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CONFORMAL IRREGULARITY FOR DENJOY DIFFEOMORPHISMS OF THE 2-TORUS

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1. Introduction. If $\alpha \in (0,1)$ is irrational, then the map $R_\alpha(x) = x+\alpha \pmod 1$ is an irrational rotation of the circle $T^1 = \mathbf{R}/\mathbf{Z}$. Every R_α-orbit is dense in T^1 (so R_α is *minimal*). In fact R_α has a unique invariant probability measure (so it is *uniquely ergodic*). This map serves as a dynamical model for an arbitrary homeomorphism of the circle; it has long been known that for any homeomorphism f of T^1 without periodic points, there is a continuous map $h : T^1 \to T^1$ such that $hf = R_\alpha h$ for some irrational α. (A good reference is [**4**].) In this case f is said to be *semi-conjugate* to R_α, and α is called the *rotation number* of f. When h is a homeomorphism, we say that f is (*topologically*) *conjugate* to R_α. This is the standard notion of equivalence for dynamical systems, due to Poincaré.

The following theorem serves as our model:

Theorem (Denjoy, 1932). *If f is a C^1 diffeomorphism of T^1 without periodic points and Df has bounded variation, then f is conjugate to an irrational rotation.*

In view of the preceding remarks, we can state this another way:

Any sufficiently regular circle diffeomorphism that is semiconjugate to a minimal translation R must in fact be conjugate to R.

In this paper we wish to address the question of whether the phenomenon of Denjoy's theorem occurs in higher dimensions; we focus on the case of the 2-torus. In dimension one, " sufficiently regular" means that Df has bounded variation. In two dimensions much less is known.

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A two-dimensional analog of a rotation of the circle is a translation $R(x,y) = (x+\alpha, y+\beta) \pmod 1$ defined on the torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. When 1, α and β are linearly independent over the rationals, then R is minimal. Furthermore any minimal translation of T^2 is of this form.

Of course we must first identify what we mean by "phenomenon of Denjoy's theorem." This is a problem because many equivalent properties on the circle are inequivalent in higher dimensions, so that *a priori* there are many inequivalent candidates for a generalized Denjoy theorem, some of which fail easily.

For example, it is easy to construct an analytic diffeomorphism of T^2 with no periodic points but which is not conjugate to a translation.

We will be guided by the following facts in one dimension. Any circle homeomorphism with no periodic points and not conjugate to a translation has the following properties:

(A) There is a monotone semiconjugacy of f to a minimal translation, and

(B) f permutes a dense collection of pairwise disjoint domains.

Furthermore, such homeomorphisms exist as C^1 diffeomorphisms (but not C^2 by Denjoy's theorem); they are called "Denjoy circle diffeomorphisms."

Recall that a *minimal set* for f is a closed nonempty f-invariant set containing no closed nonempty proper invariant subsets. A connected open set U is a *wandering domain* if no forward or backward image of U meets U. Then a Denjoy circle diffeomorphism has a unique minimal set equal to the Cantor set complementary to the union of the wandering intervals mentioned in (B) above.

There are at least three topologically distinct models for diffeomorphisms of T^2 satisfying (A) and (B).

Model 1. *The direct product of two Denjoy circle diffeomorphisms with rationally independent rotation numbers.*

This map has a Cantor minimal set which is the product of the minimal sets of the two factors. It is not known whether there exists a C^2 diffeomorphisms topologically conjugate to this model.

Model 2. *The direct product of a Denjoy circle diffeomorphism and a minimal circle translation, with independent rotation numbers.*

This map has a unique minimal set equal to the product of a Cantor set and a circle, and therefore must be topologically distinct from Model 1. It has a dense collection of wandering annuli. Is there a C^2 diffeomorphism conjugate to Model 2?

Model 3 (denoted "Denjoy type"). *A diffeomorphism obtained topologically from a minimal translation of T^2 by "blowing up one or more orbits into disks," as in the case of Denjoy circle homeomorphisms.*

This construction is more difficult to carry out, even topologically [**2**]. Recently P. McSwiggen [**8**] has constructed, for any $\alpha < 1$, an example of a $C^{2+\alpha}$ diffeomorphism of this type. This example has a minimal set which is a "Sierpinski curve," and is constructed so that a single orbit of wandering disks is dense in the torus.

This is the model we will concentrate on in the remainder of this paper. The following question arises naturally.

Question. Are there limitations on the regularity of diffeomorphisms with properties (A) and (B) and topologically conjugate to model 3?

For example, it is unknown whether there is a C^3 diffeomorphism satisfying these properties.

This paper contains answers to this question for an interpretation of "regularity" in terms of conformality rather than smoothness. (Conformality is automatic in one dimension.) Precise statements appear in the next section (Theorems 1 and 2), but we briefly describe the main results here.

Say that f has "conformal boundary values" on a disk Δ if the values of f and Df on $\partial\Delta$ agree with those of some conformal map defined on $\overline{\Delta}$. Theorem 1 says that a C^1 diffeomorphism f of T^2 of Denjoy type cannot be "conformally regular" in the following sense: f cannot have conformal boundary values on each of its wandering disks.

This is a kind of limitation on the geometric regularity of f on its minimal set: if f is of Denjoy type, it must be "conformally irregular." For example, if f is C^2, then the derivative Df cannot be the identity map on the boundary of each wandering disk (equivalently, on the whole minimal set); by lemma 2 below, f and Df would then have the same boundary values as a translation on each disk, and this contradicts Theorem 1. (Therefore Theorem 1 is a strengthening of Theorem A in [**10**].)

Theorem 2 improves "conformal" to "quasiconformal" (by means of an entirely different method of proof).

2. Statements of results.

Definitions. $\mathrm{Diff}_0^k(T^2)$ is the space of C^k diffeomorphisms of $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ that are homotopic to the identity. Equivalently, this is the space of orientation preserving C^k diffeomorphisms of $\mathbf{R}^2$ which commute with unit translations in the coordinate directions. (By C^0 diffeomorphism we mean a homeomorphism.) Recall that a map h of T^2 is *monotone* if for each x, $h^{-1}(x)$ is connected. A point x is a *trivial value* for h if the fiber $h^{-1}(x)$ is a single point.

We say that $f \in \mathrm{Diff}_0^0(T^2)$ is of *Denjoy type* (relative to the minimal translation R) if f is semi-conjugate to R by a map h (i.e., there exists a continuous map h such that $hf = Rh$) such that

(i) h is monotone, and

(ii) the set V of nontrivial values of h is nonempty and countable.

The main hypothesis here is (ii); it includes the case of Model 3 and McSwiggen's example, but excludes Models 1 and 2.

It is easy to see that the relation $hf = Rh$ alone implies that f has no periodic points, since R has none.

Since V is nonempty, h is not a homeomorphism. This implies (see Lemma 1) that f cannot be conjugate to R.

Let Γ be the closure of $T^2 - h^{-1}(V)$. The following lemma collects some basic facts about maps of Denjoy type, and is proved in Section 3. We note that for such a map we may still have $\Gamma = T^2$.

Lemma 1. *If f is of Denjoy type then f is uniquely ergodic and Γ is the unique minimal set for f. In particular Γ depends only on f and not on h. Furthermore Γ is connected, and is equal to $T^2 - \cup\{interior(h^{-1}(x)) : x \in V\}$. If $\Gamma \neq T^2$, then it is nowhere dense, and each component of its complement is an open topological disk. In any case f is not conjugate to R.*

A natural subclass of Denjoy type is composed of those homeomorphisms for which the nontrivial h-fibers are topological or even smooth 2-disks, as in McSwiggen's example. Naively, one might even imagine an example for which the nontrivial h-fibers are circular disks, with the 1-jet of f restricted to each disk boundary simply that of a uniform homothety taking one circle onto the next. Theorem 1 below implies that this is impossible if f is a C^1 diffeomorphism.

Definitions. For now we will consider T^2 as $\mathbf{C}/\mathbf{J}$, where $\mathbf{J}$ denotes the complex integers. For a homeomorphism f of Denjoy type with semiconjugacy h having nontrivial values V, let $\hat{V} = \{h^{-1}(p) : p \in V\}$, the set of nontrivial fibers of h. $\hat{V}$ is an invariant collection of closed subsets of T^2. We say f has *conformal boundary values* if, for each $E \in \hat{V}$, there is a conformal map g_E defined on E such that

$$f|_{\partial E} = g_E|_{\partial E} \text{ and } Df|_{\partial E} = Dg_E|_{\partial E}.$$

We say that Df has conformal boundary values if there exist maps g_E as above for which one merely has $Df|_{\partial E} = Dg_E|_{\partial E}$.

Theorem 1. *If $f \in \mathrm{Diff}_0^1(T^2)$ is of Denjoy type then f does not have conformal boundary values.*

The proof appears in Section 4.

Definitions. A connected set C is *rectifiably connected* if, for every $x, y \in C$ there is a connected subset $S(x, y)$ of C containing x and y such that S is a countable union of sets of finite 1-dimensional Hausdorff measure. Rectifiable curves and connected open sets are simple examples of rectifiably connected sets.

Lemma 2. *If $f, g \in C^1(\mathbf{R}^2, \mathbf{R}^2)$ and, for some connected set S, $Df|_S = Dg|_S$, then $(f-g)|_S$ is constant if either*

i) *f and g are C^2, or*

ii) *S is rectifiably connected.*

Proof. Apply corollary 1 of [**9**] to each coordinate of $f - g$. □

Lemma 2 allows us to strengthen the conclusion of theorem 1 at the cost of introducing mild hypotheses.

Corollary 1. *If $f \in \mathrm{Diff}_0^1(T^2)$ is of Denjoy type, and either f is C^2 or ∂E is rectifiably connected for each $E \in \hat{V}$, then Df does not have conformal boundary values.*

Proof. Otherwise, adjusting each map g_E by an additive constant would, by Lemma 2, show that f itself had conformal boundary values, contrary to Theorem 2. □

Remark. One can generalize the notion of "rectifiably connected" by making use of s-dimensional Hausdorff measures, $s > 1$. This weaker notion of connectivity can be combined with a Hölder differentiability condition to give the conclusion of Lemma 2, and hence Corollary 1, when neither (i) nor (ii) obtains. See [**9**].

We can improve the boundary hypotheses even further by realizing that it is actually only quasiconformality we need, not conformality.

One difficulty with Theorem 1 is that, to make sense of the condition of conformal boundary values, we have to think of the smooth manifold T^2 as supplied with a particular conformal structure, which we did by naively identifying T^2 with $\mathbf{C}/\mathbf{J}$. However, the best result along these lines should depend only on the smooth structure on T^2, as in the one-dimensional case. Since a smooth change of coordinates does not alter the dynamics of a diffeomorphism, we would like a statement with hypotheses that persist after conjugating f by a smooth diffeomorphism. Theorem 1 does not have this property.

However, any smooth structure naturally induces a unique quasiconformal structure, so we can speak of quasiconformality without imposing further choices on the manifold.

The reader unfamiliar with quasiconformal mappings in the plane should consult a text such as [**7**] for full definitions. We give a brief reminder here.

If $f : \mathbf{C} \to \mathbf{C}$ is differentiable at a point z, we denote by $K_f(z)$ the dilatation of f at z. This is defined to be the ratio of the lengths of the major to the minor axis of any ellipse obtained as the image of a circle under the linear transformation $Df(z)$. It is a fact that if f is a homeomorphism and is absolutely continuous on lines (ACL—see [**7**]), then f is differentiable almost everywhere. This justifies the definition: if U, V are open sets in the plane, and $1 \leq K < \infty$, then a homeomorphism $f : U \to V$ is called K*-quasiconformal* if f is ACL and $K_f(z) \leq K$ for almost every $z \in U$.

Definitions. Let A and B be subsets of T^2. A map $g : A \to B$ is said to be K*-quasiconformal* (K-qc) if g extends to a K-qc homeomorphism from a neighborhood of A to a neighborhood of B. Thus for open A this is the standard definition. A map is *quasiconformal* (qc) if it is K-qc for some $K < \infty$.

A family $\mathcal{F}$ of qc maps is *unformly quasiconformal* (unif. qc) if there exists $K < \infty$ such that each member of $\mathcal{F}$ is K-qc.

We say a homeomorphism f of Denjoy type has *uniformly quasiconformal boundary values* if

(i) for each $E \in \hat{V} = \{h^{-1}(p) : p \in V\}$, there is a qc map g_E defined on E such that $f|_{\partial E} = g_E|_{\partial E}$, and

(ii) the family of all possible finite compositions of the g_E's is unif. qc.

Uniform quasiconformality of the iterates of a mapping is a way of saying that the dynamics has "bounded (linear) distortion." It is worth noting the following fact, showing that diffeomorphisms of Denjoy type must exhibit abundant unbounded distortion.

Proposition 1. *Let A be a subset of T^2 and suppose that $f \in \mathrm{Diff}_0^1(T^2)$ is such that*

$$S = \bigcup_{n=-\infty}^{\infty} f^n(A)$$

is dense in T^2. If for some $K < \infty$ the iterates of f are all K-qc on A, then f is (qc) *conjugate to a translation.*

Proof. If the iterates of f are K-qc on A, then they are uniformly K^2-qc on the invariant set S, as follows. The dilatation satisfies the following elementary properties: $K_{fg}(z) \leq K_f(g(z))K_g(z)$ and $K_f(z) = K_{f^{-1}}(f(z))$.

Now if $x \in S$, then $x = f^m(y)$ for some $m \in Z, y \in A$. Hence,

$$\begin{aligned} K_{f^n}(x) = K_{f^{m+n}f^{-m}}(x) &\leq K_{f^{m+n}}(y)K_{f^{-m}}(x) \\ &= K_{f^{m+n}}(y)K_{f^m}(y) \leq K^2. \end{aligned}$$

Now by continuity of Df, the iterates of f are unif. K^2-qc on all of T^2. By a theorem of Sullivan [**11**, p. 750], there is an f-invariant conformal structure on T^2, so (via the measurable Riemann mapping theorem) f is qc conjugate to a conformal homeomorphism of T^2, which must be a translation. □

As a consequence, we have, for example,

Corollary 2. *If the iterates of a diffeomorphism f of T^2 have bounded linear distortion* (*i.e., bounded dilatation*) *at a single point with dense orbit, then f is* (qc) *conjugate to a minimal translation.*

Another consequence is that the iterates of a diffeomorphism f of Denjoy type cannot be unif. qc on $T^2 - \Gamma$ when Γ is not all of T^2. The import of the next theorem is that under a further hypothesis f cannot even have uniformly quasiconformal boundary values.

Note that $f|_\Gamma$ is a homeomorphism of Γ. We will say that f has *uniformly bounded dilatation on* Γ if $f^n|_\Gamma$ has bounded dilatation a.e., with the bound independent of n. This property is satisfied, for example, if Γ has measure zero.

Theorem 2. *If f is a* qc *homeomorphism of T^2 of Denjoy type and f has uniformly bounded dilatation on Γ, then f does not have unif.* qc *boundary values.*

We prove this result in Section 5. Since any diffeomorphism of T^2 is qc, and conformal boundary values implies uniformly bounded dilatation on Γ, this theorem implies Theorem 1. (However the proof of Theorem 1 in Section 3 makes no use of quasiconformal analysis.)

Note. Theorem 2 remains true if T^2 is merely a quasiconformal torus, without a smooth structure. The proof takes place entirely in the qc category.

3. Proof of Lemma 1.

Proof. Let $S = T^2 - \cup\{\text{interior}\,(h^{-1}(x)) : x \in V\}$; we show that $\Gamma = S$. Certainly $\Gamma \subset S$. Conversely, it suffices to show that if $x \in \partial h^{-1}(y)$ for some $y \in V$, then x is a limit of a sequence in $T^2 - h^{-1}(V)$. If not, there is a small compact disk D centered at x containing no elements of $T^2 - h^{-1}(V)$. This means that D is the countable disjoint union of the compact sets $D \cap h^{-1}(y)$, $y \in V$. This contradicts *Sierpinski's theorem* [**6**]: no compact, connected Hausdorff space is the countable union of disjoint closed subsets.

Next, let μ and ν be two invariant probability measures for f. Since μ is f-invariant, $\mu(h^{-1}(\text{point})) = 0$. Since V is countable, this means $\mu(h^{-1}(V)) = 0$. Hence h is μ-almost everywhere injective. This means that

$$(h^{-1})_* h_* \mu = \mu,$$

and similarly for ν. But $h_*\mu$ and $h_*\nu$ are R-invariant probability measures, and hence are equal by unique ergodicity of R. Therefore $\mu = \nu$ and so f is uniquely ergodic.

Since any minimal set supports an invariant probability measure, we immediately deduce that f has a unique minimal set M. By the relation $hf = Rh$, every orbit in the invariant set $T^2 - h^{-1}(V)$ is dense, so $T^2 - h^{-1}(V)$ is contained in M. Since M is closed, Γ is contained in M, and so by minimality $\Gamma = M$.

Suppose $\Gamma \neq T^2$. Then $\partial\Gamma \neq \varnothing$. Since $\partial\Gamma$ is a closed, invariant subset of Γ, it must equal Γ by minimality. Hence Γ has no interior, and so is nowhere dense.

To further describe the topology of Γ and its complement, it is convenient to consider the collection of subsets $M = \{h^{-1}(x) : x \in T^2\}$ as a "decomposition space." The decomposition space M is given a topology by declaring that a subset U of M is open if and only if $\cup U$ is open in T^2 (see [**12**]). It is a general fact (Theorem 3.4, p. 126 of [**12**]) that whenever A and B are compact metric spaces and $h : A \to B$ is a continuous map, then the decomposition space of A is homeomorphic to $h(A)$ by the homeomorphism $\hat{h}$ sending $h^{-1}(x)$ to x. In our case this means that M is homeomorphic to T^2.

Claim. *No element of $\hat{V}$ separates T^2.*

Suppose for contradiction that for some $z \in T^2, h^{-1}(z)$ separates T^2 into disjoint open sets X, Y. Pick $x \in X$, $y \in Y$. Since h is monotone, $h^{-1}(h(x))$ and $h^{-1}(h(y))$ lie in X and Y, respectively, and so are separated by $h^{-1}(z)$. This means that removal of $h^{-1}(z)$ disconnects M. This is our contradiction, since T^2 cannot be disconnected by removal of any single point, and the claim is proved.

As a consequence, if Δ is a component of the interior of an element of $\hat{V}$, then Δ is simply connected and therefore is an open topological disk. To establish this, it only remains to see that Δ is homotopically trivial in T^2. If not, then neither is the disjoint image $f(\Delta)$. In this case removal of the corresponding elements of $\hat{V}$ will disconnect M, as above. This is impossible, since removal of two points from T^2 cannot disconnect T^2.

As a result, we have established that Γ (if $\neq T^2$) is the complement of countably many disjoint open disks, and therefore is connected.

Finally: f is not conjugate to R. If there were a homeomorphism g such that $gf = Rg$, then $hg^{-1}Rg = Rh$, or $jR = Rj$, where $j = hg^{-1}$. Therefore $jR^n = R^n j$ for all n. Since R is minimal and j continuous, this forces j to be a translation, and therefore a homeomorphism. This contradicts the fact that h is not a homeomorphism. □

4. Proof of Theorem 1. The following lemma is a simple consequence of a result of A. Hinkannen (Theorem A of [**5**]). First some terminology. We say that $\mu : [0,\infty) \to [0,\infty)$ is a *majorant* if μ is continuous, nondecreasing with $\mu(0) = 0$, and

$$\mu(2t) \le 2\mu(t) \qquad \text{for all } t \in [0,\infty).$$

If f is a function defined on A, we say that μ *majorizes* f *on* A if μ is a majorant and

$$|f(x) - f(y)| \le \mu(|x-y|) \qquad \text{for all } x, y \in A.$$

Lemma 4 (Hinkannen). *Let Δ be a simply connnected domain in the plane and suppose that f is analytic in Δ and continuous on Δ^-. If μ majorizes f on $\partial\Delta$, then 74μ majorizes f on Δ^-.*

Proof of Theorem 1. We work with a lift of f to $\mathbf{R}^2$, denoted by the same letter. (As in the beginning of Section 2, $\mathbf{R}^2$ is identified with the complex plane $\mathbf{C}$ in the standard way.)

There are two cases. If no element of $\hat{V}$ has interior, then $\cup\{\partial E : E \in \hat{V}\}$ is dense in T^2, so that on a dense set Df is pointwise conformal. By continuity Df is pointwise conformal everywhere. But any C^1 mapping of the plane with everywhere a conformal derivative is necessarily holomorphic (via the Cauchy-Riemann equations, e.g., [**1**]). So f is a holomorphic diffeomorphism of T^2 and hence must be a translation. This is impossible. Therefore, we may assume that at least one (and therefore infinitely many) $E \in \hat{V}$ has interior.

Let $\mathcal{D} = \{\Delta_1, \Delta_2, ...\}$ be the collection of all the components of interiors of elements of $\hat{V}$, ordered in some arbitrary way. (By Lemma 1, each Δ_i is an open topological disk, and $\cup\mathcal{D}$ is dense in T^2.)

For each $\Delta \in \mathcal{D}$, there is a unique $E \in \hat{V}$ containing Δ (and hence ∂E contains $\partial\Delta$.) Let $g_\Delta = g_E$.

For each $\Delta \in \mathcal{D}$ we know that

$$f|_{\partial\Delta} = g_\Delta|_{\partial\Delta} \quad \text{and} \quad Df|_{\partial\Delta} = Dg_\Delta|_{\partial\Delta}. \tag{4.1}$$

Now inductively define $F_0 = f$, and for $i = 1, 2, 3, \ldots$

$$F_i = \begin{cases} F_{i-1} & \text{off } (\Delta_i)^- \\ g_{\Delta_i} & \text{on } (\Delta_i)^- \end{cases} .$$

It follows from (4.1) that each F_i agrees with f on Γ, and that each F_i is continuous. Since each g_Δ has degree one on $\partial\Delta$, $g_\Delta|_{\Delta^-}$ is injective. Hence each F_i is a homeomorphism.

By the maximum modulus principle, each partial derivative of g_Δ has its maximum modulus on $\partial\Delta$, where it agrees with the corresponding partial derivative of f.

Hence the supremum norm $\|Dg_\Delta|_{\Delta^-}\|$ is uniformly controlled over all Δ by $\|Df\|$ on T^2. This means that $\{F_i\}$ forms an equicontinuous family. Since this family is pointwise convergent, it must converge uniformly to a continuous mapping G.

Now G agrees with f on Γ and takes each Δ_i to $f(\Delta_i)$. Furthermore, G is injective on each disk Δ_i. Since f is injective, this means that G is globally injective, hence a homeomorphism.

We wish to show that G is a C^1 diffeomorphism. We accomplish this by showing that $\{F_i\}$ is Cauchy relative to the C^1 norm $|\!|\!| \cdot |\!|\!|$ defined by $|\!|\!| g |\!|\!| = \max(\|g\|, \|Dg\|)$. (Note that (4.1) implies that each F_i is C^1.) Since the space of C^1 mappings of T^2 equipped with the norm $|\!|\!| \cdot |\!|\!|$ is complete, this will show that G is C^1. Since DG has rank 2 in each Δ_i because g_{Δ_i} is conformal, and on Γ because it agrees with Df there, G must actually be a diffeomorphism.

To complete the proof of Theorem 1, we note that DG is pointwise conformal on the dense set $\cup\mathcal{D}$ and argue as before that G must therefore be a translation, which is impossible because of the wandering disks Δ_i. This is our contradiction.

It remains to show that $\{F_i\}$ is $|\!|\!| \cdot |\!|\!|$-Cauchy.

Since the F_i converge uniformly in the C^0 sense, we may concentrate on the norm of the derivatives. Write g_i for g_{Δ_i}.

Now for $i < j$,

(4.2)

$$\begin{aligned}\max_{x\in T^2}\|DF_i(x)-DF_j(x)\| &\le \max_{i\le k\le j}\ \max_{x\in\Delta_k^-}\|Dg_k(x)-Df(x)\|\\ &\le \max_{i\le k\le j}\ \max_{x\in\Delta_k^-}\Big(\|Dg_k(x)-Dg_k\big(y(x)\big)\|\\ &\qquad + \|Df\big(y(x)\big)-Df(x)\|\Big)\end{aligned}$$

where $y(x)$ is a point of $\partial\Delta_k$ minimizing the distance $\operatorname{dist}(x, y(x))$. (Recall that $Dg_k = Df$ on $\partial\Delta_k$.)

Now Df, being continuous, has some modulus of continuity $\mu(t)$. (For example $\mu(t) = \max\{\|Df(x)-Df(y)\| : x, y \in T^2 \text{ and } |x-y| \le t\}$.) By suitably increasing μ if necessary, we can assume that μ is a majorant (see Appendix), and therefore that it majorizes Df on T^2. Hence, for each k, μ majorizes Dg_k on $\partial\Delta_k$.

Since $Dg_k(x)v = g_k'(x)v$ (the lefthand side we interpret as matrix multiplication with a tangent vector v and the righthand side is multiplication of the two complex numbers $g_k'(x)$ and v), this means that

$$|g_k'(x) - g_k'(y)| \le \mu(|x-y|) \qquad \text{for all } x, y \in \partial\Delta_k.$$

Since g_k' is analytic on Δ^- we may apply Lemma 4 to conclude that $|g_k'(x) - g_k'(y)| \le 74\mu(|x-y|)$ for all $x, y \in \Delta^-$, i.e.,

$$\|Dg_k(x) - Dg_k(y)\| \le 74\mu(|x-y|) \qquad \text{for all } x, y \in \Delta^-.$$

Hence, (4.2) is less than or equal to

$$\max_{i\le k\le j}\ \max_{x\in(\Delta_k)^-}\Big(75\mu\big(\text{inradius }(\Delta_k)\big)\Big)$$

where inradius (Δ_k) is by definition the radius of the largest ball contained in Δ_k. (Note that $|x-y(x)|$ is simply the radius of the largest ball with center x contained in Δ_k.) But inradius (Δ_k) tends to zero as $k \to \infty$ since the collection $\{\Delta_k\}$ has finite total area. Therefore, $\max\{\|DF_i(x)-DF_j(x)\| : x \in T^2\}$ tends to zero as i, j tend to infinity. □

5. Proof of Theorem 2. To prove Theorem 2, we assume that f has uniformly bounded dilatation on Γ, and for contradiction that f does have unif. qc boundary values.

Suppose first that some element of $\hat{V}$ has interior. As before, let $\mathcal{D} = \{\Delta_1, \Delta_2, \cdots\}$ be the collection of the components of the interiors of the elements of $\hat{V}$, ordered arbitrarily. As in the proof of Theorem 1, we define a sequence of self maps of T^2, $\{F_n : T^2 \to T^2, n \geq 0\}$ by letting $F_0 = f$ and proceeding inductively with

$$F_i = \begin{cases} F_{i-1} & \text{off } (\Delta_i)^- \\ g_{\Delta_i} & \text{on } (\Delta_i)^-. \end{cases}$$

For each i, since g_{Δ_i} agrees with F_{i-1} on $\partial\Delta_i$, F_i is continuous. Furthermore, g_{Δ_i} has degree one on $\partial\Delta_i$ and hence is injective. Consequently, F_i is a homeomorphism. Now we choose K so large that f is K-qc, all iterates of f have dilatation bounded by K a.e. on Γ, and all possible compositions of the g's are K-qc.

We claim that each F_i is K-qc. We will argue this below, but first let us see that this allows us to prove the theorem.

To this end, we lift the F_i to automorphisms $\hat{F}_i : \mathbf{R}^2 \to \mathbf{R}^2$ which send $(0,0)$ to a point in the fundamental region $[0,1) \times [0,1)$. The $\hat{F}_i$ are then K-qc homeomorphisms of $\mathbf{C}$ which move the fundamental lattice at most a bounded distance from itself, and hence by a well known theorem (found in [**7**, Chapter II, Section 5]) the $\hat{F}_i$ form a normal family, and so do the F_i. Any limit F of the F_i is therefore a K-qc automorphism of T^2. We pick one such limiting F.

By construction, the collection of forward iterates of F is unif. K-qc on $\cup\mathcal{D}$. Since by assumption they are also unif. K-qc on Γ, it follows that the forward iterates of F are unif. K-qc on T^2.

By a theorem of Sullivan [**11**, p. 750], there exists an F-invariant conformal structure on T^2, so F is conjugate to a conformal homeomorphism of T^2, which must be a translation. This contradiction completes the proof in case some element of $\hat{V}$ has interior. If not, then by hypothesis f is unif. qc on $\Gamma = T^2$, so Sullivan's theorem gives a contradiction as before.

To establish the claim, we need the following lemma due to Bers [**3**, p. 93].

Bers' Lemma. *Let $D \subset \hat{\mathbf{C}}$ be open, and let f be a topological automorphism of* $\mathbf{C}$ *such that $f|_D$ is* qc *and $f|_{\hat{\mathbf{C}}\setminus D} = \mathrm{id}$. Then f is* qc *(and has no* qc *dilatation off of D).*

An immediate corollary of this is

Lemma 5. *If $f : T^2 \to T^2$ is a K-*qc *automorphism of the torus, $\Delta \subset T^2$ is open, and g is a K-qc map defined on Δ^- such that $g|_{\partial\Delta} = f|_{\partial\Delta}$, then the map $F : T^2 \to T^2$ defined by*

$$F = \begin{cases} f & \textit{off } \Delta^- \\ g_\Delta & \textit{on } \Delta^- \end{cases}$$

*is a K-*qc *automorphism of T^2 (and on $T^2 \setminus \Delta$, all the dilatation of F comes from f).*

Proof. Let $\Delta' = g(\Delta)$, and consider the map $h = g \circ f^{-1}$ on $(\Delta')^-$. Our hypotheses give that h is a qc automorphism of Δ' which agrees with the identity on $\partial\Delta'$. Extending h to be the identity on $T^2 \setminus \Delta'$ yields, by Bers' lemma, a qc automorphism (H) of T^2 which has no conformal dilatation off of Δ'. Our desired $F = H \circ f$ is thus quasiconformal. Since F agrees with g on Δ, and with f off Δ^-, it is K-qc except perhaps on $\partial\Delta$. But there, by the Bers lemma, H is conformal. Thus any dilatation by F on $\partial\Delta$ comes from f, so that F is K-qc everywhere. □

Applying this theorem inductively to the F_i shows that they are each K-qc, completing the proof. □

Appendix

For completeness we include a proof of the following Lemma.

Definition. A nondecreasing continuous function $\nu : [0, \infty) \to \mathbf{R}$ with $\nu(0) = 0$ is a *modulus of continuity* for a continuous function f if

$$|f(x) - f(y)| \leq \nu(|x - y|) \qquad \text{for all } x, y.$$

Lemma 6. *If $\nu(t)$ is a modulus of continuity for a function f with compact support, then there is a majorant μ such that $\mu(t) \geq \nu(t)$ for all $t \in [0, \operatorname{diam}(\operatorname{support}(f))]$.* A fortiori*, μ is also a modulus of continuity for f.*

Proof. Let d be the diameter of the support of f. Choose the smallest $k \in \mathbf{Z}$ so that $2^k \geq d$. Define $\mu(s) = \nu(2^k)$ for all $s \geq 2^k$. Now inductively define μ on $[2^j, 2^{j+1}]$, $j = k-1, k-2, k-3, \ldots$ by

$$\mu(t) = \max\big(\nu(t), \mu(2t)/2\big).$$

The reader can check that this defines a continuous and nondecreasing function on $[0, \infty)$ with $\mu(0) = 0$. Then μ is clearly a majorant and $\mu(t) \geq \nu(t)$ for all $t \leq 2^k$. □

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