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LIAPUNOV THEORY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

T.A. BURTON AND R.H. HERING

Dedicated to Paul Waltman on the occasion of his 60th birthday

1. Introduction and summary. This paper is concerned with stability properties of the zero solution of a system of functional differential equations written as

$$x'(t) = f(t, x_t) \tag{1}$$

where $f : [0,\infty) \times C_1 \to R^n$ is continuous and takes bounded sets into bounded sets. Here $(C, ||\cdot||)$ is the Banach space of continuous functions $\phi : [-h, 0] \to R^n$ with the supremum norm, h is a positive constant, C_1 is the subset of C with $||\phi|| < 1$, $f(t, 0) = 0$, and $x_t(s) = x(t+s)$ for $-h \leq s \leq 0$.

The results are based on continuous functionals $V : [0,\infty) \times C_1 \to [0,\infty)$ which are locally Lipschitz in ϕ and whose derivative along any solution of (1) satisfies $V'_{(1)}(t, x_t) \leq 0$. Such a functional is called a *Liapunov functional* for (1). We also use continuous strictly increasing functions $W : [0,\infty) \to [0,\infty)$ with $W(0) = 0$, called *wedges*. Stability definitions will be stated in the next section.

The following is the classical theorem on uniform stability (US) for the zero solution of (1). It goes back to Krasovskii [**7**; pp. 143–157].

Theorem K1. *If there is a Liapunov functional for* (1) *and wedges satisfying*

$$W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(||\phi||) \tag{i}$$

and

$$V'_{(1)}(t, x_t) \leq 0, \tag{ii}$$

then $x = 0$ is uniformly stable.

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This result has remained the standard in the literature to the present day. It has been considered to be very satisfactory because examples are readily constructed and because, when (1) is smooth enough, then it has a converse (cf. Krasovskii [**7**, pp. 146–150]). In preparation for our results on asymptotic stability, we offer a simple generalization of this result which turns out to be very convenient in applications. It may be stated as follows.

Theorem 1. *Suppose there is a Liapunov functional and wedges for* (1) *such that, for each* $\gamma > 0$*, there is a wedge* W_γ *such that*

$$\text{(i)} \qquad W_1(|\phi(0)|) \le V(t,\phi) \le W_\gamma(||\phi||) + \gamma$$

and

$$\text{(ii)} \qquad V'_{(1)}(t,x_t) \le 0.$$

Then $x = 0$ *is* US.

The basic conjecture for (1) on uniform asymptotic stability (UAS) also goes back to Krasovskii [**7**, pp. 143–157] and may be stated as follows.

Conjecture K. *If there is a Liapunov functional for* (1) *and wedges such that*

$$\text{(i)} \qquad W_1(|\phi(0)|) \le V(t,\phi) \le W_2(||\phi||)$$

and

$$\text{(ii)} \qquad V'_{(1)}(t,x_t) \le -W_3(|x(t)|),$$

then $x = 0$ *is* UAS.

The conjecture was widely believed but never proved, and the result which remained standard in the literature through 1977 (cf. Hale [**6**, p. 105]) was crippled by the Marachkov [**10**] condition that $|f(t,\phi)|$ be bounded for $||\phi|| < 1$. This result can also be gleaned from the work of Krasovskii [**7**, pp. 143 –157].

Theorem K2. *If there is a Liapunov functional, wedges, and a constant M such that*

(i) $$W_1(|\phi(0)|) \le V(t,\phi) \le W_2(||\phi||),$$

(ii) $$V'_{(1)}(t,x_t) \le -W_3(|x(t)|),$$

and

(iii) $$|f(t,\phi)| \le M \quad \text{if } t \ge 0 \quad \text{and} \quad ||\phi|| < 1,$$

then $x = 0$ is UAS.

In 1978 a step was taken [**3**] toward the conjecture. Here, we denote by

$$|\phi|_p = \left(\int_{-h}^{0} |\phi(s)|^p\, ds\right)^{1/p}, \qquad 0 < p < \infty.$$

Thus, in particular, when $0 < p < 1$, this is not a norm. The following result can also be proved with $|\phi|_2$ replaced by $|\phi|_p$ if $p \ge 1$, as has been noted in several places. We will extend it to $0 < p < \infty$ along the lines of Theorem 1.

Theorem B. *Suppose there is a Liapunov functional and wedges such that*

(i) $$W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(|\phi|_2)$$

and

(ii) $$V'_{(1)}(t,x_t) \le -W_4(|x(t)|).$$

Then $x = 0$ is UAS.

We noted that $|\phi|_2$ can be replaced by $|\phi|_p$ for $p \ge 1$, and it is known that $|\phi|_p \to ||\phi||$ as $p \to \infty$ pointwise on C_1. Thus, there has consistently been hope that the conjecture would follow from a limiting argument with Theorem B. We offer a first step in that direction in our Theorem 3 by generalizing the Marachkov condition and using a compactness argument. We do not state that result until a later section.

But, in the same vein, the next two results are well-motivated by our main example in the next section. We state the result in two ways. The first way indicates a degree of sharpness, while the second way is less cumbersome and easier to use; however, Theorem 3 is based on the first version.

Theorem 2. *Suppose that there is a Liapunov functional for* (1) *and wedges such that for each* $\xi > 0$ *there is a* W_ξ *such that*

$$W_1(|\phi(0)|) \leq V(t,\phi) \leq W_\xi(||\phi||) + \xi \tag{i}$$

and

$$V'_{(1)}(t,x_t) \leq -W_2(|x(t)|). \tag{ii}$$

Under these conditions the zero solution of (1) *is* UAS *if and only if there is a wedge* W_3*, and for each* $\gamma > 0$ *there is a* $p = p(\gamma)$ *in* $(0,\infty)$*, a wedge* W_γ*, and a positive constant* T_γ *such that* $t \geq t_0 + T_\gamma$ *implies that*

$$V(t,x_t) \leq W_3(|x(t)|) + W_\gamma(|x_t|_p) + \gamma \tag{iii}$$

for every solution $x(t,t_0,\phi)$ *of* (1) *having* $t_0 \geq 0$ *and* $||\phi|| < \delta$*, where* δ *is that of* US *for* $\varepsilon = 1$.

Theorem 2A. *Suppose that there is a Liapunov functional for* (1) *and wedges such that for each* $\gamma > 0$ *there is a* $p = p(\gamma)$ *in* $(0,\infty)$ *and a* W_γ *such that*

$$W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(|\phi(0)|) + W_\gamma(|\phi|_p) + \gamma \tag{i}$$

and

$$V'_{(1)}(t,x_t) \leq -W_3(|x(t)|). \tag{ii}$$

Then the zero solution of (1) *is* UAS.

2. The setting and an example. Under the conditions stated with (1), for each $t_0 \geq 0$ and each $\phi \in C_1$ there is at least one solution $x(t, t_0, \phi)$ of (1) for $t_0 \leq t < t_0 + \alpha$ and, if the solution remains in a closed subset of C_1, then $\alpha = \infty$. The derivative of V along a solution $x(t, t_0, \phi)$ of (1) is frequently computed by the chain rule but is formally defined by

$$V'_{(1)}(t, x_t(t_0, \phi)) = \limsup_{\delta \to 0^+} (1/\delta)\{V(t + \delta, x_{t+\delta}(t_0, \phi)) - V(t, x_t(t_0, \phi))\}.$$

Basic discussions of these matters are found, for example, in Yoshizawa [**11**, pp. 181–182].

Since $f(t, 0) \equiv 0$, $x(t) \equiv 0$ is a solution of (1), and it is said to be:

(a) *uniformly stable* (US) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $[t_0 \geq 0, ||\phi|| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$;

(b) *uniformly asymptotically stable* (UAS) if it is uniformly stable and if there is a $\lambda > 0$, and for each $\mu > 0$ there is a $T > 0$ such that $[t_0 \geq 0, ||\phi|| < \lambda, t \geq t_0 + T]$ imply that $|x(t, t_0, \phi)| < \mu$.

In preparation for our example we look at the delay equation

$$x'(t) = -a(t)x(t) + b(t)x(t - h)$$

where $h > 0$, a and b are continuous, $|b(t)| \leq K$ for K constant, $a(t) - |b(t + h)| \geq \alpha$ for some $\alpha > 0$. Then

$$V(t, x_t) = |x(t)| + \int_{t-h}^{t} |b(s + h)|\,|x(s)|\,ds$$

satisfies

$$\begin{aligned} V'(t, x_t) &\leq -a(t)|x(t)| + |b(t)|\,|x(t - h)| \\ &\quad + |b(t + h)|\,|x(t)| - |b(t)|\,|x(t - h)| \\ &\leq -\alpha|x(t)|. \end{aligned}$$

We then have

$$|x(t)| \leq V(t, x_t) \leq |x(t)| + K \int_{t-h}^{t} |x(s)|\,ds \leq (1 + Kh)||x_t||$$

and

$$V'(t, x_t) \leq -\alpha|x(t)|$$

so that Theorem K1 holds, while Theorem K2 holds only if $a(t)$ is bounded. But Theorem B holds even with $a(t)$ unbounded, and we have UAS. Interesting results of Busenberg and Cooke [**5**] also apply to such problems, as do differential techniques presented, for example, by Lakshmikantham, Matrosov, and Sivasundaram [**8**]. There are also applicable results when a function space norm appears in V', as may be seen in [**1, 2, 4**], for example. Makay [**9**] has also contributed significantly to this problem.

But difficulties can occur when the system ceases to be smooth enough. The following example illustrates the difficulties and motivates our results.

Example 1. Let $a_n(t)$ and $b_n(t)$ be continuous scalar functions on $[0,\infty)$ for $n = 0, 1, 2, 3, \ldots$, and suppose that $|x| < 1$ and that

$$\text{(i)} \qquad a_n(t) - |b_n(t+1)| \geq 0 \qquad \text{for all } n \geq 0,$$

(ii) there is an $\eta > 0$ and an $i \geq 1$ with $a_i(t) - |b_i(t+1)| \geq \eta$,

there is a sequence of constants B_n with

$$\text{(iii)} \qquad |b_n(t)| \leq B_n \quad \text{and} \quad B = \sum_0^\infty B_n < \infty,$$

for each $T > 0$ there is a sequence $\{A_n(T)\}$ with

$$\text{(iv)} \qquad 0 \leq a_n(t) \leq A_n(T) \quad \text{if} \quad 0 \leq t \leq T \quad \text{and} \quad \sum_0^\infty A_n(T) < \infty.$$

Then the zero solution of

$$\text{(E)} \qquad x'(t) = \sum_{n=0}^\infty \left\{ -a_n(t)(x(t))^{1/(2n+1)} + b_n(t)(x(t-1))^{1/(2n+1)} \right\}$$

is UAS.

Proof. Let

$$\text{(E1)} \qquad V(t, x_t) = |x(t)| + \sum_{n=0}^\infty \int_{t-1}^t |b_n(s+1)|\,|x(s)|^{1/(2n+1)}\, ds$$

so that

$$V'_{(E)}(t,x_t) \le \sum_0^\infty \Big\{ -a_n(t)|x(t)|^{1/(2n+1)} + |b_n(t)|\,|x(t-1)|^{1/(2n+1)} + |b_n(t+1)|\,|x(t)|^{1/(2n+1)} - |b_n(t)|\,|x(t-1)|^{1/(2n+1)} \Big\};$$

hence, by (i) and (ii) we have

$$V'_{(E)}(t,x_t) \le -\eta |x(t)|^{1/(2i+1)}. \tag{E2}$$

Thus, an appropriate wedge for V' is $W(r) = \eta r^{1/(2i+1)}$.

To justify this work, we note that conditions (iii) and (iv) ensure that the series in (E) converges uniformly in (t,x) for $|x| < 1$ and $0 \le t \le T$ by the Weierstrass M-test. Moreover, the series $\sum a_n x^{1/(2n+1)}$ converges to a function continuous in (t,x) and $\sum b_n x^{1/(2n+1)}(t-1)$ converges to a function continuous in $(t, x(t-1))$ for $|x| < 1$ and $0 \le t \le T$. To see this, for fixed t the series converges to a function uniformly continuous in x, while for fixed x the limit function is continuous in t; hence, it is jointly continuous in (t,x). Thus, the existence results hold. In the same way, we can differentiate the series for V term-by-term because the differentiated series converges uniformly and its terms are continuous.

In this example the conditions of the conjecture are satisfied, but not those of Theorem B. Thus, this illustrates the need for an intermediate theorem.

In fact, for each $\gamma > 0$ there exists N such that so long as we work in the set $||x_t|| \le 1$ then

$$\sum_{n=N+1}^\infty \int_{t-1}^t |b_n(s+1)|\,|x(s)|^{1/(2n+1)}\,ds \le \sum_{n=N+1}^\infty B_n < \gamma.$$

Thus,

$$\begin{aligned} |x(t)| \le V(t,x_t) &\le |x(t)| + \sum_{n=0}^N \int_{t-1}^t B_n |x(s)|^{1/(2n+1)}\,ds + \gamma \\ &\le |x(t)| + B\int_{t-1}^t |x(s)|^{1/(2n+1)}\,ds + \gamma \\ &= |x(t)| + B(|x_t|_{1/(2N+1)})^{1/(2N+1)} + \gamma \end{aligned}$$

so condition (iii) of Theorem 2 holds. As for the US, for the $\gamma > 0$ we find N with

$$V(t,x_t) \le |x(t)| + \sum_0^N \int_{t-1}^t |b_n(s+1)|\,|x(s)|^{1/(2n+1)}\,ds + \gamma$$
$$\le |x(t)| + B||x_t||^{1/(2N+1)} + \gamma$$

and the conditions of Theorem 1 hold.

3. Proof of Theorem 1. Let $\varepsilon > 0$ be given, $\varepsilon < 1$, and choose $\gamma = W_1(\varepsilon)/2$ so that there is a W_γ with

$$W_1(|\phi(0)|) \le V(t,\phi) \le W_\gamma(||\phi||) + \gamma.$$

Choose $\delta > 0$ so that $W_\gamma(\delta) + \gamma < W_1(\varepsilon)$. If $||\phi|| < \delta$, then for $t_0 \ge 0$ and $x(t) = x(t,t_0,\phi)$ we have

$$W_1(|x(t)|) \le V(t,x_t) \le V(t_0,\phi) \le W_\gamma(||\phi||) + \gamma$$
$$\le W_\gamma(\delta) + \gamma < W_1(\varepsilon),$$

so $|x(t)| < \varepsilon$, as required. □

4. Proof of Theorem 2A. We first show that the conditions of Theorem 1 are satisfied. For a given $\gamma > 0$, we have

$$W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_\gamma\left(\left(\int_{-h}^0 |\phi(s)|^p\,ds\right)^{1/p}\right) + \gamma$$
$$\le W_2(|\phi(0)|) + W_\gamma((h||\phi||^p)^{1/p}) + \gamma$$
$$\le W_\gamma^*(||\phi||) + \gamma,$$

for some W_γ^*, as required.

Thus, $x = 0$ is US and there is a $\delta_1 > 0$ such that $[||\phi|| < \delta_1, t_0 \ge 0, t \ge t_0]$ imply that $|x(t,t_0,\phi)| < 1$.

Let $\mu > 0$ be given. We must find $T > 0$ such that $[||\phi|| < \delta_1, t_0 \ge 0, t \ge t_0 + T]$ imply that $|x(t,t_0,\phi)| < \mu$. Use the US to find a $\delta > 0$ (with $\delta \le \delta_1$) so that $[||\phi|| < \delta, t_0 \ge 0, t \ge t_0]$ imply that $|x(t,t_0,\phi)| < \mu$.

Find $\delta_2 > 0$ with $W_2(\delta_2) < W_1(\delta)$. Then choose $\gamma > 0$ with

$$W_1(\delta) - W_2(\delta_2) - \gamma =: \lambda > 0.$$

For this $\gamma > 0$ find p and W_γ of Theorem 2A(i). Fix $t_0 \geq 0$ and ϕ with $||\phi|| < \delta_1$, and consider the intervals

$$I_n = [t_0 + (n-1)h, t_0 + nh],$$

$n = 1, 2, 3, \ldots$. Now consider any solution $x(t) = x(t, t_0, \phi)$. On I_n there are two possibilities for $n \geq 3$:

(a) $||x_{t_0+nh}|| < \delta$, so $|x(t)| < \mu$ for $t \geq t_0 + nh$, or

(b) there is a $t_n \in I_n$ with $|x(t_n)| \geq \delta$.

Unless there is an $s_n \in [t_n - h, t_n]$ with $|x(s_n)| = \delta_2$, then $V'_{(1)}(t, x_t) \leq -W_3(|x(t)|)$ implies that $V(t, x_t)$ decreases by at least $hW_3(\delta_2)$ on $I_{n-1} \cup I_n$. But if s_n exists, then

$$\begin{aligned} W_1(\delta) \leq W_1(|x(t_n)|) &\leq V(t_n, x_{t_n}) \leq V(s_n, x_{x_n}) \\ &\leq W_2(|x(s_n)|) + W_\gamma(|x_{s_n}|_p) + \gamma \\ &\leq W_2(\delta_2) + W_\gamma(|x_{s_n}|_p) + \gamma, \end{aligned}$$

so

$$\lambda = W_1(\delta) - W_2(\delta_2) - \gamma \leq W_\gamma(|x_{s_n}|_p).$$

This yields

$$W_\gamma^{-1}(\lambda) \leq |x_{s_n}|_p,$$

so that

$$\xi := [W_\gamma^{-1}(\lambda)]^p \leq \int_{s_n - h}^{s_n} |x(s)|^p \, ds.$$

Now

$$V'_{(1)}(t, x_t) \leq -W_3(|x(t)|) = -W_3((|x(t)|^p)^{1/p}) =: -W_5(|x(t)|^p)$$

for some W_5 which, by renaming if necessary, we shall assume to be convex downward since it is always possible to write for $u \geq 0$:

$$W_6(u) = \int_0^u W_5(s) \, ds \leq uW_5(u) \leq W_5(u)$$

since we deal with $u \le 1$. And W_6 is convex downward. Thus, for $V = V(t)$ we have

$$\begin{aligned} V(s_n) - V(s_n - h) &\le -\int_{s_n-h}^{s_n} W_5(|x(t)|^p)\,dt \\ &\le -hW_5\left(\frac{1}{h}\int_{s_n-h}^{s_n} |x(t)|^p\,dt\right) \\ &\le -hW_5(\xi/h). \end{aligned}$$

Note that ξ and W_5 depend on p; however, the dependence vanishes in the final line above. Hence, until we reach an I_n with $n \ge 3$ and (a) holding, then V decreases by the minimum of

$$hW_3(\delta_2) \quad \text{and} \quad hW_5(\xi/h) \quad \text{on} \quad I_{n-2} \cup I_{n-1} \cup I_n.$$

As

$$0 \le V(t, x_t) \le W_2(\delta) + W_\gamma(\delta) + \gamma$$

(for $\gamma = 1$, for example) there is a fixed N independent of t_0 and ϕ with (a) holding if $t \ge t_0 + Nh =: t_0 + T$. This completes the proof. □

5. Remarks on Theorem 2. The proof that the conditions of Theorem 2 yield UAS has essentially already been given in the proof since that proof dealt only with solutions, as reflected in (iii). The US follows from Theorem 1 and, here, the I_n become

$$I_n = [t_0 + T_\gamma + (n-1)h, t_0 + T_\gamma + nh].$$

The proof of the necessity of (iii) is simple to the point of disappointment. For a given $\gamma > 0$ we take $\xi = \gamma/2$ and find W_ξ. Let δ be that of US for $\varepsilon = 1$. There is then a $T > 0$ by the UAS such that

$$[t_0 \ge 0, ||\phi|| < \delta, t \ge t_0 + h + T] \quad \text{imply that} \quad W_\xi(||x_t||) < \gamma/2.$$

Hence, with $T_\gamma = T + h$ and with p and W_3 arbitrary, we have $V(t, x_t) \le W_\xi(||x_t||) + \xi < \gamma$, and (iii) is satisfied.

6. Marachkov's condition. If the convergence of $|\phi|_p \to ||\phi||$ as $p \to \infty$ were uniform on C_1, then Theorem 2 would prove the conjecture. We now give a condition to ensure that uniform convergence along solutions. In particular, we now show how to reduce condition (iii) of Theorem K2 in the context of Theorem 2.

Theorem 3. *Suppose there are wedges, a Liapunov functional, and a locally integrable function* $M : [0, \infty) \to [0, \infty)$ *such that for each* $\gamma > 0$ *there is a wedge* W_γ *such that*

(i) $$W_1(|\phi(0)|) \le V(t, \phi) \le W_\gamma(||\phi||) + \gamma,$$

(ii) $$V'_{(1)}(t, x_t) \le -W_3(|x(t)|),$$

and for $t \ge 0$ *and* $||\phi|| \le 1$ *that*

(iii) $$|f(t, \phi)| \le M(t) \quad \textit{and} \qquad \left| \int_{t_1}^{t_2} M(t)\, dt \right| \le W_4(|t_2 - t_1|) \quad \textit{for } t_1, t_2 \ge 0.$$

Then $x = 0$ *is* UAS.

Proof. By Theorem 1, $x = 0$ is US. Find the $\delta_1 > 0$ of US for $\varepsilon = 1$; we will show that the solutions of (1), written as $x_t(t_0, \phi)$ for $t \ge t_0 + h$ and $||\phi|| < \delta_1$ all lie in a compact set.

For any such solution, note that

$$|x(t_2) - x(t_1)| = \left| \int_{t_1}^{t_2} f(s, x_s)\, ds \right| \le \left| \int_{t_1}^{t_2} M(s)\, ds \right| \le W_4(|t_2 - t_1|)$$

and so the set of solutions $x_t(t_0, \phi)$ are uniformly bounded and equicontinuous. Thus, they reside in a compact set K.

Let $\gamma > 0$ be given and find W_γ of (i) with $V(t, \phi) \le W_\gamma(||\phi||) + \gamma/2$. We will show that (iii) of Theorem 2 holds. Note that if W is any wedge and if $a \ge 0$ and $b \ge 0$, then $a \ge b$ or $b \ge a$ so

$$W(a + b) \le W(2a) + W(2b).$$

Next, for the $\gamma > 0$ pick $\delta_1 > 0$ and $\delta_2 > 0$ satisfying

$$W_\gamma(2\delta_1) + W_\gamma(4\delta_2) + W_\gamma(8(1 + h)\delta_1) < \gamma/2.$$

Since the set of all δ_1 neighborhoods of K cover K, there is a finite number of points $\phi_1, \dots, \phi_n \in K$ with the property that $\phi \in K$ implies that $||\phi - \phi_i|| < \delta_1$ for some ϕ_i.

Now $|\phi_1|_p \to ||\phi_1||$ as $p \to \infty$ so there is a p_1 with $|\,|\phi_1|_p - ||\phi_1||\,| < \delta_2$ if $p \geq p_1$. Likewise, there is a p_i for each ϕ_i with this property. We let $p = \max p_i$. (Here, $p \geq 1$.)

If $x_t \in K$, then there is an i with

$$\begin{aligned}
V(t, x_t) &\leq W_\gamma(||x_t||) + \gamma/2 \\
&= W_\gamma(||x_t - \phi_i + \phi_i||) + \gamma/2 \\
&\leq W_\gamma(||x_t - \phi_i|| + ||\phi_i||) + \gamma/2 \\
&\leq W_\gamma(2||x_t - \phi_i||) + W_\gamma(2||\phi_i||) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(2||\phi_i||) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(2|\,||\phi_i|| - |\phi_i|_p\,| + 2|\phi_i|_p)\gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(4|\,||\phi_i|| - |\phi_i|_p\,|) + W_\gamma(4|\phi_i|_p) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(4\delta_2) + W_\gamma(4|\phi_i|_p) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(4\delta_2) + W_\gamma(4|\phi_i - x_t + x_t|_p) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(4\delta_2) + W_\gamma(8|\phi_i - x_t|p) + W_\gamma(8|x_t|_p) + \gamma/2 \\
&\leq W_\gamma(2\delta_1) + W_\gamma(4\delta_2) + W_\gamma(8h^{1/p}\delta_1) + W_\gamma(8|x_t|_p) + \gamma/2 \\
&\leq W_\gamma(8|x_t|_p) + \gamma \\
&\leq \overline{W}_\gamma(|x_t|_p) + \gamma
\end{aligned}$$

where $W_\gamma(8r) = \overline{W}_\gamma(r)$. In this work i depends on t, but the result is uniform for all solutions. Thus, (iii) of Theorem 2 can be satisfied and the proof is complete. □

We can substantially reduce the conditions of Theorem 3 and obtain a result on asymptotic stability. For reference here, the zero solution of (1) is

(a) *stable* if for each $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta > 0$ such that $[||\phi|| < \delta, t \geq t_0]$ imply that $|x(t, t_0, \phi)| < \varepsilon$.

The zero solution of (1) is

(b) *asymptotically stable* if it is stable and if for each $t_0 \geq 0$ there is a $\xi > 0$ such that $||\phi|| < \xi$ implies that $|x(t, t_0, \phi)| \to 0$ as $t \to \infty$.

Theorem 4. *Suppose there is a wedge* W_1 *and a Liapunov functional* V *with*

(i) $$W_1(|\phi(0)|) \le V(t,\phi), \qquad V(t,0)=0,$$

and

(ii) $$V'_{(1)}(t,x_t) \le 0.$$

If, in addition, there are wedges, a locally integrable function $M : [0,\infty) \to [0,\infty)$*, a constant* $k \ge h$*, a sequence* $\{t_n\} \uparrow \infty$*, and, for each* $\gamma > 0$ *there is a wedge* W_γ *such that for* $\phi \in C_1$ *we have*

(iii) $$V(t,\phi) \le W_\gamma(||\phi||) + \gamma \quad \text{if } t = t_n,$$

(iv) $$|f(t,\phi)| \le M(t) \quad \text{if } t \in [t_n - k, t_n],$$

(v) $$\left|\int_{s_1}^{s_2} M(t)\,dt\right| \le W_4(|s_2 - s_1|) \quad \text{for } s_1, s_2 \in [t_n - k, t_n],$$

and

(vi) $$V'_{(1)}(t,x_t) \le -W_3(|x(t)|) \quad \text{for } t \in [t_n - k, t_n].$$

Then $x = 0$ *is* AS.

Proof. Now (i) and (ii) are the classical conditions for stability. Thus, for $\varepsilon = 1$ and a given $t_0 \ge 0$, find the δ of stability, select an arbitrary ϕ with $||\phi|| < \delta$, and consider a fixed solution $x(t) = x(t,t_0,\phi)$. Because of (i), (ii) an (iii), if there is any subsequence of t_n along which $||x_t|| \to 0$, then $x(t) \to 0$. Hence, we suppose there is a $\mu > 0$ with $||x_{t_n}|| \ge \mu$ for all n. Since $k \ge h$, there is then a point $s_n \in [t_n - k, t_n]$ with $|x(s_n)| \ge \mu$; either $|x(t)| \ge \mu/2$ for all $t \in [t_n - k, t_n]$, in which case $V(t,x_t)$ decreases by at least $kW_3(\mu/2)$ on $[t_n - k, t_n]$, or there is a $q_n \in [t_n - k, t_n]$ with $|x(q_n)| = \mu/2$ and $\mu/2 \le |x(t)|$ on the interval from s_n to q_n. In the latter case,

$$\mu/2 \le |x(s_n) - x(q_n)| = \left|\int_{s_n}^{q_n} f(s,x_s)\,ds\right|$$
$$\le \left|\int_{s_n}^{q_n} M(t)|,dt\right| \le W_4(|q_n - s_n|),$$

so that

$$|q_n - s_n| \geq W_4^{-1}(\mu/2)$$

and $V(t, x_t)$ decreases by at least

$$W_4^{-1}(\mu/2)W_3(\mu/2)$$

on the interval from q_n to s_n. Thus, in any case we have $V(t, x_t) \to -\infty$ as $t \to \infty$, a contradiction to

$$0 \leq V(t, x_t) \leq W_\gamma(||x_t||) + \gamma. \qquad \square$$

Example 2. Consider the scalar equation

$$\begin{aligned} x'(t) = &-[1 + (t+1)(|\sin(t+1)| - \sin(t+1))]x(t) \\ &+ t(|\sin t| - \sin t)x(t-1) \end{aligned}$$

with

$$V(t, x_t) = |x(t)| + \int_{t-1}^{t} (s+1)(|\sin(s+1)| - \sin(s+1))|x(s)|\, ds$$

so that

$$V'(t, x_t) \leq -|x(t)|.$$

Select $t_n = (2n+1)\pi - 1$ so that for $t \in [t_n - 1, t_n]$ we have

$$x'(t) = -x(t)$$

and

$$V(t, x_t) = |x(t)|.$$

The conditions of Theorem 4 are all satisfied.

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Department of Mathematics, Southern Illinois University, Carbondale, Illinois 62901

Department of Mathematics, University of Missouri-Rolla, Rolla, Missouri 65401