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CYCLIC OPERATORS ON SHIFT COINVARIANT SUBSPACES

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ABSTRACT. Let ψ be a Blaschke product on the unit disk and denote by P_ψ the orthogonal projection of H^2 onto $H^2\theta\psi H^2$. Necessary and sufficient conditions for the adjoint $\{P_\psi T_\phi P_\psi\}^*$ of the compression of an analytic Toeplitz operator $T_\phi : F \to \phi F$ to $H^2\theta\psi H^2$ to be cyclic are given.

1. Introduction. Let H^2 and H^∞ denote the standard Hardy spaces on the unit disk $\mathbf{D} \equiv \{z \in \mathbf{C} : |z| < 1\}$. The standard unilateral shift S on H^2 is given by $S : F(z) \to zF(z)$.

For each function ϕ in H^∞, the analytic Toeplitz operator T_ϕ with symbol ϕ is a bounded linear operator on H^2 defined by $T_\phi : F \to \phi F$. The commutant of the shift operator S on H^2 is precisely the algebra $\{T_\phi : \phi \text{ is in } H^\infty\}$ of analytic Toeplitz operators. In [**8**], Wogen showed that there exists a fixed function in H^2 which is a cyclic vector for the adjoint T_ϕ^* of every analytic Toeplitz operator T_ϕ having nonconstant symbol ϕ (see Wogen [**8**, Theorem 1, p. 163]).

Let ψ be an inner function on the unit disk and denote by P_ψ the orthogonal projection of H^2 onto $H^2\theta\psi H^2$. Let ϕ be any function in H^∞. The compression $P_\psi T_\phi P_\psi$ of the analytic Toeplitz operator T_ϕ to the shift coinvariant subspace $H^2\theta\psi H^2$ is given by

$$(1)\qquad P_\psi T_\phi P_\psi : F \to P_\psi(\phi F).$$

For $\phi(z) = z$, the operator $S_\psi \equiv P_\psi T_\phi P_\psi$ is the compression of the shift operator S to $H^2\theta\psi H^2$. Sarason has shown that a bounded linear operator T on $H^2\theta\psi H^2$ commutes with S_ψ if and only if T assumes the form (1) for some function ϕ in H^∞ (see Sarason [**5**, Theorem 1, p. 179]). Nikolskii points out that it would be of interest to determine those functions ϕ in H^∞ for which $P_\psi T_\phi P_\psi$ and $\{P_\psi T_\phi P_\psi\}^*$ are cyclic (see [**4**]). Incidental results have been obtained for the case

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$\psi(z) = e^{(z+1)/(z-1)}$ (see [**6**]) and ψ an arbitrary singular inner function (see [**7**]).

The purpose of this paper is to give necessary and sufficient conditions for the adjoint $\{P_\psi T_\phi P_\psi\}^*$ of an operator in the commutant of the compression S_ψ of the shift operator to $H^2 \theta \psi H^2$ to be cyclic for the case ψ a Blaschke product.

Throughout this paper, ψ will denote a Blaschke product on the unit disk having distinct zeros $\{\lambda_n\}$ with multiplicities $\{m_n\}$ respectively and ϕ will denote an arbitrary function in H^∞.

For a complex number λ in the unit disk $\mathbf{D}$, the reproducing kernel $K_\lambda(z)$ for H^2 is given by $K_\lambda(z) \equiv 1/(1-\bar{\lambda}z)$ for all z in $\mathbf{D}$ and has the property that $F(\lambda) = \langle F, K_\lambda \rangle_{H^2}$ for each function F in H^2. An orthonormal basis $\{e_n\}$ for $H^2 \theta \psi H^2$ is provided by the Malmquist-Walsh lemma (see, for instance, Nikolskii [**4**, Malmquist-Walsh lemma, p. 116] or the proof of Theorem 3.1 in Ahern and Clark [**1**, p. 337]). The functions e_n are given explicitly as follows. Write

$$\psi(z) \equiv \prod_{k=1}^{\infty} \frac{\overline{a_k}}{|a_k|} \cdot \frac{z - a_k}{1 - \overline{a_k} z}$$

where $a_k = \lambda_1$ for $1 \le k \le m_1$, $a_k = \lambda_2$ for $m_1 + 1 \le k \le m_2, \dots$. For $\lambda_k = 0$, the term $\overline{a_k}/|a_k|$ is taken to be one. Define $\mathcal{B}_1(z) \equiv 1$, and for each integer n greater than one, let

$$\mathcal{B}_n(z) \equiv \prod_{k=1}^{n-1} \frac{\overline{a_k}}{|a_k|} \cdot \frac{z - a_k}{1 - \overline{a_k} z}.$$

Then the n^{th} basis element e_n for $H^2 \theta \psi H^2$ is given by $e_n(z) = \{1 - |a_n|^2\}^{1/2} \mathcal{B}_n(z) K_{a_n}(z)$.

2. A matrix representation for $\{P_\psi T_\phi P_\psi\}^*$. Let ψ be a Blaschke product on the unit disk having distinct zeros $\{\lambda_n\}$ with multiplicities $\{m_n\}$ respectively, and let ϕ be an arbitrary function in H^∞.

We show that the matrix representation for $\{P_\psi T_\phi P_\psi\}^*$ with respect to the basis $\{e_n\}$ assumes the form

$$\begin{pmatrix} T_1 & * & * & \cdots \\ & T_2 & * & \cdots \\ 0 & & \ddots & \end{pmatrix} \tag{2}$$

where each block T_n on the main diagonal is of the form

$$\begin{pmatrix} \alpha_n & \beta_n & * & \cdots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ 0 & & & \ddots & \beta_n \\ & & & & \alpha_n \end{pmatrix}$$

where α_n and β_n are complex numbers.

Theorem 1. A matrix representation for $\{P_\psi T_\phi P_\psi\}^*$.*Let ψ be a Blaschke product on the unit disk having distinct zeros $\{\lambda_n\}$ with multiplicities $\{m_n\}$, respectively. Let ϕ be any function in H^∞. Then the matrix representation for $\{P_\psi T_\phi P_\psi\}^*$ with respect to the basis $\{e_n\}$ for $H^2 \theta \psi H^2$ assumes the form*

$$\{P_\psi T_\phi P_\psi\}^* = \begin{pmatrix} T_1 & * & \cdots & \\ & T_2 & * & \cdots \\ & & \ddots & \vdots \\ 0 & & & \end{pmatrix}$$

where the n^{th} block T_n on the main diagonal is an $m_n \times m_n$ matrix given by

$$T_n = \begin{pmatrix} \alpha_n & \beta_n & * & \cdots & * \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & * \\ 0 & & & \ddots & \beta_n \\ & & & & \alpha_n \end{pmatrix}$$

where $\alpha_n = \overline{\phi(\lambda_n)}$ and $\beta_n = \{1 - |\lambda_n|^2\}(\overline{\lambda_n}/|\lambda_n|)\overline{\phi'(\lambda_n)}$. If $\lambda_n = 0$, then the term $\overline{\lambda_n}/|\lambda_n|$ in β_n does not occur.

Proof. For each positive integer n, define

$$\mathcal{H}_n = \operatorname{span}\left\{ e_k : \sum_{j=1}^{n-1} m_j < k \leq \sum_{j=1}^{n} m_j \right\}$$

and denote by $P_{\mathcal{H}_n}$ the orthogonal projection of $H^2\theta\psi H^2$ onto $\mathcal{H}_n$. Since $\mathcal{H}_n$ has dimension m_n, the operator $T_n \equiv P_{\mathcal{H}_n}\{P_\psi T_\phi P_\psi\}^* P_{\mathcal{H}_n}$ acting on $\mathcal{H}_n$ is an $m_n \times m_n$ matrix.

We show that the main diagonal entries of the n^{th} block T_n are all equal to $\overline{\phi(\lambda_n)}$. Let n be any positive integer and let j be any integer in $\{1,2,3,\dots,m_n\}$. Define $N \equiv j + \sum_{k=1}^{n-1} m_k$ and let ψ_N denote the analytic continuation of ψ/B_N to the unit disk. So $a_N = \lambda_n$. The j^{th} main diagonal entry of the n^{th} block T_n is given by $\langle \{P_\psi T_\phi P_\psi\}^* e_N, e_N\rangle_{H^2\theta\psi H^2}$. There exists a unique function H_N in H^2 such that $\phi e_N + \psi H_N\{1-|a_N|^2\}^{1/2}$ is in $H^2\theta\psi H^2$. Since B_N is an inner function and $\psi_N(a_n) = 0$, we have that

$$\begin{aligned}
&\langle \{P_\psi T_\phi P_\psi\}^* e_N, e_N\rangle_{H^2\theta\psi H^2}\\
&\qquad = \langle e_N, \phi e_N + \psi H_N\{1-|a_N|^2\}^{1/2}\rangle_{H^2}\\
&\qquad = \{1-|a_N|^2\}\langle B_N K_{a_N}, B_N(\phi K_{a_N} + \psi_N H_N)\rangle_{H^2}\\
&\qquad = \{1-|a_N|^2\}\langle K_{a_N}, \phi K_{a_N} + \psi_N H_N\rangle_{H^2}\\
&\qquad = \{1-|a_N|^2\}\cdot\{\overline{\phi(a_N)/(1-|a_N|^2) + \psi_N(a_N)H_N(a_n)}\}\\
&\qquad = \overline{\phi(a_N)} = \overline{\phi(\lambda_n)}.
\end{aligned}$$

We show that the super diagonal entries of the n^{th} block T_n are all equal to $\{1-|\lambda_n|^2\}(\overline{\lambda_n}/|\lambda_n|)\overline{\phi'(\lambda_n)}$ where $\overline{\lambda_n}/|\lambda_n|$ is taken to be 1 if $\lambda_n = 0$. Let n be any positive integer and let j be any integer in $\{2,3,4,\dots,m_n\}$. Define $N \equiv j + \sum_{k=1}^{n-1} m_k$. So $a_N = a_{N-1} = \lambda_n$. The $(j, j+1)$-entry of T_n is given by $\langle \{P_\psi T_\phi P_\psi\}^* e_N, e_{N-1}\rangle_{H^2\theta\psi H^2}$. There exists a unique function H_{N-1} in H^2 such that $\phi e_{N-1} + \psi H_{N-1}\{1-|a_{N-1}|^2\}^{1/2}$ is in $H^2\theta\psi H^2$. Since $\mathcal{B}_N$ is an inner function and $\langle zK_\lambda^2, F\rangle_{H^2} = \overline{F'(\lambda)}$ for all functions F in H^2 and all complex numbers λ in the unit disk (see, for instance, Nikolskii [**4**, p. 33(2)]), we have that

$$\begin{aligned}
&\langle \{P_\psi T_\phi P_\psi\}^* e_N, e_{N-1}\rangle_{H^2\theta\psi H^2}\\
&\quad = \langle e_N, \phi e_{N-1} + \psi H_{N-1}\{1-|a_{N-1}|^2\}^{1/2}\rangle_{H^2}\\
&\quad = \{1-|a_N|^2\}^{1/2}\{1-|a_{N-1}|^2\}^{1/2}\\
&\qquad \cdot \langle \mathcal{B}_{N-1}\frac{\overline{a_{N-1}}}{|a_{N-1}|}\frac{z-a_{N-1}}{1-\bar{a}_{N-1}z}K_{a_N}, \mathcal{B}_{N-1}(\phi K_{a_{N-1}} + \psi_{N-1}H_{N-1})\rangle_{H^2}
\end{aligned}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \langle (z - a_N) K_{a_N}^2, \phi K_{a_N} + \psi_{N-1} H_{N-1} \rangle_{H^2}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \{ \langle z K_{a_N}^2, \phi K_{a_N} + \psi_{N-1} H_{N-1} \rangle_{H^2} - \langle K_{a_N}^2, \overline{a_N} (\phi K_{a_N} + \psi_{N-1} H_{N-1}) \rangle_{H^2} \}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \{ \langle z K_{a_N}^2, \phi K_{a_N} + \psi_{N-1} H_{N-1} \rangle_{H^2} - \langle z K_{a_N}^2, z \overline{a_N} (\phi K_{a_N} + \psi_{N-1} H_{N-1}) \rangle_{H^2} \}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \{ \langle z K_{a_N}^2, (1 - \overline{a_N} z)(\phi K_{a_N} + \psi_{N-1} H_{N-1}) \rangle_{H^2}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \{ \overline{\frac{d}{dz} (1 - \overline{a}_N z)(\phi K_{a_N} + \psi_{N-1} H_{N-1}) \}} |_{z = a_N}$$

$$= \{1 - |a_N|^2\} \frac{\overline{a_N}}{|a_N|} \overline{\phi'(a_N)}$$

$$= \{1 - |\lambda_n|^2\} \frac{\overline{\lambda_n}}{|\lambda_n|} \overline{\phi'(\lambda_n)}. \qquad \square$$

3. Cyclicity of $\{P_\psi T_\phi P_\psi\}^*$. We give necessary and sufficient conditions for $\{P_\psi T_\phi P_\psi\}^*$ to be cyclic. The following will be of use.

Lemma 1. *Let T be a bounded linear operator on a separable Hilbert space $\mathcal{H}$. Suppose that there exists a basis for $\mathcal{H}$ such that the corresponding matrix representation for T assumes the form* (2). *If $\alpha_m \neq \alpha_n$ for $m \neq n$ and $\beta_n \neq 0$ for all positive integers n, then T is cyclic.*

Proof. Let T assume the form (2). Define the block diagonal matrix D on $\mathcal{H}$ by

$$D = \begin{pmatrix} T_1 & & 0 \\ & T_2 & \\ 0 & & \ddots \end{pmatrix}.$$

Since $\sigma T_m \cap \sigma T_n = \{\alpha_m\} \cap \{\alpha_n\} = \varnothing$, there exists a bounded linear operator X on $\mathcal{H}$ which is one-to-one and has dense range such that $TX = XD$ (see Davidson and Herrero [**2**, Proposition 2.5, p. 35]). By Lemma 3.5 of Herrero, Larson, and Wogen [**3**], it suffices to show that D is cyclic.

Since each β_n is nonzero, the operator T_n has cyclic vector $x_n \equiv (0, \dots, 0, 1)$. Define $x \equiv \oplus_{n=1}^{\infty} \delta_n x_n$, where $\{\delta_n\}$ is a sequence of positive numbers to be specified later. There exists a collection $\{p_{n,k}\}$ of polynomials such that for each pair of positive integers n and k, $p_{n,k}(T_k)$ is the identity operator, and for each j in $\{1, 2, \dots, k-1, k+1, \dots, n\}$, $p_{n,k}(T_j)$ is the zero operator. So $p_{n,k}(T)x = x_k + \oplus_{j=n+1}^{\infty} p_{n,j}(T_j)\delta_j x_j$. The sequence $\{\delta_n\}$ can be chosen to decrease to zero rapidly enough so that for each fixed positive integer k,

$$||p_{n,k}(T)x - x_k|| = \left\| \oplus_{j=n+1}^{\infty} p_{n,j}(T_j)\delta_j x_j \right\|$$

tends to zero as n tends to infinity. Hence, each x_k is in $\overline{\{p(T)x : p \text{ is a polynomial}\}}$ and so x is a cyclic vector for D. The result follows. □

If a bounded linear operator T on a separable Hilbert space $\mathcal{H}$ is cyclic, then T is cyclic on each invariant subspace of $\mathcal{H}$ having an algebraic complement which is also invariant for T.

Lemma 2. *Let T be a bounded linear operator on a separable Hilbert space $\mathcal{H}$. If $\mathcal{N}$ is a closed invariant subspace of T with an invariant complement $\mathcal{M}$, then $T \mid \mathcal{N}$ is cyclic on $\mathcal{N}$.*

Proof. Let x be any cyclic vector for T on $\mathcal{H}$. Since $\mathcal{M}$ is a complementary subspace of $\mathcal{N}$, there exist unique vectors x_1 in $\mathcal{N}$ and x_2 in $\mathcal{M}$ such that $x = x_1 + x_2$. For each positive integer n, we have that $T^n x = T^n x_1 + T^n x_2$. Since $T\mathcal{N} \subseteq \mathcal{N}$ and $T\mathcal{M} \subseteq \mathcal{M}$, $T^n x_2$ is always in $\mathcal{M}$ and so x_1 is a cyclic vector for $T \mid \mathcal{N}$ on $\mathcal{N}$. □

Using Theorem 1, we show that $\{P_\psi T_\phi P_\psi\}^*$ is cyclic if and only if ϕ is one-to-one on the set of zeros of ψ and has nonvanishing derivative on the set of multiple zeros of ψ.

Theorem 2. Cyclicity of $\{P_\psi T_\phi P_\psi\}^*$. *Let ψ be a Blaschke product on the unit disk having distinct zeros $\{\lambda_n\}$ with multiplicities $\{m_n\}$, respectively. Let ϕ be any nonconstant function in H^∞. Then $\{P_\psi T_\phi P_\psi\}^*$ is cyclic if and only if*

(i) $\phi(\lambda_m) \neq \phi(\lambda_n)$ *for* $m \neq n$

and

(ii) $\phi'(\lambda_n) \neq 0$ *whenever* $m_n \geq 2$.

Proof. Conditions (i) and (ii) are sufficient for $\{P_\psi T_\phi P_\psi\}^*$ to be cyclic by Theorem 1 and Lemma 1.

Conversely, let $\{P_\psi T_\phi P_\psi\}^*$ be cyclic. We assume that there exist distinct positive integers m and n such that $\phi(\lambda_m) = \phi(\lambda_n)$ and deduce a contradiction. By reordering the zeros of ψ, we may assume without loss of generality that there exists a positive integer N such that $\phi(\lambda_k) = \phi(\lambda_1)$ for k in $\{1, 2, \dots, N\}$ and $\phi(\lambda_k) \neq \phi(\lambda_1)$ for each integer k greater than N. By hypothesis, N is at least two. Define $\mathcal{M} \equiv \operatorname{span}\{\operatorname{Ker}(S_\psi^* - \bar{\lambda}_k)^{m_k} : 1 \leq k \leq N\}$ and $\mathcal{N} = \operatorname{span}\{\operatorname{Ker}(S_\psi^* - \overline{\lambda_k})^{m_k} : k \geq N+1\}$. Since the operators $\{P_\psi T_\phi P_\psi\}^*$ and S_ψ^* commute, the subspaces $\mathcal{M}$ and $\mathcal{N}$ are invariant for $\{P_\psi T_\phi P_\psi\}^*$. Since $\operatorname{Ker}(S_\psi^* - \bar{\lambda}_k)^{m_k} = \operatorname{span}\{z^j K_{\lambda_k}^{j+1} : 0 \leq j < m_k\}$ for each positive integer k (see Corollary 3 of [**4**, p. 82]), we have that the set of positive integers

$$\mathcal{M} + \mathcal{N} = \operatorname{span}\{z^j K_{\lambda_k}^{j+1} : k \in \mathbf{N}, 0 \leq j < m_k\}$$

$$= \operatorname{span}\{e_n : n \in \mathbf{N}\} = H^2 \theta \psi H^2$$

(see Corollary 5 of [**4**, p. 83]). Moreover, $\mathcal{M} \cap \mathcal{N} = \{0\}$ so that $\mathcal{M}$ is a complement for $\mathcal{N}$ which is invariant for $\{P_\psi T_\phi P_\psi\}^*$.

By Theorem 1, the matrix representation for $\{P_\psi T_\phi P_\psi\}^* \mid \mathcal{N}$ assumes the form

$$\begin{pmatrix} \overline{\phi(\lambda_1)} & & * & & & & & & & \\ & \ddots & & & & & & * & & \\ 0 & & \overline{\phi(\lambda_1)} & & & & & & & \\ & & & \overline{\phi(\lambda_2)} & & * & & & & \\ & & & & \ddots & & & & & \\ & & & 0 & & \overline{\phi(\lambda_2)} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \overline{\phi(\lambda_N)} & & * \\ & & & & & & & & \ddots & \\ & \mathbf{0} & & & & & & 0 & & \overline{\phi(\lambda_N)} \end{pmatrix}.$$

Since N is at least 2 and $\overline{\phi(\lambda_1)} = \overline{\phi(\lambda_2)} = \cdots = \overline{\phi(\lambda_N)}$, the eigenspace corresponding to the eigenvalue $\overline{\phi(\lambda_1)}$ is at least two-dimensional. Hence $\{P_\psi T_\phi P_\psi\}^* \mid \mathcal{N}$ is not cyclic, contradicting Lemma 2.

A similar argument holds if there exists a positive integer n for which $\phi'(\lambda_n) = 0$ with $m_n \geq 2$. $\square$

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REFERENCES

1. P.R. Ahern and D.N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.

2. K.R. Davidson and D.A. Herrero, *The Jordan form of a bitriangular operator*, J. Funct. Anal. **94** (1990), 27–73.

3. D.A. Herrero, D.R. Larson and W.R. Wogen, *Semitriangular operators*, to appear.

4. N.K. Nikolskii, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschaften, Band 273, Springer-Verlag, New York, 1980.

5. D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. **127** (1967), 179–203.

6. S.M. Seubert, *Dissipative finite convolution operators*, Houston J. Math. **16** (1990), 573–581.

7. ———, *Dissipative operators on quotient spaces of H^2*, to appear.

8. W.R. Wogen, *On some operators with cyclic vectors*, Indiana Univ. Math. J. **27** (1978), 163–171.

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