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CONSTRUCTIONS AND 3-DEFORMATIONS OF 2-POLYHEDRA AND GROUP PRESENTATIONS

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ABSTRACT. In this paper we shall study the Andrews-Curtis conjecture (AC) and its relation with some other interesting and important conjectures in low dimensional topology through special polyhedra and special presentations. An abelian monoid is constructed in the set of equivalence classes of mutually 3-deformable, contractible special polyhedra. It is shown that this abelian monoid is trivial if and only if (AC) is true. Some properties of this monoid are also discussed. Via the generalized Nielsen operations, we see the connection between special polyhedra and group presentations; hence, we derive another version of (AC). Then we prove that some cases of this version of (AC) are true.

1. Introduction. In 1964, E.C. Zeeman made the conjecture (Z) [**16**]: Every compact contractible 2-polyhedron is 1-collapsible. He also showed that (Z) implies the 3-dimensional Poincaré conjecture (3D-P). In 1965 Andrews and Curtis made their conjecture (AC) [**2**]: Every balanced, finite presentation of the trivial group can be reduced to the empty presentation by the generalized Nielsen operations. Later in 1975, P. Wright showed an equivalent formulation of (AC) [**13**]: Every compact contractible 2-polyhedron 3-deforms to a point. Because of this equivalent geometric formulation of (AC), it is easy to see that (Z) implies (AC). To understand these conjectures better and find a way to prove or disprove them, mathematicians are looking for their relationships. In 1983, Gillman and Rolfsen showed [**5**] that (Z) for thickened special polyhedra is equivalent to (3D-P). In 1987, S.V. Matveev claimed [**9**] that (Z) for unthickened special polyhedra is equivalent to (AC). In Section 2 we give most of the definitions for the future discussion in this paper. In Section 3 we construct $(\mathcal{M}, +)$ and show that it is an abelian monoid. Then we discuss the significance of $(\mathcal{M}, +)$ and the analogy between $(\mathcal{M}, +)$ and M.M. Cohen's geometric construction of the Whitehead group $\mathrm{Wh}\,(L)$ for a CW complex L. This analogy may give us some hint for calculating $(\mathcal{M}, +)$. In Section

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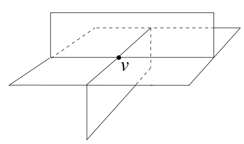
4 we derive another version of (AC) via the correspondence between the set of mutually 3-deformable contractible special polyhedra and that of mutually reducible balanced special presentations of the trivial group by the generalized Nielsen operations (GNO's). Then we show that this version of (AC) is true when the number of generators is fewer than four.

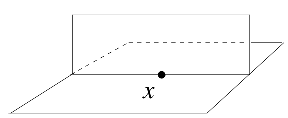
2. Preliminaries.

Definition. A compact, connected 2-polyhedron X is called *special* if X has a CW complex structure relative to which (0) each vertex $v \in X^{(0)}$ has a (closed) neighborhood in X which is homeomorphic to



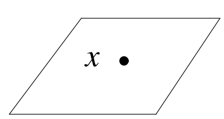
Type 3 point or singular point.

(1) each nonvertex point $x \in X^{(1)} - X^{(0)}$ has a (closed) neighborhood in X which is homeomorphic to



Type 2 point or three fin point.

Such vertices are called *singular points* and the points in $X^{(1)} - X^{(0)}$ are called *three fin points*. It follows that each point $x \in X - X^{(1)}$ has a (closed) neighborhood in X homeomorphic to



Type 1 point or manifold point

and the 1-skeleton $X^{(1)}$ is a regular graph of valence four. Each component of $X - X^{(1)}$ is called a 2-*component*. In the definition of special polyhedrons, all 2-components are required to be open 2-cells.

There are several closely related, but different, definitions and names for this concept (cf. Casler [**3**], Gillman and Rolfsen [**5**], Ikeda [**6**], Matveev [**8, 9**]). Here I use the definition and name from Matveev [**9**].

Definition. A presentation $\pi = \langle X|R\rangle$ is called *special* if the number of the total appearances of x_i and x_i^{-1}, x_i in X, in R is precisely three.

Suppose that P is a special polyhedron. By collapsing a maximal tree in its 1-skeleton, we get a polyhedron with a CW structure containing only one vertex. Then we can write a presentation $\omega = \omega(P)$ for the fundamental group of this new polyhedron via the standard procedure. Since there are many choices of maximal tree in the 1-skeleton of P, we may get many different presentations in this way, but they are all special. Since P is homotopic to this new polyhedron (see S. Young [**15**]), the fundamental group of P has at least a special presentation. Furthermore, if the special polyhedron P is contractible, we will have a balanced special presentation for the trivial group.

Definition. Let $\pi = \langle X|R\rangle$ be a finite presentation of a group, i.e.,

$$\pi = \langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$$

where n and m are nonnegative integers. The following operations on π are called the *generalized Nielsen operations*:

(I) $\pi \to \langle x_1, \dots, x_n \mid r_1, \dots, r_i x_j^{\varepsilon} x_j^{-\varepsilon}, r_{i+1}, \dots, r_m\rangle$ where $\varepsilon = \pm 1$.

(I^{-1}) is the inverse of (I), i.e., it removes inverse pairs from relators.

(II) $\pi \to \langle x_1, \dots, x_n \mid r_1, \dots, r_i', \dots, r_m\rangle$ where r_i is replaced by r_i' a cyclic permutation of r_i.

(III) $\pi \to \langle x_1, \dots, x_n \mid r_1, \dots, r_i^{-1}, \dots, r_m\rangle$.

(IV) $\pi \to \langle x_1, \dots, x_n \mid r_1, \dots, r_i r_j, r_{i+1}, \dots, r_m\rangle$, $i \neq j$.

(V) $\pi \to \langle x_1, \dots, x_n, x_{n+1} \mid r_1, \dots, r_m, x_{n+1} w\rangle$ where w is a word in $x_1, \dots, x_n$.

(V^{-1}) is the inverse of (V), i.e., it removes a generator x and a relator xw provided that x does not appear in any other relators or w.

Here, I define the generalized Nielsen operations (GNO's) as S. Young did in his thesis [**15**]. Although this definition of GNO's is somewhat different from the original one given by Andrews and Curtis, these two definitions are equivalent. We note that GNO's maintain the balanced presentations.

For later use, we will introduce a notation and quote a fact well known to the workers in this area. Given a presentation ω, we denote by $K(\omega)$ the 2-polyhedron with a CW structure obtained via standard procedure. The proof of the following can be found in P. Wright [**13**] or S. Young [**15**].

Proposition 1. *If π and ω are group presentations, then the following are equivalent*:

(1) *π can be transformed to ω by the generalized Nielsen operations.*

(2) $K(\pi) \swarrow\!\overset{3}{\wedge}\!\searrow K(\omega)$.

3. Construction and properties of $(\mathcal{M},+)$**.** Since our primary interest is in contractible polyhedra, in the following we only construct the monoid for contractible special polyhedra although the construction can be carried out in the set of all special polyhedra.

Let $\mathcal{P} = \{P^2 | P$ a contractible, compact special polyhedron$\}$. Define an equivalence relation $\sim$ on $\mathcal{P}$ as follows. For P and Q in $\mathcal{P}$, $P \sim Q$ if and only if P 3-deforms to Q. Let $\mathcal{M} = \{[P] | [P]$ the equivalence class with representative P in $\mathcal{P}\}$. We shall define an abelian monoid structure on $\mathcal{M}$.

First, we define a binary operation $+$ on $\mathcal{P}$. Let $P, Q \in \mathcal{P}$ be disjoint, and $p \in P^{(1)} - P^{(0)}$, $q \in Q^{(1)} - Q^{(0)}$. As an intermediate step we define

$$P \vdash\!\dashv Q = (P \cup I \cup Q)/p \sim 0 \& q \sim 1,$$

where I is the unit interval (Figure 1).

Take a small closed neighborhood of $p \in P^{(1)}$ which does not intersect $P^{(0)}$. We denote this neighborhood by interval notation $[p-\varepsilon, p+\varepsilon]$. Similarly for $q \in Q^{(1)}$. Now we expand $P \vdash\!\dashv Q$ by two 2-dimensional elementary expansions from arcs $[p-\varepsilon, p] \vee I \vee [q-\varepsilon, q]$ and $[p, p+\varepsilon] \vee I \vee [q, q+\varepsilon]$, respectively, where $[p-\varepsilon, p]$ and $[p, p+\varepsilon]$ are

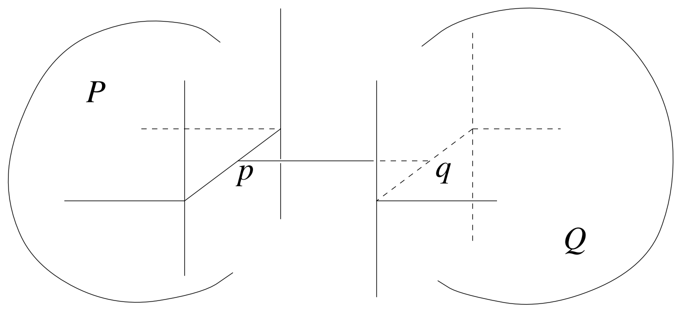


FIGURE 1.

subarcs of $[p-\varepsilon, p+\varepsilon]$, and $[q-\varepsilon, q]$ and $[q, q+\varepsilon]$ subarcs of $[q-\varepsilon, q+\varepsilon]$. After expansions, we get a band B connecting P and Q, denoted by $P \overset{\mathrm{B}}{\vdash\!\dashv} Q$. Since there are three sheets meeting along arc $[p-\varepsilon, p+\varepsilon]$ in P, we take any two of them and choose one small disk in each of the chosen sheets such that each of the two chosen disks has as part of its boundary the arc $[p-\varepsilon, p+\varepsilon]$ (Figure 2).

Similarly for $q \in Q^{(1)} \subset Q$. The two disks in P will be called $D_1(P)$ and $D_2(P)$. Likewise, $D_1(Q)$ and $D_2(Q)$ are for those two disks in Q.

Now we expand $P \overset{\mathrm{B}}{\vdash\!\dashv} Q$ by two 3-dimensional elementary expansions from disks $D_1(P) \cup B \cup D_1(Q)$ and $D_2(P) \cup B \cup D_2(Q)$, respectively, and get a solid tube connecting P and Q, denoted by $P \overset{\mathrm{T}}{\vdash\!\dashv} Q$. By collapsing $P \overset{\mathrm{T}}{\vdash\!\dashv} Q$ by two 3-dimensional elementary collapses from disks $D_1(Q)$

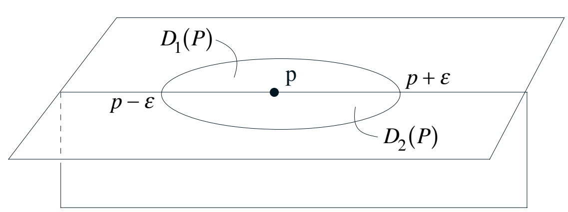


FIGURE 2.

and $D_2(P)$, respectively, we get $P+Q$, which is special with four more vertices, eight more edges (1-cells), and three more 2-components. This procedure is illustrated in four stages in Figure 3. By our construction of $P+Q$, we see that $P+Q$ 3-deforms to $P \vdash\!\dashv Q$ and hence $P+Q$ is contractible if P and Q are. In $\mathcal{P}$, this definition of $P+Q$ is not well-defined, but as far as $\mathcal{M}$ is concerned, it is, as we will show.

Now we define $+$ on $\mathcal{M}$ by $[P]+[Q]=[P+Q]$.

Temporarily, $(P+Q)_{\mathrm{pq}}$ will denote $P+Q$ obtained relative to p, q and $(P \vdash\!\dashv Q)_{\mathrm{pq}}$ will denote $P \vdash\!\dashv Q$ obtained relative to p, q. Our construction shows that $(P+Q)_{\mathrm{pq}}$ 3-deforms to $(P \vdash\!\dashv Q)_{\mathrm{pq}}$.

Lemma 2. $(P+Q)_{\mathrm{pq}}$ 3-*deforms to* $(P+Q)_{\mathrm{pq_1}}$*, where* p *in* $P^{(1)}-P^{(0)}$ *and* q, q_1 *are distinct in* $Q^{(1)}-Q^{(0)}$*.*

Proof.

Case 1. *q and q_1 lie in the same component* (1*-cell*) *of* $Q^{(1)}-Q^{(0)}$*.*

$(P \vdash\!\dashv Q)_{\mathrm{pq}} \overset{3}{\swarrow\searrow} P \cup_p \Delta \cup \overset{3}{\swarrow\searrow} (P \vdash\!\dashv Q)_{pq_1}$ (Figure 4), where Δ is the 2-cell obtained by expanding from arc $qq_1 \cup I_{\mathrm{pq}}$, where I_{pq} is the arc connecting P and Q at point p and q, and arc $qq_1 \subset Q^{(1)}$.

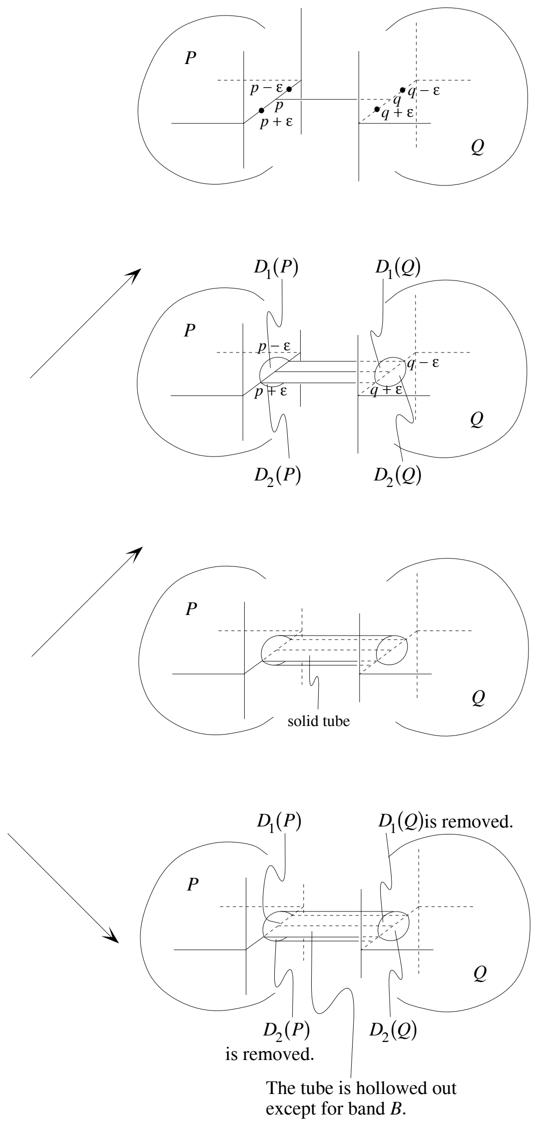
Case 2. *q and q_1 lie in different, but adjacent, components of* $Q^{(1)}-Q^{(0)}$*.*

We may extend the method developed in Case 1 to prove this case.

Case 3. *The general case.* By using Case 2 repeatedly, we get $(P \vdash\!\dashv Q)_{\mathrm{pq}} \overset{3}{\swarrow\searrow} (P \vdash\!\dashv Q)_{\mathrm{pq_1}}$, and hence $(P+Q)_{\mathrm{pq}} \overset{3}{\swarrow\searrow} (P+Q)_{\mathrm{pq_1}}$. $\square$

In fact, we can define $(P \vdash\!\dashv Q)_{\mathrm{pq}}$ for any pair of points p in P and q in Q, and by the proof of Lemma 2 we see that $(P \vdash\!\dashv Q)_{\mathrm{pq}} \overset{3}{\swarrow\searrow} (P \vdash\!\dashv Q)_{\mathrm{pq_1}}$ because we may choose a simple path in Q joining q and q_1 in place of the arc $qq_1 \subset Q^{(1)}$ as in the proof of the Lemma.

Lemma 3. $P+Q$ 3-*deforms to* $P+Q_1$ *if* Q 3-*deforms to* Q_1 *where* P*,* Q*, and* Q_1 *are special.*



P
Q
$p-\varepsilon$
p
$p+\varepsilon$
$q-\varepsilon$
q
$q+\varepsilon$
$D_1(P)$
$D_1(Q)$
$D_2(P)$
$D_2(Q)$
solid tube
$D_1(Q)$is removed.
$D_2(P)$ is removed.
The tube is hollowed out except for band B.

FIGURE 3.

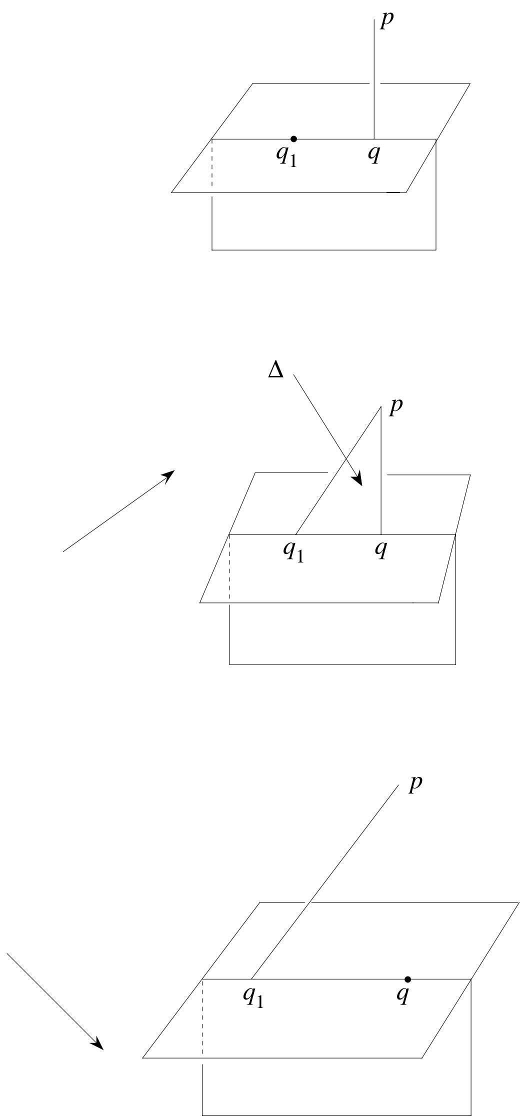


FIGURE 4.

Proof. Since $P + Q$ 3-deforms to $P \vdash\!\dashv Q$, and $P + Q_1$ 3-deforms to $P \vdash\!\dashv Q_1$, we only need to show that under the above hypothesis, $P \vdash\!\dashv Q$ 3-deforms to $P \vdash\!\dashv Q_1$. First we extend the definition of $X \vdash\!\dashv Y$ to arbitrary compact, connected polyhedra X and Y as follows: Take disjoint copies of X and Y; then join them by an arc A such that the intersection of X and A is an end point of A, and the intersection of Y and A, the other.

Since Q 3-deforms to Q_1, then there exists a finite sequence of elementary collapses and expansions such that $Q = X_0 \to X_1 \to \cdots \to X_n = Q_1$. Take a copy of P which is disjoint from each X_i, $i = 0, \dots, n$. We show inductively on i that $P \vdash\!\dashv X_i$ 3-deforms to $P \vdash\!\dashv X_{i+1}$, $i = 0, \dots, n-1$. Let $A = A_0$ and we will assume that we have defined arcs $A_0, \dots, A_i$ such that our model for $P \vdash\!\dashv X_j$ has A_j as its joining arc, $j = 0, \dots, i$, and that we have shown $P \vdash\!\dashv X_0$ 3-deforms to $P \vdash\!\dashv X_i$. We now show that $P \vdash\!\dashv X_i$ 3-deforms to $P \vdash\!\dashv X_{i+1}$.

Case 1. $X_i \overset{e}{\nearrow} X_{i+1} = X_i \cup C^k \cup B^{k+1}$. By the remark right after the proof of Lemma 2, we first adjust, via a 3-deformation, the arc A_i in $P \vdash\!\dashv X_i$ getting an arc A_{i+1} so that the intersection of B^{k+1} and A_{i+1} is empty; then expand $P \vdash\!\dashv X_i$ to $P \vdash\!\dashv X_{i+1}$ by adding B^{k+1}.

Case 2. $X_i \overset{e}{\searrow} X_{i+1}$, i.e., $X_i = X_{i+1} \cup C^k \cup B^{k+1}$. Again, by the same token, we adjust the arc A_i in $P \vdash\!\dashv X_i$ getting another arc A_{i+1} so that the intersection of B^{k+1} and A_{i+1} is empty; then collapse $P \vdash\!\dashv X_i$ to $P \vdash\!\dashv X_{i+1}$ by removing C^k and B^{k+1}. □

Proposition 4. *The operation + on $\mathcal{M}$ is well defined.*

Proof. This is an immediate consequence of Lemmas 2 and 3.

By Lemma 2, $[P+Q]$ is independent of the choice of p in $P^{(1)} - P^{(0)}$ and q in $Q^{(1)} - Q^{(0)}$. Lemma 3 shows that $[P+Q] = [P+Q_1]$ if $[Q] = [Q_1]$; hence $[P+Q] = [P_1+Q_1]$ if $[P] = [P_1]$ and $[Q] = [Q_1]$. This completes the proof. □

Definition. We define the special polyhedron $T_1(1)$ by identifying the edges of two disks as indicated below:

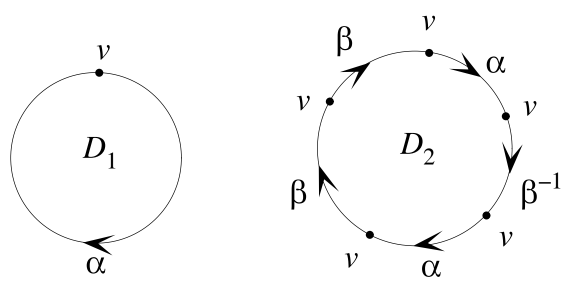


FIGURE 5.

Remark. This $T_1(1)$ is what Ikeda called Abalone [**7**], and he denoted it by $F^1_{1,1}$ [**6**]. He showed that it is a spine of the 3-ball.

Lemma 5. $T_1(1)$ *is 1-collapsible.*

Proof. For a proof, see Ikeda's paper [**6**]. □

Proposition 6. $[P] + [T_1(1)] = [P]$ *for all* $[P]$ *in* $\mathcal{M}$.

Proof. By Lemma 5, $T_1(1)$ 3-deforms to a point x. Then

$$\begin{aligned} P + T_1(1) &\overset{3}{\swarrow\!\searrow} P \vdash\!\dashv T_1(1) \\ &\overset{3}{\swarrow\!\searrow} P \vdash\!\dashv x \quad \text{(by the proof of Lemma 3)} \\ &\overset{3}{\swarrow\!\searrow} P. \qquad \square \end{aligned}$$

Theorem 7. $(\mathcal{M}, +)$ *is an abelian monoid.*

Proof. By Propositions 4 and 6, we know $+$ is well-defined and has identity $[T_1(1)]$. The associativity and commutativity of $+$ on $\mathcal{M}$ are obvious from the construction. □

In the following, we are going to discuss the significance of the monoid $(\mathcal{M}, +)$ and find a possible way to calculate it.

First we observe the following

Theorem 8. $(\mathcal{M}, +)$ *is trivial if and only if* (AC) *is true.*

Proof. This theorem is an easy corollary to P. Wright's theorem: *Every contractible compact* 2*-polyhedron* 3*-deforms to a contractible special one.*

For a proof, see P. Wright's paper [**14**], where he showed this holds even without the assumption of contractibility, although we can provide an alternate proof for this particular case. $\square$

Second, we mention an analogy between $(\mathcal{M}, +)$ and Cohen's geometric construction of the Whitehead groups. For Cohen's construction, the reader is referred to [**4**]. To see the analogy between Cohen's construction and ours, we need an intermediate construction. Let K be a contractible 2-polyhedron with a distinguished point v. Two such pairs (K, v) and (L, v) are said to be equivalent if and only if $K \overset{3}{\swarrow\!\searrow} L \ \mathrm{rel}\, v$. $[K, v]$ is the equivalence class represented by (K, v). Define an addition among those equivalence classes by $[K, v] + [L, v] = [K \cup_v L, v]$. It is easy to show that the set $(M, +)$ of all equivalence classes of contractible 2-polyhedra with a fixed distinguished point v becomes an abelian monoid under this addition and is isomorphic to $(\mathcal{M}, +)$, but Cohen's method of defining inverses, which raises dimension, does not work in our case. Our construction is obviously similar to his, but differs from his in that we have restrictions on both dimensions of polyhedra and deformations.

Since the construction of $(M, +)$ is similar to Cohen's construction of $\mathrm{Wh}\,(L)$ and $(M, +)$ is isomorphic to $(\mathcal{M}, +)$, it may be possible to use similar methods of calculating $\mathrm{Wh}\,(L)$ to calculate $(\mathcal{M}, +)$.

Third, we observe a relationship of $(\mathcal{M}, +)$ to the group. Let $\Pi = \{\pi | \pi$ a finite balanced special presentation for the trivial group$\}$. Define an equivalence relation on Π by $\pi \sim \omega$ if and only if $\pi \overset{\mathrm{GNO}}{\longrightarrow} \omega$ where "$\pi \overset{\mathrm{GNO}}{\longrightarrow} \omega$" means that the presentation π can be transformed to the presentation ω by a finite sequence of generalized Nielsen operations. It is readily seen that this is an equivalence relation on Π. Let $\Gamma = \{[\pi] | [\pi]$ equivalence class of $\pi \in \Pi\}$. Following the pattern of what we did for

$(\mathcal{M}, +)$, we shall give a monoid structure to Γ. First we define a binary operation $+$ on Π by $\pi + \omega = \langle X, Y \mid R, S\rangle$ where $\pi = \langle X \mid R\rangle$ and $\omega = \langle Y \mid S\rangle$ are in Γ, and X and Y are disjoint. Obviously, $\pi + \omega = \langle X, Y \mid R, S\rangle$ is a finite balanced special presentation for the trivial group; hence $\pi + \omega$ is in Π. Extending this operation to Γ, we define $[\pi] + [\omega] = [\pi + \omega]$. Γ with this operation $+$ is denoted $(\Gamma, +)$. To see that $(\Gamma, +)$ is a monoid, we define $f : (\mathcal{M}, +) \to (\Gamma, +)$ by sending $[P]$ to $[\omega(P)]$, where $\omega(P)$ was a balanced special presentation obtained from P by standard procedure. Then it is easy to show that f is a bijection and maintains the operations; hence, $(\Gamma, +)$ is an abelian monoid isomorphic to $(\mathcal{M}, +)$.

In $(\Gamma, +)$ the identity class is $[\langle x \mid x^2x^{-1}\rangle]$ because $T_1(1)$ 3-deforms to the dunce hat which is $K(\langle x \mid x^2x^{-1}\rangle)$. Since $(\mathcal{M}, +)$ is trivial if and only if (AC) is true, we have that $(\Gamma, +)$ is trivial if and only if (AC) is true. So here we get another version of (AC): Every balanced special presentation for the trivial group can be reduced to the empty presentation by the GNO's. We refer to this version of (AC) as the special version of (AC).

4. Special version of (AC). Here we will give a direct algebraic proof of the equivalence of the two versions, i.e., the original version in [**2**] and the special version named above.

Theorem 9. *Any finite presentation can be reduced to a special one by finitely many generalized Nielsen operations. Furthermore, a balanced presentation is reduced by this procedure to a balanced special presentation.*

Proof. Let $\pi = \langle x_1, \dots, x_n \mid r_1, \dots, r_m\rangle$ be a finite group presentation. Let k_i denote the total number of appearances of $x_i^{\varepsilon_i}$, $\varepsilon_i = \pm 1$, in $\{r_1, \dots, r_m\}$, i.e., the appearances of x_i in $\{r_1, \dots, r_m\}$ plus that of x_i^{-1} in $\{r_1, \dots, r_m\}$. For simplicity, we refer to this as the total appearances of x_i in $\{r_1, \dots, r_m\}$ hereafter. We may assume at the start that there are no inverse pairs $x_i^{\varepsilon_i}x_i^{-\varepsilon_i}$ in any r_j, where $\varepsilon_j = \pm 1$ for all i, for if there were such a pair, we could use (I^{-1}) to get rid of it.

It is easy to see that for $k_1 = 0, 1, 2, 3$, we can reduce π by a couple of GNO's to a new presentation such that each of x_1 and a possible new generator has exactly three appearances in total. From now on, we shall assume $k_1 \geq 4$. In this case,

$$\begin{aligned}\pi &\longrightarrow \langle x_1, \dots, x_n, a_{1,1} \mid r_1, \dots, r_m, a_{1,1}x_1^{-1}\rangle \quad \text{(by using (V) once)}\\ &\longrightarrow \langle x_1, \dots, x_n, a_{1,1} \mid r'_1, \dots, r'_m, a_{1,1}x_1^{-1}\rangle\\ &\qquad \text{(by using (I)–(V) several times)}\end{aligned}$$

where $\{r'_j, j = 1, \dots, m\}$ is obtained from $\{r_j, j = 1, \dots, m\}$ by replacing all but two $x_1^{\varepsilon_1}$ by $a_{1,1}^{\varepsilon_1}$. So at this point, the total appearances of x_1 in the new set of relators are three and the total appearances of $a_{1,1}$ in the new set of relators are $k_1 - 1$ while k_i, $i = 2, \dots, n$ are fixed. Since $k_1 \geq 4$, $k_1 - 1 \geq 3$. If $k_1 - 1 = 3$, then we can now turn to k_2 and start from the very beginning. If $k_1 - 1 \geq 4$, we just repeat the above procedure by introducing $a_{1,2}$ as a generator, and $a_{1,2}a_{1,1}^{-1}$ as a relator. By doing this $k_1 - 3$ times, we shall reach the point where in the new set of relators the total appearances of each of x_1, $a_{1,1}, \dots, a_{1,k_1-3}$ are three while all k_i, $i = 2, \dots, n$, remain unchanged.

Now we repeat for k_2, then k_3, and so on. Finally, we will get the desired special presentation.

Furthermore, from the above argument, we see that whenever we introduce a new generator, we introduce a new relator. So if we start with a balance presentation, we shall end up with a balanced special presentation. $\square$

For example, we may apply the method given in the proof of Theorem 9 to reduce the presentation $\langle x, y \mid x^3yx^{-2}y^{-1}, y^3xy^{-2}x^{-1}\rangle$ to a balanced special presentation of ten generators. This presentation has been suggested as a counterexample to (AC) (cf. [**10**] and [**15**]).

By Theorem 9, it is easy to see the equivalence of the original (AC) and the special version of (AC).

Since given a balanced special presentation we do not know whether it represents the trivial group, so we rephrase (AC) as follows and denote it as (AC)$'$: Let $\pi = \langle x_1, \dots, x_n \mid r_1, \dots, r_n\rangle$ be a balanced special presentation. Then π is either nontrivial, or may be reduced to the empty presentation by the generalized Nielsen operations.

Theorem 10. (AC)$'$ *is true if* $n = 1, 2, 3$.

Proof. The case $n = 1$ is very easy and is left to the reader. The proof of the cases $n = 2, 3$ are based on a discussion of several subcases.

Let π be a balanced special presentation with two or three generators. Given r as a relator in π, denote by $l(r)$ the total number of appearances of generators in r, e.g., $l(r_1 r_1^{-2}) = 3$. There is no loss of generality in assuming that $l(r_1) \leq l(r_2)$ where r_1, r_2 are the two relators in π for the case $n = 2$, and $l(r_1) \leq l(r_i)$, $i = 2, 3$, where r_1, r_2, and r_3 are the three relators in π for the case $n = 3$.

In each of the two cases $n = 2, 3$, we need to consider the following three cases $l(r_1) = 1, 2$, and 3. The case $l(r_1) = 1$ is easy to verify for both $n = 2$ and 3. The case $l(r_1) = 2$ is divided into three subcases, and the case $l(r_1) = 3$, four subcases in both cases $n = 2$ and 3. The verification of these cases and subcases is left to the interested reader. $\square$

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