ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

UNITARY UNITS IN GROUP RINGS OF GROUPS OF ORDER 16

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Let $U(\mathbf{Z}G)$ be the group of units of an integral group ring $\mathbf{Z}G$ and $U_1(\mathbf{Z}G)$ the subgroup of units of augmentation 1. Ritter and Sehgal [**8**] have shown that the bicyclic and Bass cyclic units generate a subgroup of finite index in $U(\mathbf{Z}G)$ for many nilpotent groups G. The restrictions are on the 2-Sylow subgroups of G. They also showed that the first difficulties arise with nonabelian groups of order 16 and that the result is false for $P = \langle a, b \mid a^4 = 1 = b^4, ba = a^3b\rangle$ and $Q_{16} = \langle a, b \mid a^8 = 1, a^4 = b^2, ba = a^7b\rangle$, but true [**9**] for the dihedral group $D_{16} = \langle a, b \mid a^8 = 1 = b^2, ba = a^7b\rangle$. It was later shown [**5, 6**] that D_{16} is the only indecomposable group of order 16 for which this result holds.

The notion of unitary units in group rings was first studied systematically by Bovdi [**1**] and Bovdi and Sehgal [**2**] characterized those groups G with the property that all bicyclic units in $\mathbf{Z}G$ are unitary. It turns out that there are five non-Hamiltonian groups of order 16 which satisfy this property–namely D_{16}, P, Q_{16}, $D = \langle a, b, c \mid a^2 = b^2 = c^4 = 1, ac = ca, bc = cb, ba = c^2ab\rangle$ and $D_8 \times C_2$ where $D_8 = \langle a, b \mid a^4 = 1 = b^2, ba = a^3b\rangle$ and $C_2 = \{1, c\}$ is the cyclic group of order 2.

In this note we show that if G is one of the five groups just listed, the Bass cyclic and unitary units will together generate a subgroup of finite index in $U(\mathbf{Z}G)$. In fact, we prove that when G is one of P, D or $D_8 \times C_2$, there is a torsion-free normal complement for G in $U_1(\mathbf{Z}G)$ which consists entirely of unitary units satisfying $u^{-1} = u^f$. It follows that, in each of these cases, a finite set of unitary units plays the same role that the bicyclic units play in $\mathbf{Z}D_8$ [**3, 7**] and $\mathbf{Z}S_3$ [**4**].

1. Preliminaries. Throughout, we will follow the notation of [**11**].

We will need the following easily proved observation.

Received by the editors on July 5, 1992.
This research was supported in part by NSERC grant A8775.

Lemma 1. *Let H be a finite normal subgroup of G. If u is a unit in $\mathbf{Z}(G/H)$ and u can be written in the form $1+|H|\overline{\beta}$, $\beta \in \mathbf{Z}G$, then u can be lifted to the unit $1+\hat{H}\beta$ in $\mathbf{Z}G$, where $\hat{H}$ denotes the sum of all elements of H.*

It may be interesting to note that a weaker version of Lemma 1 applies to all units in $\mathbf{Z}(G/H)$ whenever G is finite.

Lemma 2. *Let H be a normal subgroup of a finite group G. If u is a unit in $\mathbf{Z}(G/H)$, then there exists a natural number n such that u^n can be lifted back to a unit in $\mathbf{Z}G$.*

Proof. Let e denote the idempotent $\hat{H}/|H|$ in the group algebra $\mathbf{Q}G$. Note that

$$\mathbf{Z}G \subseteq \mathbf{Z}Ge \oplus \mathbf{Z}G(1-e) \subseteq \mathbf{Q}Ge \oplus \mathbf{Q}G(1-e) = \mathbf{Q}G.$$

Also note that $\mathbf{Z}Ge \cong \mathbf{Z}(G/H)$. Letting α be any preimage of u in $\mathbf{Z}G$, this isomorphism identifies u with αe. Now $\alpha e \oplus 1-e$ is a unit in $\mathbf{Z}Ge \oplus \mathbf{Z}G(1-e)$. But it follows from [**11**] that $U(\mathbf{Z}G)$ is of finite index in $U(\mathbf{Z}Ge \oplus \mathbf{Z}G(1-e))$, so $\alpha^n e \oplus (1-e)$ is in $\mathbf{Z}G$ for some natural number n. The result follows. □

Lemma 2 has the following corollary. The "central" case was also proved in [**10**], but using a different argument. The case where "central" is not assumed can also be obtained directly from Higman's theorem.

Corollary 3. *Let H be a normal subgroup of a finite group G. If $\mathbf{Z}(G/H)$ contains a nontrivial* (*central*) *unit, then $\mathbf{Z}G$ contains a nontrivial* (*central*) *unit.*

Proof. Recall [**11**] that if $\mathbf{Z}(G/H)$ contains a nontrivial (central) unit, then it contains a nontrivial (central) unit of infinite order. The result then follows immediately from Lemma 2. □

Next we recall some basic definitions and results about bicyclic and unitary units.

For an element $a \in G$ of finite order n, write $\hat{a} = 1 + a + \cdots + a^{n-1}$. If $a, b \in G$, $o(a) < \infty$, then $u_{a,b} = 1 + (1-a)b\hat{a}$ is a unit with inverse $1 - (1-a)b\hat{a}$. The elements $u_{a,b}$ are called the bicyclic units of $\mathbf{Z}G$.

If $f : G \to U(\mathbf{Z}) = \{\pm 1\}$ is a homomorphism, for each $x = \sum \alpha_g g$ in $\mathbf{Z}G$ we put $x^f = \sum \alpha_g f(g) g^{-1}$. A unit u in $\mathbf{Z}G$ is called f-unitary if $u^{-1} = u^f$ or $u^{-1} = -u^f$. Note that the group of f-unitary units always contains $\pm G$ and is equal to $\pm G$ when f is trivial.

The following theorem of Bovdi and Sehgal [**2**] gives necessary and sufficient conditions for a group G to have the property that all of the bicyclic units in $\mathbf{Z}G$ are f-unitary for some f. Since all of our groups will be finite, we specialize their theorem to that case.

Theorem 4. *Let A be the kernel of a nontrivial orientation homomorphism $f : G \to \{\pm 1\}$, where G is finite. The bicyclic units of $\mathbf{Z}G$ are all f-unitary if and only if either G is Hamiltonian or G is a non-Hamiltonian group which contains an element $b \neq 1$ such that one of the following conditions is fulfilled*:

(1) *A is an abelian group, the order of b divides 4 and $bab^{-1} = a^{-1}$ for all $a \in A$.*

(2) *A is a Hamiltonian 2-group, $b^2 = 1$, G is the semidirect product of A and $\langle b \rangle$, and every subgroup of A is normal in G.*

(3) *A is a Hamiltonian 2-group, b is of order 4 and G is the direct product of a Hamiltonian 2-subgroup of A and $\langle b \rangle$.*

Corollary 5. *If G is one of $D_8, D_{16}, P, Q_{16}, D$ or $D_8 \times C_2$, then there exists an orientation homomorphism $f : G \to \{\pm 1\}$ such that every bicyclic unit in $\mathbf{Z}G$ is f-unitary.*

Proof. For D_8, D_{16} and Q_{16}, use $A = \langle a \rangle$ in part (1) of Theorem 4. For $D_8 \times C_2$, use $A = \langle a, c \rangle$ in part (1). For P, use $A = \langle a, b^2 \rangle$ in part (1). For D, use $A = \langle ab, ac \rangle$ in part (2). □

Finally, if $a \in G$ is of order n, i is relatively prime to n and $m = \phi(n)$, then

$$u = (1 + a + \cdots + a^{i-1})^m + \frac{1 - i^m}{n} \hat{a}$$

is a Bass cyclic unit of $\mathbf{Z}G$. Note that if we choose α such that $i\alpha + n\beta = 1$ and $0 < \alpha < n$, then the inverse of u as defined above is given by

$$u^{-1} = (1 + a^i + \cdots + a^{(\alpha-1)i})^m + \frac{1-\alpha^m}{n}\hat{a}.$$

We will need the fact that, for an element a of order 8, a Bass cyclic unit is $(1+a+a^2)^4 - 10\hat{a} = 1 + (-9-6a+6a^3)(1-a^4)$.

2. Main results. Our first observation follows immediately from [**7**].

Theorem 6. *Let $f : P \to \{\pm 1\}$ be the orientation homomorphism with kernel $\langle a, b^2\rangle$. Then in $U_1(\mathbf{Z}P)$, P has a torsion-free normal complement consisting entirely of f-unitary units satisfying $u^{-1} = u^f$.*

Proof. In [**7**], it was shown that the following nine elements, obtained by applying Lemma 1 to a set of units in $\mathbf{Z}D_8$, generate a torsion-free normal complement for P in $U_1(\mathbf{Z}P)$.

$$\begin{aligned}
v_1 &= 1 + (1-a^2)(1+b^2)(a+b)\\
v_2 &= 1 + (1-a^2)(1+b^2)(-a+ab)\\
v_3 &= 1 + (1-a^2)(1+b^2)(13a+5b-12ab)\\
v_4 &= 1 + (1-a^2)(1+b^2)(17a+15b-8ab)\\
v_5 &= 1 + (1-a^2)(1+b^2)(-125a-44b+117ab)\\
v_6 &= 1 + (1-a^2)(1+b^2)(149a+51b-140ab)\\
v_7 &= 1 + (1-a^2)(1+b^2)(-2+a-2ab)\\
v_8 &= 1 + (1-a^2)(1+b^2)(-8-19a-14b+15ab)\\
v_9 &= 1 + (1-a^2)(1+b^2)(-2-7a-4b+6ab).
\end{aligned}$$

Since each of these elements satisfies $u^{-1} = u^f$, the result follows. □

Next we turn our attention to Q_{16}. The following was proved in [**5**].

Proposition 7. *In $U_1(\mathbf{Z}Q_{16})$, Q_{16} has a torsion-free normal complement which is the direct product of an infinite cyclic group and a free group of rank 9.*

The free group of rank 9 referred to in the above is obtained in the same way as with $\mathbf{Z}P$, namely by applying Lemma 1 to a set of units in $\mathbf{Z}D_8$. Since all such units in $\mathbf{Z}D_8$ are unitary, it again follows that we have a set of f-unitary units in $\mathbf{Z}Q_{16}$ (with respect to the orientation homomorphism with kernel $\langle a\rangle$).

It can be seen by following the isomorphism used in the proof of Proposition 7 in [**5**] that the infinite cyclic group referred to is generated by the central unit $u = a^4(1 + (1 + a - a^3)(1 - b^2))$. Note that u is not unitary, but $u^2 = b^2(1 + (-9 - 6a + 6a^3)(1 - b^2))$ and $1 + (-9 - 6a + 6a^3)(1 - b^2)$ is a Bass cyclic unit.

We have proved

Theorem 8. *Let $f : Q_{16} \to \{\pm 1\}$ be the orientation homomorphism with kernel $\langle a\rangle$. Then the Bass cyclic and f-unitary units generate a subgroup of finite index in $U(\mathbf{Z}Q_{16})$.*

Next we turn our attention to $\mathbf{Z}D$.

Let $\Gamma(2)$ denote the principal congruence subgroup modulo 2 of the Picard group. That is, $\Gamma(2)$ is obtained by factoring out

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

from the group of determinant 1 matrices of the form

$$\begin{pmatrix} 1+2a & 2b \\ 2c & 1+2d \end{pmatrix}$$

where a, b, c, d are Gaussian integers. The following characterization of $U(\mathbf{Z}D)$ appeared in [**6**].

Proposition 9. *In $U_1(\mathbf{Z}D)$, D has a torsion-free normal complement $V = \{u = 1 + (1 - c^2)\alpha \mid \alpha \in \Delta_{\mathbf{Z}}(D), u \text{ a unit}\}$. V is isomorphic to the subgroup of $\Gamma(2)$ consisting of those matrices*

$$\begin{pmatrix} 1+2a & 2b \\ 2c & 1+2d \end{pmatrix}$$

for which $b+c$ is divisible by 2. One such isomorphism maps

$$1+(1-c^2)(\alpha_0+\alpha_1 c+(\beta_0+\beta_1 c)a+(\gamma_0+\gamma_1 c)b+(\delta_0+\delta_1 c)ab)$$

to the matrix

$$\begin{pmatrix} 1+2(\alpha_0-\delta_1)+2(\alpha_1+\delta_0)i & 2(\gamma_0-\beta_1)+2(\gamma_1+\beta_0)i \\ 2(\gamma_0+\beta_1)+2(\gamma_1-\beta_0)i & 1+2(\alpha_0+\delta_1)+2(\alpha_1-\delta_0)i \end{pmatrix}.$$

Let u be a unit in V as described in the statement of Proposition 9. The matrix representation of u^{-1} is

$$\begin{pmatrix} 1+2(\alpha_0+\delta_1)+2(\alpha_1-\delta_0)i & 2(-\gamma_0+\beta_1)+2(-\gamma_1-\beta_0)i \\ 2(-\gamma_0-\beta_1)+2(-\gamma_1+\beta_0)i & 1+2(\alpha_0-\delta_1)+2(\alpha_1+\delta_0)i \end{pmatrix}.$$

With respect to the orientation map $f: D \to \{\pm 1\}$ with kernel $\langle ab, ac\rangle$

$$\begin{aligned} u^f &= 1+(1-c^2)(\alpha_0-\alpha_1 c^3-\beta_0 a+\beta_1 c^3 a-\gamma_0 b+\gamma_1 c^3 b+\delta_0 c^2 ab-\delta_1 cab) \\ &= 1+(1-c^2)(\alpha_0+\alpha_1 c-\beta_0 a-\beta_1 ca-\gamma_0 b-\gamma_1 cb-\delta_0 ab-\delta_1 cab). \end{aligned}$$

The matrix representation of u^f is

$$\begin{pmatrix} 1+2(\alpha_0+\delta_1)+2(\alpha_1-\delta_0)i & 2(-\gamma_0+\beta_1)+2(-\gamma_1-\beta_0)i \\ 2(-\gamma_0-\beta_1)+2(-\gamma_1+\beta_0)i & 1+2(\alpha_0-\delta_1)+2(\alpha_1+\delta_0)i \end{pmatrix}.$$

Hence, $u^f = u^{-1}$ for all such units u, and we have proved

Theorem 10. *Let $f: D \to \{\pm 1\}$ be the orientation homomorphism with kernel $\langle ab, ac\rangle$. Then in $U_1(\mathbf{Z}D)$, D has a torsion-free normal complement consisting entirely of f-unitary units satisfying $u^{-1} = u^f$.*

Note that the above argument uses Proposition 9 instead of simply examining a list of generators, as was done in the proof of Theorem 6. The problem is that although such a list does appear in [**3**], it unfortunately contains a few errors; for example, the first generator listed is not unitary.

Next we will prove

Theorem 11. *Let* $f : D_8 \times C_2 \to \{\pm 1\}$ *be the orientation homomorphism with kernel* $\langle a, c\rangle$. *Then in* $U_1(\mathbf{Z}(D_8 \times C_2))$, $D_8 \times C_2$ *has a torsion-free normal complement consisting entirely of* f*-unitary units satisfying* $u^{-1} = u^f$.

Proof. In [**5**], it was shown that a torsion-free normal complement for $D_8 \times C_2$ in $U_1(\mathbf{Z}(D_8 \times C_2))$ is generated by the bicyclic units of $\mathbf{Z}D_8$, all of which satisfy $u^{-1} = u^f$, and

$$K = \{u = 1 + (1 - a^2)(1 - c)\alpha \mid \alpha \in \mathbf{Z}(D_8 \times C_2), u \text{ a unit}\}.$$

It was also shown that

$$K \cong \left\{ A = \begin{pmatrix} 1+4c & 8d \\ 4e & 1+4f \end{pmatrix} \mid c, d, e, f \in \mathbf{Z}, \det A = 1 \right\}$$

via the isomorphism which maps $u = 1 + (1 - a^2)(1 - c)(\alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ab)$ to the matrix

$$\begin{pmatrix} 1 + 4(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3) & 8(-\alpha_1 + \alpha_2) \\ 4(\alpha_1 + \alpha_3) & 1 + 4(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3) \end{pmatrix}.$$

For any such u, the matrix representing u^{-1} is

$$\begin{pmatrix} 1 + 4(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3) & 8(\alpha_1 - \alpha_2) \\ 4(-\alpha_1 - \alpha_3) & 1 + 4(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3) \end{pmatrix}.$$

We also have

$$\begin{aligned} u^f &= 1 + (1 - a^2)(1 - c)(\alpha_0 + \alpha_1 a^3 - \alpha_2 b - \alpha_3 ab) \\ &= 1 + (1 - a^2)(1 - c)(\alpha_0 - \alpha_1 a - \alpha_2 b - \alpha_3 ab). \end{aligned}$$

The matrix representing u^f is therefore

$$\begin{pmatrix} 1 + 4(\alpha_0 + \alpha_1 - \alpha_2 + \alpha_3) & 8(\alpha_1 - \alpha_2) \\ 4(-\alpha_1 - \alpha_3) & 1 + 4(\alpha_0 - \alpha_1 + \alpha_2 - \alpha_3) \end{pmatrix}.$$

We conclude that $u^{-1} = u^f$, and the result follows. □

It is interesting to note that we were unable to decide in [**5**] whether or not the bicyclic units of $\mathbf{Z}(D_8 \times C_2)$ generate a subgroup of finite index.

Finally, we recall again that Ritter and Sehgal [**9**] showed that the Bass cyclic and unitary (in fact, the Bass cyclic and bicyclic) units of $\mathbf{Z}D_{16}$ generate a subgroup of finite index. Note that the central unit $1+(1+a-a^3)(1-a^4)$, seen earlier in $\mathbf{Z}Q_{16}$, shows that the Bass cyclic and unitary units do not generate a torsion-free normal complement of D_{16} in $U_1(\mathbf{Z}D_{16})$.

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