniformly for $t \in [0,T]$.

For equation (2.17) we can do similar things and get

Theorem 2.4. *Assume that* (2.17) *has a unique solution u^ρ on* $\mathbf{R}^+$ *for each $\rho > 0$ with v^ρ given on* $\mathbf{R}^-$*. Also, let assumption* (2.1)$'$ *be satisfied. In addition, assume that there is a constant G such that*

$$\begin{gathered} ||v^\rho(0,\cdot)||_{L^2}, ||v_t^\rho(0,\cdot)||_{L^2}, \frac{2}{\rho}\int_0^1\int_0^{v_x^\rho(0,x)}\phi(\tau)\,d\tau\,dx, \\ \frac{1}{\rho}||\phi(v_x^\rho(s,\cdot)_x||_{L^2} \le G, \qquad s\le 0, \rho>0. \end{gathered}$$

Then zero is the unique solution on $\mathbf{R}$ *of* (2.17) *with $\rho = 0$, and for every $T > 0$,*

$$u^\rho(t,\cdot)\to 0 \quad in\ L^2([0,1],\mathbf{R}) \qquad as\ \rho\to 0,$$

uniformly for $t \in [0,T]$.

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