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AN ANALOGUE OF THE STABILIZATION MAP FOR REGULAR Z_p ACTIONS

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ABSTRACT. Several authors have used the stabilization technique introduced by Boardman to analyze the fixed point structure of a smooth action of a compact Lie group on smooth manifolds. Using regular $\mathbf{Z}_p$ actions, defined herein, and the bordism groups of regular $\mathbf{Z}_p$ actions, we show that if there is a regular action of $\mathbf{Z}_p$ on a smooth manifold and the manifold is not a boundary in the Thom oriented cobordism group mod the ideal of those elements all of whose Pontrjagin numbers are 0 mod p, then the manifold must have some component of its fixed point set of dimension at least half that of the ambient manifold. As a coda to this, we study the relationship between this stabilization map and the factorization of the cyclotomic polynomials over $\mathbf{Z}_p$.

1. Introduction. We want to know how the size of the fixed point set of an action of a group on a manifold affects the bordism class of the ambient manifold in the appropriate bordism ring. For example, we know that if $\mathbf{Z}_2$ acts freely on M, or if $(\mathbf{Z}_2)^k$ acts on M without fixed points, then the manifold bounds in the Thom unoriented bordism ring, $\mathfrak{N}_*$. Also, Ossa [**14**] has shown that if S^1 acts without fixed points on a closed smooth oriented manifold then a suitable multiple of this manifold bounds equivariantly, and the manifold represents a torsion element in Ω_*.

What can be proved if the fixed point set is not empty? The first result appears in Conner-Floyd [**3**, Theorem 27.1], and this dealt with involutions.

Theorem 1.1. *For all $k \in \mathbf{Z}$, $k \geq 0$, there exists $\phi(k) \in \mathbf{Z}$, $\phi(k) \geq 0$, such that if (T, M^n) is a smooth involution on a closed manifold with $\dim(M^{\mathbf{Z}_2}) \leq k$ and $n = \dim(M) > \phi(k)$, then M^n bounds in $\mathfrak{N}_*$.*

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They inquired as to a suitable estimate for $\phi(k)$. Boardman [**2**] showed that one can take $\phi(k) = 5k/2$ and that this is the best possible general result; *i.e.*, there exists a nonbounding manifold of dimension 5 on which there is an action of $\mathbf{Z}_2$ which has a nonbounding component of its fixed point set of dimension 2. In 1978 Kosniowski and Stong [**13**] and, independently, this author [**15**] strengthened this result by showing that the manifold in question actually bounds equivariantly.

$\mathbf{Z}_2$ actions are nice objects with which to work, having a particularly nice representation structure. One direction that was followed in this direction was to look at semifree S^1 actions—actions in which the isotropy group is either the whole group or the identity subgroup. These are a particularly nice oriented analogues of involutions on unoriented manifolds. Similar questions were considered by Kawakubo [**8**] and the author [**16**].

Theorem 1.2. *Let T be a smooth orientation preserving semifree S^1-action on the closed oriented smooth manifold M^n. Let k be the maximum of the nonbounding components of the fixed point set of T on M^n. If $[M^n] \neq 0$ in $\Omega \otimes \mathbf{Q}$ then $k \geq n/2$.*

The oriented analogue of the example in the unoriented case is still present. There is a ten-dimensional oriented manifold with semifree S^1-action with a four-dimensional fixed point set. This manifold is a torsion element though.

This, of course, skips a large class of groups about which a lot has become known. An analogous approach to orientation preserving $\mathbf{Z}_p$-actions for p an odd prime would seem appropriate. Much has been done in the area of fixed set dimension and what this determines about the ambient manifold by Tom Dieck [**5, 6, 7**] and Kosniowski [**10, 11**]. The approaches are quite distinct. The elegant work of Tom Dieck is by an analysis *via* equivariant homotopical cobordism, and that of Kosniowski is arrived at from the direction of the *G-Signature Theorem* approach.

The purpose of this paper is to study the case of regular $\mathbf{Z}_p$ actions, defined in Section 2. Suffice it to say that if one restricts a semifree S^1-action to the action of $\mathbf{Z}_p$ that is naturally included—as $\mathbf{Z}_p$ is included as the p-th roots of unity—then one has a regular $\mathbf{Z}_p$ action.

In Section 3 we analyze the bordism group of regular $\mathbf{Z}_p$-actions on manifolds, $R_*(\mathbf{Z}_p)$ and calculate how the image of the bordism group of semifree S^1-actions, $\mathrm{SF}_*(S^1)$ sits in it. Next in Section 4 an endomorphism Γ of degree $+2$ is defined on $\widehat{R}_*(\mathbf{Z}_p)$, a quotient of $R_*(\mathbf{Z}_p)$. We show that it satisfies certain nice properties analogous to the map defined in [**1, 3, 12, 15** and **16**].

In Section 5 we study an ideal of $\widehat{R}_*(\mathbf{Z}_p)$ described by Γ and use it to set up the machinery to prove the main theorem in Section 6. This is completely analogous to the proof technique in [**15**] and [**16**]. Our main result is:

Theorem 1.3. *Let T be a smooth orientation preserving regular $\mathbf{Z}_p$ action on M^n and let k be the fixed point dimension of T on M^n. If $[M^n] \neq 0$ in $\Omega/I(p)$, then $k \geq n/2$.*

This says that some component of the fixed point set has dimension at least half that of the ambient manifold.

In Section 7 we look at the connection between the mapping Γ and its role in the algebra $\widehat{R}_*(\mathbf{Z}_p)$ and the factorization of $t^n - 1$ by the cyclotomic polynomials in one variable over the integers $\mod p$.

All manifolds are assumed to be smooth. All mappings of our manifolds are assumed to be smooth and orientation preserving. If M^n is a closed oriented manifold, then we will use $-M^n$ to denote the same manifold given the opposite orientation. Throughout the remainder p will denote an odd prime.

2. Regular Z_p-actions. Let (T, M^n) be a smooth action of $\mathbf{Z}_p$ on the smooth closed oriented manifold M^n; *i.e.*, T is a smooth map of M^n to itself of period p, $T^p = 1_M$. We identify $\mathbf{Z}_p$ with the p^{th} roots of unity $\{\exp(2\pi ik/p) \mid k = 0, 1, 2, \ldots, p-1\}$. Let $F(T)$ denote the fixed point set of T on M^n—we shall use F when there is no possibility of confusing the author. Each component of the fixed point set is orientable. The normal bundle to each component, F_i, of the fixed point set has a canonical $\mathbf{Z}_p$-invariant decomposition into subbundles

$$\lambda_i = \sum_{k=1}^{(p-1)/2} \lambda(k)$$

where $E(\lambda(k))$, the total space, has a unique complex structure so that $\mathbf{Z}_p$ acts by multiplication in the fibers by $\exp(2\pi ik/p)$.

We can then canonically orient F_i so that the orientation of a fiber followed by that of F_i gives the orientation of $E(\lambda_i)$, where $E(\lambda_i)$ has the orientation of a tubular neighborhood of F_i in M^n.

When $\lambda_i = \lambda(k)$ for some *fixed* k and for all components, we say that the action of $\mathbf{Z}_p$ is *regular*. Notice that we are requiring that there be only one representation of the group $\mathbf{Z}_p$ which is the same at every fixed point. For the purposes of this paper and to make the analysis look nicer, we will take $k = 1$ throughout the paper. The results hold true for any choice of k. Regular actions were considered by Kawakubo in [**8**].

Following the now standard outline from Conner-Floyd, [**3**], it will come as no surprise that we can define a bordism relation for regular $\mathbf{Z}_p$-actions. From this point onward all actions of $\mathbf{Z}_p$ are assumed to be regular. We say that the regular actions (T_1, M_1^n) and (T_2, M_2^n) are *isomorphic* if there is an equivariant orientation preserving diffeomorphism of M_1^n onto M_2^n. A regular $\mathbf{Z}_p$ action (T, M^n) *bounds* if there is a regular $\mathbf{Z}_p$ action (τ, W^{n+1}) on a compact manifold with boundary so that $(\tau|_\partial W^{n+1})$ is isomorphic to (T, M^n). Given a regular $\mathbf{Z}_p$ action (T, M^n), define $-(T, M^n) = (T, -M^n)$ to be (T, M^n) with the opposite orientation. Two regular $\mathbf{Z}_p$ actions are *bordant* if the disjoint union $(T_1, M_1^n) \sqcup -(T_2, M_2^n)$ bounds. This clearly forms an equivalence relation and denote the equivalence, or bordism, class of (T, M^n) by $\{T, M^n\}$. The collection of these bordism classes forms an abelian group under disjoint union, and will be denoted by $R_n(\mathbf{Z}_p)$. The direct sum

$$R_*(\mathbf{Z}_p) = \sum_{n=0}^{\infty} R_n(\mathbf{Z}_p)$$

is a graded ring with the product given by the Cartesian product:

$$\{T_1, M_1\} \times \{T_2, M_2\} = \{T_1 \times T_2, M_1 \times M_2\}.$$

We will also use the augmentation map $\varepsilon\colon R_n(\mathbf{Z}_p) \to \Omega_n$ given by forgetting the action: $\varepsilon(\{T, M^n\}) = [M^n]$.

Using the notation of [**3**] we let $\Omega_n(\mathbf{Z}_p)$ denote the bordism group of *free* $\mathbf{Z}_p$ actions. Any free action of $\mathbf{Z}_p$ is clearly regular, so if $[T, M^n] \in \Omega(\mathbf{Z}_p)$ then it also represents an element in $R_n(\mathbf{Z}_p)$. Let $\alpha\colon \Omega_n(\mathbf{Z}_p) \to R_n(\mathbf{Z}_p)$ denote this correspondence, which is clearly a homomorphism.

For the regular $\mathbf{Z}_p$ action (T, M^n) let F^{n-2k} denote the union of the $2k$ codimensional components of the fixed point set of T oriented as mentioned above. Let $\xi_k\colon E_k \to F^{n-2k}$ denote the normal complex bundle to F^{n-2k}. Define a map $\nu\colon R_n(\mathbf{Z}_p) \to \sum_k \Omega_{n-2k}(BU(k))$ by

$$\nu(\{T, M^n\}) = \sum_k [\xi_k \to F^{n-2k}] = \sum_k [\xi_k].$$

Note that since we have fixed the actions to have only one representation, we have tremendously simplified the fixed point data. In general there are $(p-1)/2$ conjugate representations that must be considered. This map ν is the standard fixed point data map of Conner-Floyd [**3**] restricted to regular actions. It, too, is a homomorphism.

Let $\partial_1\colon \sum_{k=0}^{n/2} \Omega_{n-2k}(BU(k)) \to \Omega_{n-1}(\mathbf{Z}_p)$ denote the usual boundary map. Let $\xi_k\colon E_k \to B^{n-2k}$ be a k dimensional complex vector bundle. $S(\xi_k)$ denotes the unit sphere bundle of ξ_k, the set of all vectors in each fiber of norm 1. $\mathbf{Z}_p$ acts on $S(\xi_k)$ by restriction of the complex multiplication in each fiber to the unit sphere. Denote this action on $S(\xi_k)$ by φ_k. Clearly this action is free, so we set $\partial'([\xi_k]) = [\varphi_k, S(\xi_k)]$. Define ∂_1 to be the sum of the ∂' and put $\partial_1(\Omega_n) = \partial_1(\Omega_n(BU(0))) = 0$.

There is a canonical augmentation map $\varepsilon\colon \Omega_n(\mathbf{Z}_p) \to \Omega_n$ given by $\varepsilon([T, M^n]) = [M^n/\mathbf{Z}_p] = [M^n/T] \in \Omega_n$. Let $\widetilde{\Omega}(\mathbf{Z}_p) = \ker \varepsilon$. This map splits by a mapping $s\colon \Omega_n \to \Omega_n(\mathbf{Z}_p)$ given by

$$s([M^n]) = [\varphi, \underbrace{M^n \sqcup \cdots \sqcup M^n}_{p \text{ copies}}] = [\varphi, M^n \times \mathbf{Z}_p]$$

where φ acts by permuting the copies of M^n cyclically. Easily, $\varepsilon s = 1$ and $\Omega_n(\mathbf{Z}_p)$ splits as

$$\Omega_n(\mathbf{Z}_p) = \Omega_n \oplus \widetilde{\Omega}_n(\mathbf{Z}_p).$$

The following theorem is found in Kawakubo, [**8**].

Theorem 2.1 [Kawakubo]. *The sequence*

$$0 \longrightarrow \Omega_n \xrightarrow{\alpha|} R_n(\mathbf{Z}_p) \xrightarrow{\nu} \sum_{k=0}^{n/2} \Omega_{n-2k}(BU(k)) \xrightarrow{\partial_1} \widetilde{\Omega}_{n-1}(\mathbf{Z}_p) \longrightarrow 0$$

is exact.

Note that the mapping $\alpha|$ in the above sequence is nothing but $\alpha[M^n] = \{\varphi, M^n \times \mathbf{Z}_p\}$ where φ permutes the factors of M^n cyclically.

Now, let us consider the bordism groups of S^1-actions as in [**12**]. We let $\mathrm{SF}_n(S^1)$ denote the bordism group of semifree S^1-actions on n-dimensional manifolds. Analogous to the above regular $\mathbf{Z}_p$-actions there is a natural homomorphism

$$\tilde{\nu}\colon\ \mathrm{SF}_n(S^1) \to \sum_k \Omega_{n-2k}(BU(k))$$

and

$$\tilde{\partial}\colon\ \sum_k \Omega_{n-2k}(BU(k)) \to \Omega_{n-1}(S^1),$$

where $\Omega_n(S^1)$ to denotes the bordism group of free S^1-actions on n-dimensional manifolds. We find the following theorem no surprise, see [**18**].

Theorem 2.2 [Uchida]. *The sequence*

$$0 \longrightarrow SF_n(S^1) \xrightarrow{\tilde{\nu}} \sum_{k=0}^{n/2} \Omega_{n-2k}(BU(k)) \xrightarrow{\tilde{\partial}} \Omega_{n-1}(S^1) \longrightarrow 0$$

is split exact.

Since we have chosen $k = 1$ for our representation of $\mathbf{Z}_p$ at every fixed point, when we restrict a semifree S^1 action to the subgroup $\mathbf{Z}_p$

acting as the p^{th} roots of unity, we get a regular $\mathbf{Z}_p$-action. Thus, we have natural restriction homomorphisms $\rho_1\colon \mathrm{SF}_n(S^1) \to R_n(\mathbf{Z}_p)$ and $\rho_2\colon \Omega_n(S^1) \to \tilde{\Omega}_n(\mathbf{Z}_p)$ for free actions. Putting Theorems 2.1 and 2.2 together with these maps gives us the following commutative diagram.

$$
\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & \mathrm{SF}_n(S^1) & \xrightarrow{\tilde{\nu}} & \displaystyle\sum_{k=0}^{n/2} \Omega_{n-2k}(BU(k)) & \xrightarrow{\tilde{\partial}} & \Omega_{n-1}(S^1) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \rho_1} & & \Downarrow & & \downarrow{\scriptstyle \rho_2} & & \\
0 & \longrightarrow & \Omega_n & \xrightarrow{\alpha|} & R_n(\mathbf{Z}_p) & \xrightarrow{\nu} & \displaystyle\sum_{k=0}^{n/2} \Omega_{n-2k}(BU(k)) & \xrightarrow{\partial_1} & \widetilde{\Omega}_{n-1}(\mathbf{Z}_p) & \longrightarrow & 0
\end{array}
$$

Figure 2.3

3. Analysis of $R_n(Z_p)$. Let $I(p) \subset \Omega_*$ denote the ideal consisting of those bordism classes all of whose Pontrjagin numbers are divisible by p. From [**4**, Theorem 50.3] we get the following theorem.

Theorem 3.1. *There exists a sequence of regular $\mathbf{Z}_p$-actions (T, M_0^{2k}) for $k > 0$ that satisfy*

(1) $F(T, M_0^{2n}) = \sqcup_{k=1}^{n}(-1)^{k-l} M_0^{2(n-k)}$, *where M_0^0 is p distinct points.*

(2) $\{[M_0^0], [M_0^4], [M_0^8], \dots\}$ *generates $I(p)$.*

(3) $\{\tilde{\partial}\nu(\{T, M_0^{2k}\})\}_{k>0}$ *is an Ω_* base for the free Ω_*-module* $\ker(\rho_2)$.

Let $\mathfrak{I}_*$ be the submodule of $R_*(\mathbf{Z}_p)$ generated by $\{\{T, M_0^{2k}\}\}_{k>0}$. The following lemma is due to Kawakubo [**9**], but its proof is included here for local completeness.

Lemma 3.2. *There is a direct sum decomposition*

$$R_n(\mathbf{Z}_p) = \mathfrak{I}_n \oplus \rho_1(\mathrm{SF}_n(S^1)) \oplus \alpha\Omega_n.$$

Proof. We know that the top sequence in the commutative diagram 2.3 splits. The splitting map is easily defined $s_1\colon \Omega_{n-1} \to \sum_k \Omega_{n-2k}(BU(k))$. $\Omega_*(S^1)$ is a free Ω_*-module on $\alpha_{2k-1} = [\varphi, S^{2k-1}]$, $k \geq 1$, where $S^{2k-1} \subset \mathbf{C}^k$ as the elements of norm 1 and S^1 act by multiplication in each factor. If $m \in \Omega_{n-1}(S^1)$ then we can write m as

$$m = \sum_{k=0}^{(n-1)/2} [V^{2k}]\alpha_{n-2k-1}.$$

Then define

$$s_1(m) = \sum_{k=0}^{(n-1)/2} [V^{2k}]\theta_0^{n/2-k},$$

where $\theta_0^{n/2-k}$ is the trivial complex $(n/2-k)$-plane bundle over a point.

Let $x \in R_n(\mathbf{Z}_p)$. Then $0 = \partial\nu(x) = \rho_2(\tilde{\partial}\nu(x))$ so that $\tilde{\partial}\nu(x) \in \ker \rho_2$. From [**4**] we know that we can use the manifolds postulated in Theorem 3.1 to write anything in the kernel of ρ_2 as

$$\tilde{\partial}\nu(x) = \sum_{k=0}^{n/2+1} [M_0^{2k}]\alpha_{n-2k-1}.$$

Then, $s_1\tilde{\partial}\nu(x) = \sum_{k=0}^{n/2+1}[M_0^{2k}]\theta_{n-2k-1} = \nu(z)$ for some unique $z \in \mathfrak{I}_n$.

Now

$$\begin{aligned}\tilde{\partial}\nu(x-z) &= \tilde{\partial}\nu(x) - \tilde{\partial}\nu(z)\\ &= \tilde{\partial}\nu(x) - \tilde{\partial}s_1\tilde{\partial}\nu(x) = 0.\end{aligned}$$

Thus, there is a unique $y \in \mathrm{SF}_n(S^1)$ so that $\tilde{\nu}(y) = \nu(x-z)$.

Then, $\nu(x-z-\rho(y)) = \nu(x-z) - \nu\rho(y) = \nu(x-z) - \tilde{\nu}(y) = 0$. Thus, again there is a unique $w \in \Omega_n$ so that $\alpha(w) = x - z - \rho(y)$. This gives us that each element in $R_n(\mathbf{Z}_p)$ can be written uniquely as

$$x = z + \rho(y) + \alpha(w),$$

and this gives us our direct sum decomposition. □

Let $\widehat{R}_n(\mathbf{Z}_p) = R_n(\mathbf{Z}_p)/\alpha\Omega_n$. This gives us a short exact sequence in the above commutative diagram 2.3, and we shall make use of this group later.

4. Definition of the map Γ. Let $\lambda = \exp(2\pi i/p)$. Let M^n be an oriented manifold together with an action of $\mathbf{Z}_p$ which extends to an action of S^1. Let $S^3 = \{(z_0, z_1) \in \mathbf{C}^2 \mid \|(z_0, z_1)\| = 1\}$. The circle acts freely on $M^n \times S^3$ by

$$(t, (m, z_0, z_1)) \mapsto (tm, tz_0, tz_1),$$

for $t \in S^1$. Let $\Gamma(M) = (M \times S^3)/S^1$. $\mathbf{Z}_p$ acts on $\Gamma(M)$ by

$$(\lambda, [m, z_0, z_1]) \mapsto [\lambda m, \lambda z_0, z_1].$$

The action extends to an action of S^1 on $\Gamma(M)$, so we can define $\Gamma^n(M)$ inductively: $\Gamma^n(M) = \Gamma(\Gamma^{n-1}(M))$ and $\Gamma^0(M) = M$. This map so defined raises dimension by 2. From our above remarks, this map is defined on most of $R_n(\mathbf{Z}_p)$. It is not defined on $\mathfrak{I}_n$ or $\alpha\Omega_n$. However, the construction of the manifolds that generate $\mathfrak{I}_*$ is related to the map Γ.

Let us generalize the construction of these manifolds $\{(T, M_0^{2k})\}$ from [**3**]. Let (T, M) be a regular action of $\mathbf{Z}_p$ on a manifold and let F denote its fixed point set, with normal bundle $\xi\colon E \to F$. $\mathbf{Z}_p$ acts regularly on the manifold $M^n \times S^1$ by the diagonal action $\sigma \times T$ where σ is multiplication in S^1 by λ. By [**3**, 35.1] this action is cobordant to the action, T_M, on $M^n \times S^1$ given by $T_M(x, z) = (x, \lambda z)$, or $T_M = 1 \times \sigma$.

We add a trivial line bundle to the normal bundle ξ, $\xi \oplus \mathbf{C}$. The diagonal map $T \times T_1$ on $E(\xi \oplus \mathbf{C})$, where T_1 is multiplication by λ restricts to a free $\mathbf{Z}_p$ action, T', on the sphere bundle, $S(\xi \oplus \mathbf{C})$. From [**3**, 35.2] $(T_M, M \times S^1)$ is freely cobordant to $(T', S(\xi \oplus \mathbf{C})$. Thus, there is a manifold with boundary, V^{n+2}, with free $\mathbf{Z}_p$-action, τ, satisfying

$$\partial(\tau, V^{n+2}) = (T_M, M \times S^1) \sqcup -(T', S(\xi \oplus \mathbf{C})).$$

Let W^{n+2} be the manifold V^{n+2} *closed up*, *i.e.*,

$$(\tau, W^{n+2}) = (T_M, M \times D^2) \cup_\partial (\tau, V^{n+2}) \cup_\partial (T', D(\xi \oplus \mathbf{C}),$$

where

(1) the action of $\mathbf{Z}_p$ on D^2 is the obvious extension of the mapping on S^1;

(2) the action of $\mathbf{Z}_p$ on the disk bundle, $D(\xi \oplus \mathbf{C})$ is the restriction of the diagonal map to the disk bundle, and is the extension of T' to the disk bundle;

(3) the disk and the disk bundle are glued to V^{n+2} along their respective boundaries.

Put $\Gamma(T, M^n) = (\tau, W^{n+2})$. From [**3**, Theorem 42.7] if the action comes from an S^1 action then these two definitions of Γ agree up to equivariant diffeomorphism. Thus, they are the same on the image of $\mathrm{SF}_n(S^1)$ in $R_n(\mathbf{Z}_p)$. This is the essentially the same definition that was used in [**3**] to construct the manifolds which generate $\mathfrak{I}_*$, so that this definition of Γ will be defined on $\mathfrak{I}_*$ as well.

It is not well-behaved, because it is not well-defined. We had to make a choice of manifold when we chose the free coboundary for the two ends. The bordism class of any free action on a closed oriented manifold is divisible by p in $\mathcal{O}_*^{SO}(\mathbf{Z}_p)$. The indeterminancy of this map thus in our case must lie in $\alpha\Omega_*$. Hence, on $\widehat{R}_n(\mathbf{Z}_p) = \mathfrak{I}_n \oplus \rho_1(\mathrm{SF}_n(S^1)$ we have a well-defined endomorphism $\Gamma\colon \widehat{R}_*(\mathbf{Z}_p) \to \widehat{R}_*(\mathbf{Z}_p)$ of degree $+2$.

Lemma 4.1. *This endomorphism Γ satisfies the following properties.*

(1) Γ *is well-defined.*

(2) Γ *is additive.*

(3) Γ *is an Ω_*-module map.*

(4) *If $x, y \in \widehat{R}_*(\mathbf{Z}_p)$ and $\varepsilon\colon \widehat{R}_*(\mathbf{Z}_p) \to \Omega_*$ is the augmentation map, then*

$$\begin{aligned} \Gamma(xy) &= \Gamma(x) \cdot y + \varepsilon(x) \cdot \Gamma(y) \\ &= x \cdot \Gamma(y) + \Gamma(x) \cdot y. \end{aligned} \tag{4.2}$$

Proof. This consists for the most part of checking the normal data and using the fact that $\nu\colon \widehat{R}_n(\mathbf{Z}_p) \to \sum_k \Omega_{n-2k}(BU(k))$ is monic. Note this is $\widehat{R}_n(\mathbf{Z}_p)$.

(1) This is done.

(2) The fact that Γ is additive follows from the fact that the Cartesian product distributes over disjoint union; i.e., that

$$(M_1 \sqcup M_2) \times S^1 = (M_1 \times S^1) \sqcup (M_2 \times S^1).$$

(3) The Ω_*-module structure on $R_n(\mathbf{Z}_p)$ is given by $[V^k]\{T, M^n\} = \{T \times 1, M^n \times V^k\}$. To see that $\Gamma([V^k]\{T, M^n\}) = [V^k]\Gamma(\{T, M^n\})$ we just check the normal data. The fixed point set of $T \times 1$ on $M^n \times V^k$ is $F \times V^k$ with normal bundle $\xi \times 0$, where 0 denotes the 0-plane bundle over V^k. Thus, the normal data for $\Gamma([V^k]\{T, M^n\})$ is

$$((\xi \times 0) \oplus \mathbf{C}) \sqcup -\mathbf{C} \quad \text{over} \quad (F \times V^k) \sqcup -(M^n \times V^k).$$

The normal data for $[V^k]\Gamma(\{T, M^n\})$ is

$$((\xi \times 0) \oplus \mathbf{C}) \sqcup -\mathbf{C}) \times 0 \quad \text{over} \quad (F \sqcup -M^n) \times V^k.$$

Passing to the bordism level finishes this.

(4) We will check that $\Gamma(xy) = x \cdot \Gamma(y) + \Gamma(x) \cdot \varepsilon(y)$. Assume that x and y are represented by manifolds (T_1, M_1) and (T_2, M_2), respectively, and let F_i with normal bundle ξ_i be the fixed data for T_i on M_i. The fixed point data for $\Gamma(\{T_1, M_1\} \cdot \{T_2, M_2\})$ is

$$((\xi_1 \times \xi_2) \oplus \mathbf{C}) \sqcup -\mathbf{C} \quad \text{over} \quad (F_1 \times F_2) \sqcup -(M_1 \times M_2).$$

The fixed point data for $x \cdot \Gamma(y)$ is

$$((\xi_1 \times \xi_2) \oplus \mathbf{C}) \sqcup -(\xi_1 \oplus \mathbf{C}) \quad \text{over} \quad (F_1 \times F_2) \sqcup -(F_1 \times M_2).$$

The fixed point data for $\Gamma(x) \cdot \varepsilon(y)$ is

$$(\xi_1 \oplus \mathbf{C}) \sqcup -\mathbf{C} \quad \text{over} \quad (F_1 \times M_2) \sqcup -(M_1 \times M_2).$$

The result follows when we pass to the bordism level, noting that the $\xi_1 \oplus \mathbf{C}$ over $F_1 \times M_2$ have opposite orientations. □

5. An ideal of $\widehat{R}_*(Z_p)$. Let $\mathfrak{A} = \{x + \Gamma(x) \mid \varepsilon(x) = 0\} \subset \widehat{R}_*(\mathbf{Z}_p)$. It is easy to check that if $x + \Gamma(x) \in \mathfrak{A}$ and $y \in \widehat{R}_*(\mathbf{Z}_p)$ then

$$\begin{aligned} xy + \Gamma(xy) &= xy + \varepsilon(x)\Gamma(y) + \Gamma(x) \cdot y \\ &= (x + \Gamma(x))y, \end{aligned}$$

and $\mathfrak{A}$ is an ideal of $\widehat{R}_*(\mathbf{Z}_p)$. For simplicity let $\mathcal{M}_*^R = \sum_{k=0}^{[\frac{n}{2}]} \Omega_{n-2k}(BU(k))$.

Lemma 5.1. *$\widehat{R}_*(\mathbf{Z}_p)/\mathfrak{A} \approx \mathcal{M}_*^R/(1+\theta_0)$, as rings.*

Proof. By the commutative diagram 2.3 and from [**17**], we know that $\mathcal{M}_*^R \approx \Omega_*[\theta_0, \theta_1, \ldots]$, where $\theta_n = [\eta_n \to \mathbf{C}\mathrm{P}(n)]$ is the bordism class of the canonical line bundle over $\mathbf{C}\mathrm{P}(n)$.

The proof of this lemma is entirely analogous to the proof of Theorem 3.1 of [**12**]. I will include an outline here, again for local completeness. If $\pi\colon \mathcal{M}_*^R \to \mathcal{M}_*^R/(1+\theta_0)$ is the quotient map, it is necessary to show that $\pi\nu\colon \widehat{R}_*(\mathbf{Z}_p) \to \mathcal{M}_*^R/(1+\theta_0)$ is onto. This is easily done by noting that in the quotient ring, we have identified the trivial 1-plane bundle $M^n \times D^1$ with M^n. We thus have identified all trivial r-plane bundles over M^n with the trivial 0-plane bundle over M^n, which is hit by $\{1, M^n\}$, under ν.

To see that $\mathfrak{A} \subset \ker(\pi\nu)$ is a matter of the construction of Γ and the choice that $\varepsilon(x) = 0$. It is easy to check that

$$\nu(x + \Gamma(x)) = [\xi \to F] \cdot (1+\theta_0),$$

where ξ is the normal bundle to the fixed set, F, of x. The proof of the other inclusion is a construction of an element in $\mathfrak{A}$ and requires the use of the Smith homomorphism and the splitting. It is exactly like the previous proof and will not be reproduced here. □

It follows from the remarks at the beginning of the previous proof, that $\widehat{R}_*(\mathbf{Z}_p)/\mathfrak{A}$ is a polynomial ring over Ω. Put $\Lambda_R(\mathbf{Z}_p) = \widehat{R}_*(\mathbf{Z}_p)/\mathfrak{A}$ and let

$$\Omega_*\{\{\Theta\}\} = \left\{ \sum_{k=0}^{\infty} [V^k]\Theta^k \mid [V^k] \in \Omega_k \right\}$$

denote the ring of homogeneous power series over Ω_*. Define a map $\psi\colon \widehat{R}_*(\mathbf{Z}_p) \to \Omega_*\{\{\Theta\}\}$ by

$$\psi(x) = \sum_{k \geq 0} (-i)^k \varepsilon(\Gamma^k(x))\Theta^{n+2k},$$

where $x \in \widehat{R}_*(\mathbf{Z}_p)$. ψ is additive by the additivity of ε and Γ. For $x \in \widehat{R}_*(\mathbf{Z}_p)$

$$\begin{aligned}\psi(x+\Gamma(x)) &= \sum_{k\geq 0}(-1)^k\varepsilon(\Gamma^k(x))\Theta^{n+2k} \\ &\quad+\sum_{k\geq 0}(-1)^k\varepsilon(\Gamma^k(\Gamma(x)))\Theta^{n+2k+2} \\ &= \varepsilon(x)\Theta^n.\end{aligned}$$

From this it follows that $\psi(\mathfrak{A})=0$. Once again the product formula for Γ and the fact that ε is multiplicative yields that ψ is multiplicative. We have just shown the following:

Lemma 5.2. *ψ is additive and multiplicative and $\psi(\mathfrak{A})=0$.*

Thus, ψ induces a well-defined, additive, multiplicative homomorphism $\bar{\psi}\colon \Lambda_R(\mathbf{Z}_p)\to\Omega_*\{\{\Theta\}\}$.

Note that the mapping $\widehat{R}_n(\mathbf{Z}_p)\to\Lambda_R(\mathbf{Z}_p)$ is a monomorphism for any fixed $n\geq 0$. The quotient ring $\mathcal{M}_*^R/(1+\theta_0)$ can easily be identified with $\Omega_*(BU)=MSO_*(BU)$, for in the quotient $[\xi]$ is identified with $[\xi\oplus\theta_0]$, *i.e.*, we are stabilizing by adding trivial bundles. The stabilization on $\widehat{R}_*(\mathbf{Z}_p)$ requires that we identify elements in differing dimensions. While $\Lambda_R(\mathbf{Z}_p)$ and $\Omega_*(BU)$ are no longer graded algebras, the map $\nu\colon \widehat{R}_n(\mathbf{Z}_p)\to\mathcal{M}_n^R$ induces a mapping $\overline{\nu}\colon \Lambda_R(\mathbf{Z}_p)\to\Omega_*(BU)$ which is an isomorphism by Lemma 5.1.

In lieu of the grading on $\Lambda_R(\mathbf{Z}_p)$ we will introduce two filtrations. First if (T,M^n) is a regular $\mathbf{Z}_p$-action, with non-empty fixed point set, we can consider it as belonging to $\Lambda_R(\mathbf{Z}_p)$. The first filtration will be *geometric*. The fixed point filtration of $\{T,M^n\}$ is defined to be $\mathrm{fil}_{FP}(\{T,M^n\})=k$ if the maximum of the dimensions of the nonbounding components of the fixed point set is k.

The second filtration is *algebraic*, based on the mapping of $\widehat{R}_n(\mathbf{Z}_p)$ to $\Omega\{\{\Theta\}\}$. Put the ψ-filtration to be $\mathrm{fil}_\psi(\{T,M^n\})=n+2j$ if $\varepsilon(\Gamma^j(\{T,M^n\}))\neq 0$ but $\varepsilon(\Gamma^i(\{T,M^n\}))=0$ for all $i<j$. This consists of picking off the lowest power of Θ which has a nonzero coefficient in the power series $\bar{\psi}(\{T,M^n\})$.

These filtrations behave well as is noted in [**15**].

Lemma 5.3. *Let* $x, y \in \Lambda_R(\mathbf{Z}_p)$,

(1) $\operatorname{fil}_{FP}(xy) = \operatorname{fil}_{FP}(x) + \operatorname{fil}_{FP}(y)$ *for* $x \neq 0$ *and* $y \neq 0$,

(2) $\operatorname{fil}_{\psi}(xy) = \operatorname{fil}_{\psi}(x) + \operatorname{fil}_{\psi}(y)$ *for* $x \neq 0$ *and* $y \neq 0$,

(3) $\operatorname{fil}_{FP}(x + y) = \max\{\operatorname{fil}_{FP}(x), \operatorname{fil}_{FP}(y)\}$ *if* x *and* y *have no monomials in common*,

(4) $\operatorname{fil}_{\psi}(x + y) = \min\{\operatorname{fil}_{\psi}(x), \operatorname{fil}_{\psi}(y)\}$ *if* x *and* y *have no monomials in common.*

So, given a set of polynomial generators for $\Lambda_R(\mathbf{Z}_p)$, it follows that the fixed point filtration of a polynomial in the generators will be the maximum of the fixed point filtrations of its terms, while the ψ-filtration of the same polynomial will be the minimum of the ψ-filtrations of its terms.

6. Proof of main theorem. To prove the main result, it is necessary to produce a set of polynomial generators for $\Lambda_R(\mathbf{Z}_p)$ as a polynomial ring over Ω_*. It is actually better to study a quotient of $\Lambda_R(\mathbf{Z}_p)$. We could choose to study the mapping $\bar{\psi}\colon \Lambda_R(\mathbf{Z}_p) \otimes \mathbf{Z}_p \to (\Omega_* \otimes \mathbf{Z}_p)\{\{\Theta\}\}$ or $\bar{\psi}\colon \Lambda_R(\mathbf{Z}_p)/\mathfrak{I}_* \to \Omega_*/I(p)\{\{\Theta\}\}$. In either case the result is the same for regular actions.

We look at $\Lambda_R(\mathbf{Z}_p)/\mathfrak{I}_*$ as a polynomial ring over $\Omega_*/I(p)$. In this case we see that we only need to take as generators the generators for the semifree S^1 actions and restrict these actions to the included regular $\mathbf{Z}_p$-action. In doing so, we do not lose any information of real interest. Recall that the generators of $\mathfrak{I}_*$ are those manifolds that are the generators of the ideal of all manifolds all of whose Pontrjagin numbers are divisible by p. Thus, $\bmod p$ we do not gain any information from $\mathfrak{I}_*$. Furthermore, the fixed point set of each of those elements is completely understood. Saying this, we deduce the following lemma from [**16**].

Lemma 6.1. *There are classes* $\{\tau, M^{4n}\} \in \Lambda_R(\mathbf{Z}_p)$ *for each* $n \geq 0$ *so that*

(1) $\{[M^{4n}\}_{n=0}^{\infty}$ *generates* $\Omega_*/I(p)$ *as a polynomial over* $\mathbf{Z}_p$;

(2) $\{\{\tau, M^{4n}\}\}_{n=0}^{\infty}$ *generates* $\Lambda_R(\mathbf{Z}_p)/\mathfrak{I}_*$ *as a polynomial ring over* $\Omega_*/I(p)$.

These are the manifolds $M^{4n} = \mathbf{C}\mathrm{P}(2n)$ with the regular $\mathbf{Z}_p$-action

$$\tau_n(\lambda, [z_0; z_1; \dots ; z_n; z_1'; z_2'; \dots ; z_n']) = [z_0; z_1; \dots ; z_n; \lambda z_1'; \lambda z_2'; \dots ; \lambda z_n'].$$

Note that for these generators the filtrations have the value

$$\mathrm{fil}_{FP}(\{\tau_n, M^{4n}\}) = 2n$$
$$\mathrm{fil}_{\psi}(\{\tau_n, M^{4n}\}) = 4n.$$

Proof of Theorem 1.3. Let (T, V^n) be a regular $\mathbf{Z}_p$ action that is not in the image of α and assume that $[V^n] \neq 0$ in $\Omega_*/I(p)$. Thus, $\mathrm{fil}_{\psi}(\{T, V^n\}) = n$. Also, we have $\mathrm{fil}_{FP}(\{T, V^n\}) = k$. Write $\{T, V^n\}$ as a polynomial in $\Lambda_R(\mathbf{Z}_p)/\mathfrak{I}_*$ in terms of the generators $\{\tau_i, M^{4i}\}$. Now, for any monomial X in these generators, from Lemma 5.3 and the above remark, we have that $\mathrm{fil}_{\psi}(X) \leq 2\mathrm{fil}_{FP}(X)$. $\{T, V^n\}$ can be uniquely expressed as a sum of distinct monomials in $\Lambda_R(\mathbf{Z}_p)/\mathfrak{I}_*$. From the remarks following Lemma 5.3, $\mathrm{fil}_{\psi}(\{T, V^n\})$ is the minimum of the ψ-filtrations of these monomials, while $\mathrm{fil}_{FP}(\{T, V^n\})$ is the maximum of the fixed point filtrations of these monomials. Thus, $n = \mathrm{fil}_{\psi}(\{T, V^n\}) \leq 2 \cdot \mathrm{fil}_{FP}(\{T, V^n\}) = 2k$. This is the stated result. □

This theorem tells us that for at least one m with $0 \leq m \leq 2k$

$$\varepsilon(\Gamma^m(\{T, M^n\})) \notin I(p).$$

Since $k < n$ it follows that $0 \leq m \leq k$.

Corollary 6.2. *Let T be a smooth orientation preserving regular $\mathbf{Z}_p$ action on a closed smooth manifold M^n and let* $\mathrm{fil}_{FP}(\{T, M^n\}) = k$. *If $n > 2k$, then $\{T, M^n\} \in \mathfrak{I}_n$.*

7. Coda. Assume that $n + m \leq N$ and assume that $\varepsilon(\Gamma^k(x)) = 0$ for $0 \leq k < N$ with $x \in \widehat{R}_*(\mathbf{Z}_p)$.

Lemma 7.1. *Under these assumptions*

$$(x-\Gamma(x))\sum_{j=0}^{n-1}\binom{n}{j}\Gamma^j(x)=x^2-\Gamma^n(x^2).$$

This follows from the repeated use of the product formula for Γ and the fact that the augmentations are zero for all $j < n+1$.

We can recognize $\mathcal{M}_*^R$ as a polynomial ring over $\mathbf{Z}_p$. To see how the polynomial $1-\theta^n$ factors in the polynomial ring $\mathbf{Z}_p[\theta]$ we only need look to cyclotomic field extensions. Any standard text in algebraic number theory will contain the following lemmas, cf. [**19**].

Recall once again that p denotes an odd prime. Consider the polynomial t^n-1 over the rationals, $\mathbf{Q}$, and let ζ_n denote the primitive n^{th} root of unity. The minimal polynomial for ζ_n is the n^{th} cyclotomic polynomial, denoted by $\mathbf{\Phi}_n(t)$. The following are consequences of the definition.

$$\begin{aligned}\mathbf{\Phi}_n(t)&=\prod_{(n,m)=1}(t-\zeta_n^m).\\ t^n-1&=\prod_{d\mid n}\mathbf{\Phi}_d(t),\end{aligned}\tag{7.2}$$

where $d \mid n$ means that d divides n.

$$\mathbf{\Phi}_n(t)=(t^n-1)/\prod_{\substack{d\mid n\\ d\neq n}}\mathbf{\Phi}_d(t).$$

Theorem 7.3. *If $p \nmid n$, then p factors in $\mathbf{Q}(\zeta_n)$ into the product of r distinct prime ideals of degree f, where $rf=\phi(n)$ and f is the smallest positive integer such that $p^f \equiv 1 \pmod{n}$, where ϕ is the Euler ϕ-function.*

Corollary 7.4. *If $p \nmid n$, then p splits completely in $\mathbf{Q}(\zeta_n)$ if and only if $p \equiv 1 \pmod{n}$.*

Theorem 7.5. *If $p \mid n$, write $n = p^k n'$ with $(p, n') = 1$. Then p factors in $\mathbf{Q}(\zeta_n)$ in the form*

$$p\mathbf{O}_{\mathbf{Q}(\zeta_n)} = (\mathfrak{B}_1 \dots \mathfrak{B}_r)^{\phi(p^k)}$$

where $\mathfrak{B}_1, \dots, \mathfrak{B}_r$ are distinct prime ideals in the ring of integers, $\mathbf{O}_{\mathbf{Q}(\zeta_n)}$, of degree f with $rf = \phi(n')$ and f being the smallest positive integer such that $p^f \equiv 1 \pmod{n'}$.

Theorem 7.6. $\mathbf{O}_{\mathbf{Q}(\zeta_n)} = \mathbf{Z}[\zeta_n]$.

Theorem 7.7. *If p factors in $\mathbf{Z}[\zeta_n]$ as $p\mathbf{O}_{\mathbf{Q}(\zeta_n)} = \left(\prod_{i=0}^{r} \mathfrak{B}_i\right)^{\phi(p^k)}$ with the above restrictions on r, f, and the ideals $\mathfrak{B}_i$, then*

$$\mathbf{\Phi}_n(t) \equiv \left(\prod_{i=1}^{r} p_i(t)\right)^{\phi(p^k)} \mod p,$$

with the $p_i(t)$ being distinct and irreducible and $\deg(p_i(t)) = f$ for $i = 1, \dots, r$.

Proof. By Theorem 7.6 $\mathbf{O}_{\mathbf{Q}(\zeta_n)} = \mathbf{Z}[\zeta_n]$ and this is a Dedekind domain. Thus, $\mathfrak{B}_i$ being prime is maximal in $\mathbf{Z}[\zeta_n]$ and, hence, $\mathfrak{K}_i = \mathbf{Z}[\zeta_n]/\mathfrak{B}_i$ is an extension field of $F_p = \mathbf{Z}/p\mathbf{Z}$. By assumption, $[\mathfrak{K}_i : F_p] = f$. Take $\overline{\zeta}_n \in \mathfrak{K}_i$ and let $p_i(Y)$ be its minimal polynomial. Now, $p_i(t)^{\phi(p^k)}$ divides $\mathbf{\Phi}_n(t) \mod p$. Doing this for all i, we have our result. □

Thus, from Theorem 7.3 and Theorem 7.7 we have that

$$t^n - 1 = \prod_{d|n} \left[\prod_{i_d=1}^{r_d} p_{i_d}(t)\right]^{\phi(p^{k_d})} \mod p$$

where $\deg(p_{i_d}(t)) = f_d$, $i_d = 1, \dots, r_d$; $d = p^{s_d} d'$; and $r_d f_d = \phi(d')$ with f_d being the smallest positive integer with $p^{f_d} \equiv 1 \pmod{d'}$ for each d which divides n.

Now, we look at what happens with Γ.

Lemma 7.8. $\Gamma^n(x)\Gamma^m(x) = \Gamma^{n+m}(x^2)$.

Proof. This is a simple double induction argument on n and m using the product formula for Γ and the assumptions about the augmentations: $n+m \leq N$ and $\varepsilon(\Gamma^k(x)) = 0$ for $0 \leq k < N$ with $x \in \widehat{R}_*(\mathbf{Z}_p)$. $\square$

Lemma 7.9. $\Gamma(x)\Gamma^n(x^r) = \Gamma^{n+1}(x^{r+1})$.

Lemma 7.10. $\Gamma^n(x^r)\Gamma^m(x^s) = \Gamma^{n+m}(x^{r+s})$.

These both follow by induction and the product formula.

Corollary 7.11. *Under the above assumptions* $[\Gamma(x)]^n = \Gamma^n(x^n)$.

Now divide the positive integers into two disjoint collections. Let $n \in \mathfrak{C}_0$ if $n = p^k$ for some $k \geq 1$; otherwise, $n \in \mathfrak{C}_1$.

Theorem 7.12 [The Freshman Theorem]. *If* $n \in \mathfrak{C}_0$, *then* $(\Gamma(x) - x)^n = \Gamma^n(x^n) - x^n$ *if* $\varepsilon(\Gamma^j(x)) = 0$ *for* $0 \leq j < n$.

Proof. Expanding the left hand side we have

$$(\Gamma(x) - x)^n = \sum_{j=0}^{n} \binom{n}{j} (\Gamma(x))^j x^{n-j}.$$

Since n is a power of p, this reduces to

$$\begin{aligned}(\Gamma(x) - x)^n &= (\Gamma(x))^n - x^n \\ &= \Gamma^n(x^n) - x^n,\end{aligned}$$

by Corollary 7.11. $\square$

Let $n \in \mathfrak{C}_1$ and let $f(t)$ be a polynomial over $\mathbf{Z}_p$ defined by $f(t) = \sum_{i=0}^n a_i t^i$, $a_i \in \mathbf{Z}_p$. Define a polynomial operator $f(\Gamma)$ by

$$f(\Gamma)(x) = \left(\sum_{i=0}^{n} a_i \Gamma^i\right)(x) = \sum_{i=0}^{n} a_i \Gamma^i(x).$$

If $g(\Gamma)$ is another polynomial operator, $g(\Gamma)(x) = \sum_{i=0}^m b_i \Gamma^i(x)$, define the product of these two polynomial operators in the obvious fashion:

$$f(\Gamma) \cdot g(\Gamma) = fg(\Gamma) = \sum_{i=0}^{n+m} c_i \Gamma^i,$$

where

$$c_i = \sum_{k=0}^{i} a_k b_{i-k} \in \mathbf{Z}_p.$$

By Lemmas 7.8, 7.9, and 7.10 we easily see that

$$[f(\Gamma)(x)] \cdot [g(\Gamma)(x)] = fg(\Gamma)(x^2).$$

Pulling all of this information together, we have the following result.

Theorem 7.13. *If* $n \in \mathfrak{C}_1$ *and if* $t^n - 1 = \prod_{d|n} [\prod_{i_d=1}^{r_d} g_{i_d}(t)]^{\phi(p^{k_d})}$ mod p *with the* $g_{i_d}(t)$ *not necessarily distinct for different values of* d *and with the above restrictions on* r_d, f_d, *and* k_d, *then*

$$\Gamma^n(x^k) - x^k = \prod_{d|n} \left[\prod_{i_d=1}^{r_d} (g_{i_d}(\Gamma))(x) \right]^{\phi(p^{k_d})} \in \widehat{R}_*(\mathbf{Z}_p)$$

where $k = \sum_{d|n} r_d \phi(p^{k_d})$.

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