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COUPLED ELASTIC AND VISCOELASTIC RODS

WOLFGANG DESCH AND RONALD GRIMMER

Dedicated to Paul Waltman on the occasion of his 60th birthday

ABSTRACT. We examine the spectrum for equations for longitudinal vibrations in coupled elastic and viscoelastic rods. A fractional derivative model is used for the viscoelastic rod. We show that, except in an exceptional case, the spectrum asymptotically decomposes into two sets corresponding to the elastic and viscoelastic parts, respectively. Thus, the model can be said to decouple.

1. Preliminaries. In this paper we formulate a linear model for coupled elastic and viscoelastic rods and examine the spectrum of this model. The model is the standard one for a one-dimensional elastic rod, while we employ a fractional derivative type model for the viscoelastic rod. Fractional derivative models for viscoelastic materials have been used frequently in recent years in a variety of studies. A representative list of references includes [**1**–**6**]. We note particularly the paper [**4**] by Desch and Miller where they examine equations with singular kernels. They show that a fractional derivative model for a viscoelastic rod can be written in the more usual form of a partial differential integral equation with a singular kernel. In particular, there are two types of kernels possible. One which we refer to as "strong" while the other is "weak." This, of course, refers to the type of singularity of the kernel at the origin. The treatment of each of these types has similar motivation, but the details are sufficiently different to require that they be considered separately. In each case we will treat the problem as a perturbation problem and show that a characteristic equation has solutions in a prescribed region with the aid of Rouché's theorem.

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The equation of interest is given by

$$\begin{aligned} \rho_1 v_t &= \frac{\partial}{\partial x}\sigma, \\ \sigma &= \int_{-\infty}^{t} E_1 \frac{\partial}{\partial x} v(s,x)\,ds, \end{aligned} \qquad x<0, \tag{1.1$-$}$$

$$\begin{aligned} \rho_2 v_t &= \frac{\partial}{\partial x}\sigma, \\ \sigma &= \int_{-\infty}^{t} g(t-s) \frac{\partial}{\partial x} v(s,x)\,ds, \end{aligned} \qquad x>0. \tag{1.1+}$$

In addition, at the interface one has the continuity conditions

$$v(t,0-) = v(t,0+), \qquad \sigma(t,0-) = \sigma(t,0+).$$

At the boundary, we have the boundary conditions

$$v(t,\pm 1) = 0.$$

Here ρ_1, ρ_2, σ, and v are densities, stress and velocity, respectively. E_1 is Young's modulus for the elastic material and $g(t)$ is the relaxation function for the viscoelastic material.

If we now take Laplace transforms we obtain the equation

$$\begin{aligned} \rho_1 \lambda \hat{v} &= \frac{\partial}{\partial x}\hat{\sigma}, \\ \hat{\sigma} &= \frac{E_1}{\lambda}\frac{\partial}{\partial x}\hat{v}, \end{aligned} \qquad x<0, \tag{1.2$-$}$$

$$\begin{aligned} \rho_2 \lambda \hat{v} &= \frac{\partial}{\partial x}\hat{\sigma}, \\ \hat{\sigma} &= \hat{g}(\lambda)\frac{\partial}{\partial x}\hat{v}, \end{aligned} \qquad x>0, \tag{1.2+}$$

with the additional conditions

$$\hat{v}(\lambda,0-) = \hat{v}(\lambda,0+), \qquad \hat{\sigma}(\lambda,0-) = \hat{\sigma}(\lambda,0+) \tag{1.3}$$

and

$$\hat{v}(\lambda, \pm 1) = 0. \tag{1.4}$$

From (1.2) we are able to determine the characteristic equation which will be at the center of our investigations in this paper. First, we consider (1.2−) with $x < 0$. This yields the second order differential equation

$$(\rho_1/E_1)\lambda^2 \hat{v} = \frac{\partial^2}{\partial x^2}\hat{v}, \qquad \hat{v}(\lambda, -1) = 0.$$

This equation together with (1.2−) lead to the solutions for $\hat{v}$ and $\hat{\sigma}$,

$$\hat{v}(\lambda, x) = k \sinh(\sqrt{\rho_1/E_1}\lambda(x+1))$$

$$\begin{aligned} \hat{\sigma}(\lambda, x) &= k(E_1/\lambda)\cosh(\sqrt{\rho_1/E_1}\lambda(x+1))(\lambda\sqrt{\rho_1/E_1}) \\ &= k\sqrt{E_1\rho_1}\cosh(\sqrt{\rho_1/E_1}\lambda(x+1)). \end{aligned} \tag{1.5}$$

Evaluating $\hat{v}$ and $\hat{\sigma}$ at the interface, $x = 0$, and solving for k yields the relation

$$\hat{v}(\lambda, 0) = \frac{\hat{\sigma}(\lambda, 0)}{\sqrt{\rho_1 E_1}} \tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right). \tag{1.6}$$

Consideration of (1.2+) for $x > 0$ gives the second order equation

$$\rho_2 \lambda \hat{v} = \hat{g}(\lambda)\frac{\partial^2}{\partial x^2}\hat{v}, \qquad \hat{v}(\lambda, 1) = 0,$$

which together with (1.2+) yields

$$\begin{aligned} \hat{v}(\lambda, x) &= k \sinh(\sqrt{\rho_2\lambda/\hat{g}(\lambda)}(x-1)) \\ \hat{\sigma}(\lambda, x) &= k\sqrt{\rho_2\lambda\hat{g}(\lambda)}\cosh(\sqrt{\rho_2\lambda/\hat{g}(\lambda)}(x-1)). \end{aligned} \tag{1.7}$$

Evaluating $\hat{v}$ and $\hat{\sigma}$ at the interface $x = 0$ and proceeding as above now yields

$$\hat{v}(\lambda, 0) = -\frac{\hat{\sigma}(\lambda, 0)}{\sqrt{\rho_2\lambda\hat{g}(\lambda)}} \tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right). \tag{1.8}$$

Finally, (1.6), (1.8) and continuity considerations at the interface $x = 0$ lead us to the characteristic equation

$$\frac{1}{\sqrt{\rho_1 E_1}} \tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) + \frac{1}{\sqrt{\rho_1 \lambda \hat{g}(\lambda)}} \tanh\left(\sqrt{\frac{\rho_2 \lambda}{\hat{g}(\lambda)}}\right) = 0, \tag{1.9}$$

or, equivalently,

$$\sqrt{\rho_1 E_1} \coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) + \sqrt{\rho_2 \lambda \hat{g}(\lambda)} \coth\left(\sqrt{\frac{\rho_2 \lambda}{\hat{g}(\lambda)}}\right) = 0. \tag{1.10}$$

As we mentioned earlier, we shall assume a fractional derivative model. Such models have a constitutive equation relating the stress σ and the strain ε of the form

$$\sigma(t) + \beta \frac{\partial^\nu}{\partial t^\nu} \sigma(t) = E\left(\varepsilon(t) + \alpha \frac{\partial^\nu}{\partial t^\nu} \varepsilon(t)\right), \tag{1.11}$$

with positive constants α, β, ν and E, with $0 \le \beta < \alpha$ and $0 < \nu < 1$. The fractional derivative is defined by the relation

$$\frac{\partial^\nu}{\partial t^\nu} \sigma(t) = \frac{1}{\Gamma(1-\nu)} \frac{\partial}{\partial t} \int_{-\infty}^{t} (t-s)^{-\nu} \sigma(s)\, ds. \tag{1.12}$$

In [**4**] it is noted that if $\beta = 0$, then

$$\begin{aligned} \sigma(t) &= E\varepsilon(t) + \frac{E\alpha}{\Gamma(1-\nu)} \int_{-\infty}^{t} (t-s)^{-\nu} \frac{\partial}{\partial t} \varepsilon(s)\, ds \\ &= \int_{-\infty}^{t} (E + E\alpha_1 (t-s)^{-\nu}) \frac{\partial}{\partial t} \varepsilon(s)\, ds. \end{aligned} \tag{1.13}$$

Here we have gathered the expression involving the gamma function and the α into α_1. It is this model which we call the *strong fractional derivative model*. If β is not zero the situation is a bit more complicated. In this case it is shown in [**4**] that

$$\sigma(t) = \int_{-\infty}^{t} \left(E + E\left(\frac{\alpha}{\beta} - 1\right) R(t-s)\right) \frac{\partial}{\partial s} \varepsilon(s)\, ds, \tag{1.14}$$

where $R(t)$ satisfies the equation

$$R(t) + \int_0^t \frac{1}{\beta\Gamma(\nu)}(t-s)^{\nu-1}R(s)\,ds = 1. \tag{1.15}$$

$R(t)$ is completely monotone and is bounded at the origin, but $R'(t)$ is unbounded near the origin. Because this case is "less singular" than the other, we call this the *weak fractional derivative model*. We shall see that there is a difference in the spectrum of the models (1.2) if g corresponds to the strong or weak fractional derivative model.

We shall first state the results for each of the fractional derivative models and then in the last section the proofs of the results will be presented.

2. The strong fractional derivative model. Let us now assume that the model is the strong fractional derivative model given by (1.13). That is, $\hat{g}$ is given by

$$\hat{g}(\lambda) = \frac{E_2}{\lambda} + \frac{\alpha E_2}{\lambda^\nu}, \qquad 0 < \nu < 1, \quad E_2, \alpha > 0. \tag{2.1}$$

We first note that if the rods were not coupled and instead of the interface condition (1.3), the condition

$$v(t,0) = 0 \quad \text{or} \quad \hat{v}(\lambda, 0) = 0 \tag{2.2}$$

was imposed on the elastic rod, we would obtain the characteristic equation

$$\frac{1}{\sqrt{\rho_1 E_1}} \tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = 0, \tag{2.3}$$

for the elastic rod. In addition, if for the viscoelastic rod a stress-free boundary condition

$$\hat{\sigma}(\lambda, 0) = 0 \tag{2.4}$$

were imposed, we would obtain from (1.2+) the characteristic equation

$$\sqrt{\rho_2 \lambda \hat{g}(\lambda)} \coth\left(\sqrt{\frac{\rho_2 \lambda}{\hat{g}(\lambda)}}\right) = 0, \tag{2.5}$$

for the viscoelastic rod.

The main result of this section states that the solutions λ of the characteristic equation (1.9)–(1.10) which have large magnitude are close to the solutions of the equations (2.3) and (2.5). In this sense, the model can be said to decouple into its component elastic and viscoelastic parts. One interpretation of this is that, because of the strong singularity, the viscoelastic material acts instantaneously like a rigid material at the interface, reflecting the vibrations from the elastic material directly back into the elastic material.

In order to clearly state our result concerning the *strong fractional derivative model* we first need an elementary lemma concerning the uncoupled model.

Because $\bar{\lambda}$ is a solution to the characteristic equation whenever λ is a solution, we may restrict our attention to λ with $\operatorname{Im}(\lambda) > 0$, and we shall usually do so in the future, often without further comment.

Lemma 2.1. *The characteristic equation for the uncoupled elastic rod,* (2.3)*, is satisfied by* λ_n *which satisfy*

$$\sqrt{\frac{\rho_1}{E_1}}\lambda_n = n\pi i. \tag{2.6}$$

The characteristic equation for the uncoupled viscoelastic rod, (2.5)*, is satisfied by* λ *for which*

$$\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}} = \left(\frac{2n+1}{2}\right)\pi i$$

or, equivalently,

$$\frac{\rho_2\lambda}{\hat{g}(\lambda)} = -\left(\frac{(2n+1)^2\pi^2}{4}\right). \tag{2.7}$$

If $\operatorname{Im}(\lambda) > 0$*, such* λ *satisfy* $\arg(\lambda) \to \pi/(1+\nu)$ *as* $|\lambda| \to \infty$.

Proof. As $\tanh(xi) = i\tan(x)$, (2.6) is readily apparent. Similarly, (2.7) is also apparent.

It is clear that the solutions of

$$\frac{\rho_2 \lambda^{1+\nu}}{\alpha E_2} + \left(\frac{(2n+1)^2 \pi^2}{4} \right) = 0$$

will occur along the line in the second quadrant with $\arg(\lambda) = \pi/(1+\nu)$. As $(\rho_2 \lambda / \hat{g}(\lambda)) \sim (\rho_2 \lambda^{1+\nu})/\alpha E_2$ the result follows. □

Our result for the *strong fractional derivative model* states that the spectrum is asymptotic to that of the decoupled model.

Theorem 2.2. *Assume the strong fractional derivative model given by* (1.13) *so that g satisfies* (2.1). *The solutions λ of the characteristic equation* (1.9) (*equivalently* (1.10)) *are asymptotic to the solutions of either* (2.6) *or* (2.7) *as $|\lambda| \to \infty$.*

3. The weak fractional derivative model. Now we consider the constitutive equation relating the stress σ and the strain ε

$$\sigma(t) + \beta \frac{\partial^\nu}{\partial t^\nu} \sigma(t) = E_2 \left(\varepsilon(t) + \alpha \frac{\partial^\nu}{\partial t^\nu} \varepsilon(t) \right), \tag{3.1}$$

when $0 < \beta < \alpha$. We recall from the discussion in the introduction that

$$\sigma(t) = \int_{-\infty}^{t} \left(E_2 + E_2 \left(\frac{\alpha}{\beta} - 1 \right) R(t-s) \right) \frac{\partial}{\partial t} \varepsilon(s)\, ds, \tag{3.2}$$

where $R(t)$ satisfies the equation

$$R(t) + \int_0^t \frac{1}{\beta \Gamma(\nu)} (t-s)^{\nu-1} R(s)\, ds = 1. \tag{3.3}$$

In this case, g satisfies

$$g = E_2 \left(1 + \left(\frac{\alpha}{\beta} - 1 \right) R \right).$$

A short calculation shows that the Laplace transform of g satisfies

$$\hat{g}(\lambda) = \frac{E_2}{\lambda} \left(\frac{1 + \alpha \lambda^\nu}{1 + \beta \lambda^\nu} \right). \tag{3.4}$$

We shall see that there are two essentially different cases in the weak fractional derivative model. These correspond to whether

$$\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2} \neq 1 \tag{3.5}$$

or

$$\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2} = 1. \tag{3.6}$$

The second case we call *impedance matching* and we shall show that in this case the spectrum is quite different than in the first case. In particular, the indication from the spectrum in this case is that vibrations from the elastic material strike the interface between the two materials and enter the viscoelastic material without a related wave of reflection. This is a feature of stress waves at the interface of two elastic media when the *characteristic impedance* of each material match. For elastic materials the characteristic impedance is defined to be the product of the mass density and the velocity. For a wave equation of the form (1.1−) the characteristic impedance would then be $\sqrt{\rho_1 E_1}$. We thus see that (3.6) is the statement of impedance matching if $\alpha = \beta = 1$ so that we are returned to the elastic case for $x > 0$. A discussion of these matters may be found in Kolsky [**7**].

Theorem 3.1. *Suppose that* (3.5) *is valid. That is, consider the case in which we do not have impedance matching. Then the spectrum is asymptotically composed of two sets. One set of solutions of the characteristic equation tends to a vertical line* $\lambda_0 + bi$ *in the complex plane as* $|\lambda| \to \infty$. *Here* $\lambda_0 < 0$ *is the solution of*

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_0\right) = -\sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}}$$

or

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_0\right) = -\sqrt{\frac{\alpha\rho_2 E_2}{\beta\rho_1 E_1}},$$

depending on whether $\beta\rho_1 E_1$ *is less than or greater than* $\alpha\rho_2 E_2$. *The other set of solutions of the characteristic equation tends asymptotically* *to a curve in the complex plane which is symmetric with respect to the real axis, and if* $\mathrm{Im}\,(\lambda) > 0$ *then* $\mathrm{Re}\,(\lambda) \to -\infty$ *and* $\mathrm{Im}\,(\lambda) \to \infty$ *and the argument of* λ *along the curve satisfies* $\arg(\lambda) \to \pi/2$.

Theorem 3.2. *Suppose that* (3.6) *is valid. That is, we have impedance matching. If* λ_n *is a sequence of solutions of the characteristic equation with* $|\lambda_n| \to \infty$, *then* $\mathrm{Re}\,(\lambda_n) \to -\infty$.

4. Proofs of the theorems.

Proof of Theorem 2.2. We shall proceed in three steps. First we shall show that as $n \to \infty$ there is a solution λ_n to (1.9) which is arbitrarily close to a solution of the uncoupled elastic rod given by (2.6). Then we shall show that as $n \to \infty$ there are solutions of (1.9) which are close to the solutions of (2.7) as $n \to \infty$. Finally, we shall show that there are no other solutions of (1.9).

As noted in Lemma 2.1, for λ which satisfies

$$\sqrt{\frac{\rho_1}{E_1}}\lambda = n\pi i,$$

we have

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = 0. \tag{4.1}$$

Now

$$\lambda\hat{g}(\lambda) = \lambda\left(\frac{E_2}{\lambda} + \frac{\alpha E_2}{\lambda^\nu}\right) = E_2 + E_2\alpha\lambda^{1-\nu}$$

and so $|\lambda\hat{g}(\lambda)| \to \infty$ if $|\lambda| \to \infty$. Thus,

$$\frac{1}{\sqrt{\lambda\hat{g}(\lambda)\rho_2}} \to 0 \qquad \text{as } |\lambda| \to \infty. \tag{4.2}$$

We wish to demonstrate that $\tanh(\sqrt{(\lambda\rho_2/\hat{g}(\lambda))})$ is bounded for $\lambda = r + is$ with $|r| \leq M < \infty$ and $|s| \to \infty$. First note that if z_1 and z_2 are complex with $(z_1/z_2) \to 1$ then $\arg(z_1) - \arg(z_2) \to 0$. Then, as $(\lambda\rho_2/\hat{g}(\lambda))/((\rho_2\lambda^{1+\nu})/\alpha E_2) \to 1$ when $\lambda = r + is$ with $|r| \leq M < \infty$ and $|s| \to \infty$, if follows that

$$\tanh(\sqrt{(\lambda\rho_2/\hat{g}(\lambda))}) \to 1 \tag{4.3}$$

as $\lambda^{1+\nu}$ has argument $\pi(1+\nu)/4$ in the limit and $0<\nu<1$. Now, utilizing (4.2) and (4.3) as well as the periodicity of $\tanh(a+bi)$ in b we see we can choose circles about the zeros of $\tanh(\sqrt{(\rho_1/E_1)}\lambda)$ on the imaginary axis with small radius so that on these circles

$$\left|\frac{1}{\sqrt{\rho_2\lambda\hat{g}(\lambda)}}\tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right)\right| < \left|\frac{1}{\sqrt{\rho_1 E_1}}\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right)\right|. \tag{4.4}$$

(In fact the radii of these circles may be chosen to be decreasing.) By Rouché's theorem, we know we can solve the characteristic equation in the interior of each of these circles if s is sufficiently large.

We now consider the viscoelastic part of the spectrum. Consider again the characteristic equation

$$\frac{1}{\sqrt{\rho_1 E_1}}\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = -\frac{1}{\sqrt{\rho_2\lambda\hat{g}(\lambda)}}\tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) \tag{4.5}$$

or

$$\sqrt{\rho_1 E_1}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = -\sqrt{\rho_2\lambda\hat{g}(\lambda)}\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right). \tag{4.6}$$

Now, recalling Lemma 2.1, we know that

$$\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) = 0 \tag{4.7}$$

whenever

$$\frac{\rho_2\lambda}{\hat{g}(\lambda)} = -\left(\frac{(2n+1)^2\pi^2}{4}\right)$$

and that such λ satisfy $\arg(\lambda)\to\pi/(1+\nu)$ as $|\lambda|\to\infty$.

Aroung such λ_n which satisfy (4.7) we wish to construct a circle so that on the boundary we have an estimate of the form

$$\left|\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right)\right| > \frac{\sqrt{\rho_1 E_1}}{|\sqrt{\rho_2\lambda\hat{g}(\lambda)}|}(1+\varepsilon) \tag{4.8}$$

with some $\varepsilon > 0$, in order to again invoke Rouché's theorem. Estimating, we see

$$
\begin{aligned}
\frac{\sqrt{\rho_1 E_1}}{|\sqrt{\rho_2 \lambda \hat{g}(\lambda)}|} &= \left|\sqrt{\frac{\rho_1 E_1}{\rho_2 E_2 (1+\alpha\lambda^{1-\nu})}}\right| \\
&= \sqrt{\frac{\rho_1 E_1}{\rho_2 E_2}} \left|\sqrt{\frac{1}{1+\alpha\lambda^{1-\nu}}}\right| \\
&\le \sqrt{\frac{2\rho_1 E_1}{\rho_2 E_2 \alpha}}\, |\lambda^{(\nu-1)/2}|,
\end{aligned}
\tag{4.9}
$$

if $|\lambda|^{1-\nu} > 1/(2\alpha)$.

The centers of the circles of interest are λ_n where

$$
\rho_2 \frac{\lambda_n^{1+\nu}}{E_2(\lambda_n^{\nu-1}+\alpha)} = -\left(\frac{(2n+1)^2\pi^2}{4}\right). \tag{4.10}
$$

As $n \to \infty$,

$$
\lambda_n^{1+\nu} \sim -\frac{\alpha E_2 \pi^2}{4\rho_2}(2n+1)^2
$$

or $|\lambda_n| \sim K(2n+1)^{2/(1+\nu)}$ where K is constant. Now write $z = \sqrt{(\lambda\rho_2/\hat{g}(\lambda))}$ and $z_n = \sqrt{(\lambda_n\rho_2/\hat{g}(\lambda_n))}$. From the Taylor series for $\coth(z)$, we obtain

$$
\coth(z) = -(z-z_n)/\sinh^2(z_n) + O((z-z_n)^2)
$$

if z is close to z_n. As λ_n is chosen so that $\coth(z_n) = 0$, we have $\cosh(z_n) = 0$ and $|\sinh(z_n)| = 1$. Thus,

$$
|\coth(z)| \ge |z-z_n|/2 \qquad \text{if } |z-z_n| \text{ is small.} \tag{4.11}
$$

Now

$$
\begin{aligned}
|z-z_n| &= \left|\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}} - \sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right| \\
&= \sqrt{\frac{\rho_2}{E_2}}\left|\sqrt{\frac{\lambda^2}{1+\alpha\lambda^{1-\nu}}} - \sqrt{\frac{\lambda_n^2}{1+\alpha\lambda_n^{1-\nu}}}\right.
\end{aligned}
\tag{4.12}
$$

Further,

$$
\begin{aligned}
(4.13)\qquad & \left|\sqrt{\frac{\lambda^2}{1+\alpha\lambda^{1-\nu}}}-\sqrt{\frac{\lambda_n^2}{1+\alpha\lambda_n^{1-\nu}}}\right| \\
&= \left|\lambda^{(1+\nu)/2}\sqrt{\frac{\lambda^{1-\nu}}{1+\alpha\lambda^{1-\nu}}}-\lambda_n^{(1+\nu)/2}\sqrt{\frac{\lambda_n^{1-\nu}}{1+\alpha\lambda_n^{1-\nu}}}\right| \\
&= \Bigg|(\lambda^{(1+\nu)/2}-\lambda_n^{(1+\nu)/2})\sqrt{\frac{\lambda^{1-\nu}}{1+\alpha\lambda^{1-\nu}}} \\
&\qquad + \lambda_n^{(1+\nu)/2}\left(\sqrt{\frac{\lambda^{1-\nu}}{1+\alpha\lambda^{1-\nu}}}-\sqrt{\frac{\lambda_n^{1-\nu}}{1+\alpha\lambda_n^{1-\nu}}}\right)\Bigg| \\
&\geq |\lambda^{(1+\nu)/2}-\lambda_n^{(1+\nu)/2}|\left|\sqrt{\frac{\lambda^{1-\nu}}{1+\alpha\lambda^{1-\nu}}}\right| \\
&\qquad - |\lambda_n|^{(1+\nu)/2}\left|\sqrt{\frac{\lambda^{1-\nu}}{1+\alpha\lambda^{1-\nu}}}-\sqrt{\frac{\lambda_n^{1-\nu}}{1+\alpha\lambda_n^{1-\nu}}}\right| \\
&= |\lambda_n|^{(1+\nu)/2}\left|\left(\frac{\lambda}{\lambda_n}\right)^{(1+\nu)/2}-1\right|\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}\right| \\
&\qquad - |\lambda_n|^{(1+\nu)/2}\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}-\sqrt{\frac{1}{\lambda_n^{\nu-1}+\alpha}}\right|
\end{aligned}
$$

As $(d/dz)(z+\alpha)^{-1/2} = -\alpha^{-3/2}/2$ at $z=0$ and $(d/dz)z^{\nu-1} = (\nu-1)$ at $z=1$ we have for large $|\lambda|$, $|\lambda_n|$,

(4.14)

$$
\begin{aligned}
\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}-\sqrt{\frac{1}{\lambda_n^{\nu-1}+\alpha}}\right| &\leq \alpha^{-3/2}|\lambda^{\nu-1}-\lambda_n^{\nu-1}| \\
&\leq \alpha^{-3/2}|\lambda_n^{\nu-1}|\,|(\lambda/\lambda_n)^{\nu-1}-1| \\
&\leq 2\alpha^{-3/2}|\nu-1|\,|\lambda_n|^{\nu-1}|(\lambda/\lambda_n)-1| \\
&\leq 2\alpha^{-3/2}|\nu-1|\,|\lambda_n|^{\nu-2}|\lambda-\lambda_n|.
\end{aligned}
$$

Similarly, as

$$\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}} \to \alpha^{-1/2}$$

we have

$$
\begin{aligned}
(4.15)\quad & \left|\left(\frac{\lambda}{\lambda_n}\right)^{(1+\nu)/2} - 1\right|\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}\right| \\
&\geq \left(\left|\frac{1+\nu}{2}\right|\left|\left(\frac{\lambda}{\lambda_n}\right)-1\right| - O\left(\left(\left(\frac{\lambda}{\lambda_n}\right)-1\right)^2\right)\right)\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}\right| \\
&\geq |(1+\nu)/4|\,|\lambda_n|^{-1}|\lambda-\lambda_n|\alpha^{-1/2}.
\end{aligned}
$$

Now $\nu \in (0,1)$ so $\nu - 2 < -1$; thus estimate (4.15) dominates the estimate given in (4.14) for $|\lambda_n|$ large, and collecting the estimates (4.12), (4.13) and (4.15), we obtain

$$
(4.16)\quad
\begin{aligned}
|z - z_n| &\geq \sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|1+\nu|\,|\lambda_n|^{(\nu-1)/2}|\lambda-\lambda_n|/8 \\
&\geq \sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|1+\nu|\,|\lambda|^{(\nu-1)/2}|\lambda-\lambda_n|/16.
\end{aligned}
$$

Examining (4.8) and (4.9) we see that to establish (4.8) we need

$$
(4.17)\quad |z - z_n| \geq 2(1+\varepsilon)\sqrt{\frac{2\rho_1 E_1}{\rho_2 E_2 \alpha}}|\lambda|^{(\nu-1)/2}.
$$

Thus, (4.8) will be satisfied if

$$
(4.18)\quad 2(1+\varepsilon)\sqrt{\frac{2\rho_1 E_1}{\rho_2 E_2 \alpha}} \leq \sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|1+\nu|\,|\lambda-\lambda_n|/16.
$$

That is,

$$
|\lambda - \lambda_n| \geq 32(1+\varepsilon)\sqrt{2\rho_1 E_1}/|1+\nu|\rho_2 \equiv R.
$$

Thus, for circles about λ_n of radius R, we can implement (4.8) to get

$$
\begin{aligned}
\left|\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right)\right| &\geq \frac{\sqrt{\rho_1 E_1}}{|\sqrt{\rho_2\lambda\hat{g}(\lambda)}|}(1+\varepsilon) \\
&\geq \frac{\sqrt{\rho_1 E_1}}{|\sqrt{\rho_2\lambda\hat{g}(\lambda)}|}\left|\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right)\right|
\end{aligned}
$$

as

$$\left|\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right)\right| \to 1.$$

Then, by Rouché's theorem, the characteristic equation

$$\sqrt{\rho_1 E_1}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = -\sqrt{\rho_2\lambda\hat{g}(\lambda)}\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right)$$

has precisely one solution in the disk of radius R around λ_n if n is chosen sufficiently large.

We now will show that (asymptotically) there are no solutions of the characteristic equation ((4.5) or (4.6)) other than the ones we have found.

Let λ_n be a sequence of solutions of the characteristic equation with $|\lambda_n| \to \infty$. that is,

$$-\sqrt{\rho_1 E_1}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_n\right) = \sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}\coth\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right).$$

Assume first that $\mathrm{Re}\,(\lambda_n) \to -\infty$. We will show that there is a fixed radius N so that $\coth(\sqrt{\rho_2\lambda/\hat{g}(\lambda)})$ has a zero in the circle about λ_n if λ_n is sufficiently large.

Define w_n by

$$w_n = -\sqrt{\rho_1 E_1}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_n\right). \tag{4.19}$$

Notice that as $\mathrm{Re}\,(\lambda_n) \to -\infty$, $w_n \to \sqrt{\rho_1 E_1}$. Now

$$\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) = \left[\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) - \frac{w_n}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}}\right] + \frac{w_n}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}}.$$

The expression in the brackets has a zero at λ_n. Thus, $\coth(\sqrt{\rho_2\lambda/\hat{g}(\lambda)})$ has a zero in the circle about λ_n with radius N if we can show the estimate

$$\left|\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) - \coth\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right)\right| > \left|\frac{w_n}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}}\right| \tag{4.20}$$

on the boundary of the disk. Again we write

$$z = \sqrt{(\lambda\rho_2/\hat{g}(\lambda))} \quad \text{and} \quad z_n = \sqrt{(\lambda_n\rho_2/\hat{g}(\lambda_n))}.$$

If $|\lambda_n - \lambda|$ stays bounded while $|\lambda_n| \to \infty$, we have from (4.12) and (4.13)

$$\begin{aligned}\sqrt{\frac{E_2}{\rho_2}}|z - z_n| &= \left|\sqrt{\frac{\lambda_n^2}{1+\alpha\lambda_n^{1-\nu}}} - \sqrt{\frac{\lambda^2}{1+\alpha\lambda^{1-\nu}}}\right| \\ &\le |\lambda_n|^{(1+\nu)/2}\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}} - \sqrt{\frac{1}{\lambda_n^{\nu-1}+\alpha}}\right| \\ &\quad + |\lambda_n^{(1+\nu)/2} - \lambda^{(1+\nu)/2}|\left|\sqrt{\frac{1}{\lambda^{\nu-1}+\alpha}}\right|.\end{aligned}$$

We now estimate, much as we did previously, assuming that $|\lambda_n - \lambda|$ is bounded by N and $|\lambda_n| \to \infty$.

$$\begin{aligned}&\le \alpha^{-3/2}|\lambda^{\nu-1} - \lambda_n^{\nu-1}|\,|\lambda_n|^{(1+\nu)/2} + 2\alpha^{-1/2}|\lambda_n^{(1+\nu)/2} - \lambda^{(1+\nu)/2}| \\ &\le \alpha^{-3/2}|\lambda_n|^{(3\nu-1)/2}|\left|\left(\frac{\lambda}{\lambda_n}\right)^{\nu-1} - 1\right| \\ &+ 2\alpha^{-1/2}|\lambda_n|^{(\nu+1)/2}\left|\left(\frac{\lambda}{\lambda_n}\right)^{(\nu+1)/2} - 1\right| \\ &\le 2\left(\alpha^{-3/2}|\lambda_n|^{(3\nu-1)/2}|\nu-1| + 2\alpha^{-1/2}|\lambda_n|^{(\nu+1)/2}\left|\frac{\nu+1}{2}\right|\right)\left|\frac{\lambda}{\lambda_n} - 1\right| \\ &= 2(\alpha^{-3/2}|\lambda_n|^{(3\nu-3)/2}|\nu-1| + \alpha^{-1/2}|\lambda_n|^{(\nu-1)/2}|\nu+1|)|\lambda - \lambda_n| \\ &\le 4\alpha^{-1/2}|\lambda_n|^{(\nu-1)/2}|\nu+1|\,|\lambda - \lambda_n|\end{aligned}$$

which $\to 0$ when $|\lambda_n - \lambda|$ remains bounded and $|\lambda_n| \to \infty$. Thus, $|z - z_n| \to 0$ and we may estimate the difference $|\coth(z_n) - \coth(z)|$ by Taylor's formula.

$$|\coth(z) - \coth(z_n)| \ge |\sinh^2(z_n)|^{-1}|z - z_n| + o(|z - z_n|)$$

and as

$$\coth(z_n) = \frac{w_n}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}} \to 0,$$

we have
$$\frac{-1}{\sinh^2(z_n)} = 1 - \coth^2(z_n) \to 1,$$
and thus
$$|\coth(z) - \coth(z_n)| \geq |z - z_n|/2.$$
For $|z - z_n|$ we have from (4.16)
$$|z - z_n| \geq \sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|1+\nu|\,|\lambda_n|^{(\nu-1)/2}|\lambda - \lambda_n|/8.$$
Thus, we have
$$|\coth(z) - \coth(z_n)| \geq \sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|1+\nu|\,|\lambda_n|^{(\nu-1)/2}|\lambda - \lambda_n|/16.$$
On the other hand, we have
$$\left|\frac{w_n}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}}\right| \leq \frac{2\sqrt{E_1\rho_1}}{\sqrt{E_2\rho_2(1+\alpha\lambda_n^{1-\nu})}} \leq \frac{4\sqrt{E_1\rho_1}}{\sqrt{E_2\alpha\rho_2}}\lambda_n^{(\nu-1)/2}.$$
Therefore, the estimate (4.20) holds, if
$$\begin{aligned}|\lambda - \lambda_n| = N &\geq 4\sqrt{\frac{E_1\rho_1}{E_2\alpha\rho_2}}\left(\sqrt{\frac{\rho_2}{E_2}}\alpha^{-1/2}|\nu+1|/16\right)^{-1}\\ &= \frac{64}{|\nu+1|}\frac{\sqrt{E_1\rho_1}}{\rho_2}.\end{aligned}$$
We have proved that if λ_n is a pole of the coupled problem sufficiently large and $\mathrm{Re}\,(\lambda_n) \to -\infty$, then in the disk about λ_n with radius N there is a zero of $\coth(\sqrt{\rho_1\lambda/\hat{g}(\lambda)})$.

Now suppose that λ_n are poles of the coupled problem with $\mathrm{Re}\,(\lambda_n)$ bounded. Then
$$\frac{1}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}}\tanh\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right) \to 0$$
since
$$\frac{1}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda_n)}} \to 0,$$

and

$$\tanh\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right)$$

is bounded because of (4.3). We therefore have that

$$\frac{1}{\sqrt{\rho_1 E_1}}\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) \to 0.$$

By the periodicity of the tanh in the imaginary direction we can infer that the distance of λ_n to the next zero of $\tanh(\sqrt{(\rho_1/E_1)}\lambda)$ goes to zero, that is, the λ_n approach the imaginary axis. □

Proof of Theorem 3.1. Recall first the characteristic equation

$$\frac{1}{\sqrt{\rho_1 E_1}}\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = -\frac{1}{\sqrt{\rho_2\lambda\hat{g}(\lambda)}}\tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right). \tag{4.21}$$

We consider first the possibility that (4.21) has solutions λ_n, with $|\lambda_n| \to \infty$ but $\mathrm{Re}\,(\lambda_n)$ is bounded. Referring to (3.4),

$$\begin{aligned}\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}} &= \sqrt{\frac{\rho_2\lambda^2(1+\beta\lambda^\nu)}{E_2(1+\alpha\lambda^\nu)}} \\ &= \lambda\sqrt{\frac{\rho_2}{E_2}}\sqrt{\frac{\beta}{\alpha}} + \lambda\sqrt{\frac{\rho_2}{E_2}}\frac{1}{2}\left(\frac{\beta}{\alpha}\right)^{-1/2}\left(\frac{1+\beta\lambda^\nu}{1+\alpha\lambda^\nu} - \frac{\beta}{\alpha}\right) \\ &\quad + \lambda\sqrt{\frac{\rho_2}{E_2}}O\left(\frac{1+\beta\lambda^\nu}{1+\alpha\lambda^\nu} - \frac{\beta}{\alpha}\right)^2 \\ &\equiv z\end{aligned}$$

using the notation from earlier. We are interested in estimating the real part of z when $|\lambda| \to \infty$ but $\mathrm{Re}\,(\lambda)$ is bounded.

$$\begin{aligned}
\mathrm{Re}\,(z) &= O(1) + \mathrm{Re}\left(\sqrt{\frac{\rho_2}{E_2\alpha\beta}}\frac{\lambda(\alpha-\beta)}{1+\alpha\lambda^\nu}\right) + O\left(\left|\frac{\lambda}{\lambda^{2\nu}}\right|\right) \\
&= O(1) + O(|\lambda|^{1-2\nu}) + \sqrt{\frac{\rho_2}{E_2\alpha\beta}}\frac{(\alpha-\beta)}{\alpha}\mathrm{Re}\,(\lambda^{1-\nu}) \\
&\quad + O\left(\left|\frac{\lambda}{1+\alpha\lambda^\nu} - \frac{\lambda}{\alpha\lambda^\nu}\right|\right) \\
&= O(1) + O(|\lambda|^{1-2\nu}) + O\left(\frac{\lambda}{\lambda^\nu + \alpha\lambda^{2\nu}}\right) \\
&\quad + \sqrt{\frac{\rho_2}{E_2\alpha\beta}}\frac{(\alpha-\beta)}{\alpha}|\lambda|^{1-\nu}\mathrm{Re}\,(i^{1-\nu}).
\end{aligned}$$

It is clear that the last term is dominant and tends to ∞, so

$$\tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat g(\lambda)}}\right) \to 1 \quad \text{and} \quad \frac{1}{\sqrt{\rho_2\lambda\hat g(\lambda)}} \to 1\Big/\left(\sqrt{\frac{\rho_2E_2\alpha}{\beta}}\right).$$

Thus, in order to solve the characteristic equation (4.21), we shall want to solve

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right) = -\sqrt{\frac{\beta\rho_1E_1}{\alpha\rho_2E_2}}. \tag{4.22}$$

We first assume that

$$\frac{\beta\rho_1E_1}{\alpha\rho_2E_2} < 1. \tag{4.23}$$

Now, denote by λ_0 the real solution of

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_0\right) = -\sqrt{\frac{\beta\rho_1E_1}{\alpha\rho_2E_2}}. \tag{4.24}$$

Choose an $\varepsilon > 0$ small, and consider the ε disks about the points

$$k_n \equiv \lambda_0 + (n\pi i)\sqrt{\frac{E_1}{\rho_1}}. \tag{4.25}$$

By periodicity of tanh in the imaginary direction, we can assume that on the boundary of each disk,

$$\left|\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_0\right) + \sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}}\right| > M > 0. \tag{4.26}$$

In addition, as $|\lambda| \to \infty$ but $|\mathrm{Re}\,(\lambda)|$ is bounded,

$$\left|\sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}} - \frac{\sqrt{E_1\rho_1}}{\sqrt{\rho_2\lambda\hat{g}(\lambda)}}\tanh\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right)\right| \to 0. \tag{4.27}$$

This implies that there is a zero of the characteristic equation (4.21) in the ε disk about k_n if n is sufficiently large by Rouché's theorem as before.

We now consider the case where

$$\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2} > 1. \tag{4.28}$$

If we note that

$$\tanh(a + \pi i/2) = \frac{1}{\tanh(a)}$$

we see that we must look for solutions of the characteristic equation near

$$k_n \equiv \lambda_0 + (n + 1/2)\pi i\sqrt{\frac{E_1}{\rho_1}}.$$

The argument is exactly as above to show that there are solutions of the characteristic equation near k_n as n increases to infinity.

We consider now the characteristic equation of the form

$$\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) = -\sqrt{\frac{\rho_1 E_1}{\rho_2\lambda\hat{g}(\lambda)}}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right). \tag{4.29}$$

The term on the right hand side of (4.29) converges to

$$\sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}}$$

when $|\lambda| \to \infty$ along with $\mathrm{Re}\,(\lambda) \to -\infty$. Therefore, we consider λ_n which satisfies

$$\coth\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right) - \sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}} = 0. \tag{4.30}$$

We first note that the λ_n must satisfy

$$\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}} = r + n\pi i \tag{4.31}$$

where, depending on the size of the square root, r is either real or $\mathrm{Im}\,(r) = \pi/2$. Using the equation for $\hat{g}$ given in (3.4) and (4.31) yields

$$\lambda_n\sqrt{\frac{\rho_2}{E_2}}\sqrt{\frac{(1+\beta\lambda_n^\nu)}{(1+\alpha\lambda_n^\nu)}} = r + n\pi i. \tag{4.32}$$

Now

$$\frac{d}{dx}\sqrt{\frac{x+\beta}{x+\alpha}} = \frac{(\alpha-\beta)\sqrt{x+\alpha}}{2(x+\alpha)^2\sqrt{x+\beta}} \tag{4.33}$$

and so, setting $x = 0$, we see that (4.32) can be written

$$\lambda_n\sqrt{\frac{\rho_2}{E_2}}\left[\left(\frac{\beta}{\alpha}\right)^{1/2} + \frac{\alpha-\beta}{2\alpha^{3/2}\beta^{1/2}}\lambda_n^{-\nu} + O(\lambda_n^{-2\nu})\right] = r + n\pi i$$

or

$$\sqrt{\frac{\rho_2\beta}{E_2\alpha}}\lambda_n + \frac{\sqrt{\rho_2}(\alpha-\beta)}{2\sqrt{E_2}\alpha}\lambda_n^{1-\nu} + O(\lambda_n^{1-2\nu}) = r + n\pi i. \tag{4.34}$$

As the real part of $r + n\pi i$ is bounded, we see that $\vartheta_n = \arg\lambda_n \to \pi/2$. Taking real parts in (4.34) we see that

$$\sqrt{\frac{\rho_2\beta}{E_2\alpha}}|\lambda_n|\cos(\vartheta_n) + \frac{\sqrt{\rho_2}(\alpha-\beta)}{2\sqrt{E_2}\alpha}|\lambda_n|^{1-\nu}\cos(\vartheta_n(1-\nu)) + o|\lambda^{1-\nu}| = 0$$

and so

$$\sqrt{\frac{\rho_2\beta}{E_2\alpha}}|\lambda_n|^\nu \cos(\vartheta_n) + \frac{\sqrt{\rho_2}(\alpha-\beta)}{2\sqrt{E_2}\alpha}\cos(\vartheta_n(1-\nu)) + o(1) = 0.$$

Now, recalling that $\vartheta_n \to \pi/2$, we obtain

$$\begin{aligned} &-\left(\vartheta_n - \frac{\pi}{2}\right) + o\left(\vartheta_n - \frac{\pi}{2}\right) \\ &\qquad + |\lambda_n|^{-\nu}\sqrt{\frac{\alpha}{\beta}}\frac{(\alpha-\beta)}{2\alpha}\cos(\pi(1-\nu)/2) + o(|\lambda_n|^{-\nu}) = 0. \end{aligned}$$

Thus,

$$\vartheta_n = \frac{\pi}{2} + |\lambda_n|^{-\nu}\sqrt{\frac{\alpha}{\beta}}\frac{(\alpha-\beta)}{2\alpha}\cos(\pi(1-\nu)/2) + o(|\lambda_n|^{-\nu})$$

and

$$\begin{aligned} \lambda_n &= |\lambda_n|e^{i\vartheta_n} \\ &\sim i|\lambda_n|\exp\left(i\sqrt{\frac{\alpha}{\beta}}\frac{(\alpha-\beta)}{2\alpha}\cos(\pi(1-\nu)/2)|\lambda_n|^{-\nu}\right) + o(|\lambda_n|^{1-\nu}). \end{aligned}$$

From (4.34), we see that

$$|\lambda_n| = \sqrt{\frac{E_2\alpha}{\rho_2\beta}}n\pi + o(|\lambda_n|).$$

Thus, the λ_n will be, roughly speaking, on a curve

$$\operatorname{Re}\lambda_n = -K|\lambda_n|^{1-\nu}.$$

We will now show that, for an arbitrarily small radius ε and sufficiently large n, the ε circle about λ_n contains a zero of the characteristic equation. For this purpose we rewrite (4.29)

$$\begin{aligned} 0 =&\left[\coth\left(\sqrt{\frac{\rho_2\lambda}{\hat{g}(\lambda)}}\right) - \coth\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right)\right] \\ &+\left[\sqrt{\frac{\beta\rho_1E_1}{\alpha\rho_2E_2}} + \sqrt{\frac{\rho_1E_1}{\rho_2\lambda\hat{g}(\lambda)}}\coth\left(\sqrt{\frac{\rho_1}{E_1}}\lambda\right)\right]. \end{aligned} \tag{4.35}$$

The first bracketed term in (4.35) has the unique zero λ_n in the ε-circle. Thus, we must show that the second bracketed term is smaller in magnitude than the first on the boundary of the ε-disk. As the second term goes to zero, it is sufficient to show that for fixed (though small) ε the first term is bounded away from 0 as $\lambda_n \to \infty$. Again, we put

$$\begin{aligned} z &= \sqrt{\frac{\rho_2 \lambda}{\hat{g}(\lambda)}} \\ &= \lambda \sqrt{\frac{\rho_2}{E_2}} \sqrt{\frac{\lambda^{-\nu} + \beta}{\lambda^{-\nu} + \alpha}}, \end{aligned}$$

and

$$z_n = \lambda_n \sqrt{\frac{\rho_2}{E_2}} \sqrt{\frac{\lambda_n^{-\nu} + \beta}{\lambda_n^{-\nu} + \alpha}},$$

If we can show that $|z - z_n|$ stays sufficiently small we may rely on the Taylor's series expansion of $\coth(z)$ to get

$$\begin{aligned} |\coth(z) - \coth(z_n)| &\geq \left| \frac{(z - z_n)}{\sinh^2(z_n)} + o(z - z_n) \right| \\ &\geq |1 - \coth^2(z_n)|\,|z - z_n|/2 \\ &= \left| 1 - \frac{\beta \rho_1 E_1}{\alpha \rho_2 E_2} \right| |z - z_n|/2 \\ &= M_1 |z - z_n|. \end{aligned} \tag{4.36}$$

Notice that $M_1 \neq 0$ since we have assumed that we do not have impedance matching. That is, (3.5) is valid. We now wish to estimate $|z - z_n|$. We first write

$$\begin{aligned} |z - z_n| &= \sqrt{\frac{\rho_2}{E_2}} \left| \lambda \sqrt{\frac{\lambda^{-\nu} + \beta}{\lambda^{-\nu} + \alpha}} - \lambda_n \sqrt{\frac{\lambda_n^{-\nu} + \beta}{\lambda_n^{-\nu} + \alpha}} \right| \\ &= \sqrt{\frac{\rho_2}{E_2}} \Bigg| (\lambda - \lambda_n) \sqrt{\frac{\lambda^{-\nu} + \beta}{\lambda^{-\nu} + \alpha}} \\ &\qquad + \lambda_n \left(\sqrt{\frac{\lambda^{-\nu} + \beta}{\lambda^{-\nu} + \alpha}} - \sqrt{\frac{\lambda_n^{-\nu} + \beta}{\lambda_n^{-\nu} + \alpha}} \right) \Bigg|. \end{aligned} \tag{4.37}$$

Now using (4.33) we have

$$(4.38)\quad |\lambda_n|\left|\sqrt{\frac{\lambda^{-\nu}+\beta}{\lambda^{-\nu}+\alpha}}-\sqrt{\frac{\lambda_n^{-\nu}+\beta}{\lambda_n^{-\nu}+\alpha}}\right|$$

$$\begin{aligned}
&= |\lambda_n|\left[\frac{|\beta-\alpha|}{\alpha^{3/2}\beta^{1/2}}|\lambda^{-\nu}-\lambda_n^{-\nu}|+o(|\lambda^{-\nu}-\lambda_n^{-\nu}|)\right]\\
&\leq |\lambda_n|^{1-\nu}\frac{|\beta-\alpha|}{\alpha^{3/2}\beta^{1/2}}\left|1-\left(\frac{\lambda}{\lambda_n}\right)^{-\nu}\right|\\
&\leq 2|\nu|\,|\lambda_n|^{1-\nu}\frac{|\beta-\alpha|}{\alpha^{3/2}\beta^{1/2}}\left|1-\frac{\lambda}{\lambda_n}\right|\\
&\leq 2|\nu|\,|\lambda_n|^{-\nu}\frac{|\beta-\alpha|}{\alpha^{3/2}\beta^{1/2}}|\lambda-\lambda_n|.
\end{aligned}$$

The first term in (4.37) behaves like $|\lambda-\lambda_n|$ times a constant. Therefore, the first term in (4.37) is dominant and

$$2\sqrt{\frac{\rho_2\beta}{E_2\alpha}}|\lambda-\lambda_n|\geq|z-z_n|\geq\sqrt{\frac{\rho_2\beta}{E_2\alpha}}|\lambda-\lambda_n|/2.$$

In particular, if the radius of the disk is small, $|z-z_n|$ stays small and we can use (4.36). Then

$$|\coth(z)-\coth(z_n)|\geq M_1\sqrt{\frac{\rho_2\beta}{E_2\alpha}}|\lambda-\lambda_n|/2$$

is bounded away from zero on the boundary of the disk.

Applying Rouché's theorem again, we have proved that there is a set of zeros of the characteristic equation approaching the λ_n as $|\lambda_n|\to\infty$. This concludes the proof. □

Proof of Theorem 3.2. Consider the possibility that $\{\lambda_n\}$ is a sequence of solutions of the characteristic equation with $|\lambda_n|\to\infty$ and $|\mathrm{Re}\,(\lambda_n)|\leq M<\infty$. Thus,

$$(4.39)\quad \frac{1}{\sqrt{\rho_1E_1}}\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_n\right)=-\frac{1}{\sqrt{\rho_2\lambda_n\hat{g}(\lambda)}}\tanh\left(\sqrt{\frac{\rho_2\lambda_n}{\hat{g}(\lambda_n)}}\right).$$

The argument at the beginning of the proof of Theorem 3.1 shows that

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\lambda_n\right) \to -\sqrt{\frac{\beta\rho_1 E_1}{\alpha\rho_2 E_2}} = -1. \tag{4.40}$$

Now write

$$\lambda_n = \mu_n + m_n\sqrt{\frac{E_1}{\rho_1}}\pi i$$

where μ_n is a bounded sequence and m_n is a sequence of integers. By choosing a subsequence if necessary, we can assume that $\mu_n \to \mu$ as $n \to \infty$, μ a finite complex number such that

$$\tanh\left(\sqrt{\frac{\rho_1}{E_1}}\mu\right) = -1$$

which is impossible. Thus, in the impedance matching case there are no solutions λ_n of the characteristic equation with $|\lambda_n| \to \infty$ while $\mathrm{Re}\,(\lambda_n)$ remains bounded.

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Institut für Mathematik, Universität Graz, A-8010, Graz, Austria

Department of Mathematics, Southern Illinois University, Carbondale, IL 62901