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VELOCITY DEPENDENT BOUNDARY CONDITIONS FOR THE DISPLACEMENT IN A ONE DIMENSIONAL VISCOELASTIC MATERIAL

KENNETH KUTTLER

Introduction. Conservation of mass and momentum in one dimension may be written as

$$\rho_0(X) = \rho(t, X)\frac{\partial y}{\partial X}, \tag{0.1a}$$

$$\rho_0(X)\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial X}, \tag{0.1b}$$

where v is the velocity, $\rho_0(X)$ is the initial density, X is the material coordinate, y the spatial coordinate, and σ is the stress. These equations do not depend on the material under consideration. It is convenient to introduce the Lagrange mass variable [**17**], x defined by

$$x = \int_0^X \rho_0(z)\,dz.$$

Then writing (0.1b) in terms of the Lagrange mass variable,

$$\frac{\partial v}{\partial t} = \frac{\partial \sigma}{\partial x}. \tag{0.2}$$

It is assumed in this paper that $\sigma = -(P + q)$ where P is pressure and q is the part of the stress due to viscosity. We also assume internal energy is constant so that it makes sense to let

$$P = P(V)$$

where V is the specific volume, $V = \rho^{-1}$ for ρ the density.

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We write the stress in terms of a pressure which is a function of specific volume because we do not exclude the possibility that the material is a gas or a fluid. In fact, we are interested in the possibility of phase changes in the material which could be a solid, liquid or gas. The function of state, P, is therefore not required to be monotone so that for some values of pressure, a number of different values for specific volume are possible in the steady state. It seems more traditional to use spatial coordinates when dealing with fluids but our use of the material coordinate in this context is certainly not unique [**17**].

Let $u = y - X$ be the displacement. Then from (0.1a)

$$V(t,x) = \frac{\partial u}{\partial x} + V_0(x). \tag{0.3}$$

The viscous stress is proportional to the spatial velocity gradient, $\partial v/\partial y$, the case $q = -\delta(\partial v/\partial y)$ where δ is constant, being a well known example called Navier Stokes viscosity. Changing to the Lagrange mass variable this becomes

$$-\frac{\delta}{V}\frac{\partial v}{\partial x}$$

so in this paper q is of the form

$$-\alpha(V)\frac{\partial v}{\partial x}. \tag{0.4}$$

where $\alpha(V) > 0$. Typically α will not be bounded above and also $\lim_{V\to\infty} \alpha(V) = 0$. If we take α to be constant, the mathematical treatment is much easier, but this assumption, which might be suitable for solids, is not appropriate for a general material undergoing phase changes involving gasses or liquids because if one assumes this and changes to spatial coordinates, the viscous stress is of the form $-(\delta/\rho)(\partial v/\partial y)$. This says that for a given value of the spatial velocity gradient, $\partial v/\partial y$, the viscous stress is large when the density is small and small when the density is large.

Writing (0.2) in terms of the displacement yields the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = (-P(V) + \alpha(V)u_{tx})_x, \tag{0.5}$$

where $V = u_x + V_0$. Letting the Lagrange mass variable lie in the interval $[0,1]$, it would be of interest to consider initial-boundary value problems for (0.5). The most extensively discussed [**4, 5, 13, 14, 17, 18**] boundary conditions for this problem are those in which the displacement is given to be 0 at each end of the interval $[0,1]$. That is, the ends of the material are fixed. But it also seems reasonable to specify the stress at the ends of the interval. This leads to boundary conditions of the form

$$-P(V(t,0)) + \alpha(V(t,0))u_{tx}(t,0) = k_0, \tag{0.6a}$$

$$-P(V(t,1)) + \alpha(V(t,1))u_{tx}(t,1) = k_1, \tag{0.6b}$$

where k_0 and k_1 are given. In several earlier papers [**7, 8, 9, 10**] k_i is taken to be a negative constant. This corresponds to forces that tend to compress the material. In [**8**], the k_i were also allowed to depend on t. However, it would be of interest to allow these k_i to depend not just on t but also on the velocity of the ith end of the material and to eliminate all restrictions on the sign of k_i. The consideration of such boundary conditions is the main goal of this paper.

For example, consider a standard oil pump consisting of machinery on the surface of the ground and a very long "sucker rod" which is moved up and down by gravity and this machinery. We treat this "sucker rod" as a one dimensional continuum. The machinery exerts a force on the upper end of the rod which depends on displacement of that end and possibly on velocity while the bottom end experiences a force which is directed upward when moving down, and down when the end is moving up.

There is another reason for allowing the prescribed traction forces to depend on velocity. Imagine that a constant force is applied to one end of the material and that the end is moving in the direction of the applied force at high speed. it follows that the power supplied by whatever produces this force must be very large, but real machines typically do not have limitless power. Thus, without some estimate on the velocity of the end, a constant applied force is overly idealized. For weak solutions of (0.5) and (0.6), the best that can be said is that the velocity of each end is in $L^2(0,T)$ [**9**]. Regularity theorems are available in the case where the initial velocity and specific volume are in $H^1(0,1)$ [**10**], but even in this setting, the estimates for the velocity of each end can involve very large constants, especially when the viscosity is small.

The exact description of the traction forces are in (3.2) and (5.6) of this paper. The idea is to consider a given traction force as a sum of two forces, one which acts opposite in direction to the velocity (like air resistance), and the other force being supplied by something that has finite power. Thus one of these produces dissipation while the other can amplify the motion somewhat but does not have limitless power.

If the boundary conditions for (0.5) involve a given constant compressive force at one end while the other end is fixed, it can be shown [**9**] that our viscous damping causes decay to a steady state even for a non-monotone equation of state just as in [**15**]. Furthermore, the very interesting result of [**15**] that discontinuous steady states can occur as long time limits even if the initial data is smooth is also obtained in [**9**] along with the result that discontinuous specific volumes are never obtained as long time limits for any boundary conditions unless the equation of state is non-monotone. The boundary conditions in the present paper are more general so one could ask for minimal conditions under which decay to a steady state will be assured. It will be necessary to restrict the boundary conditions from those considered here, but such a project is a topic for another paper.

Our approach to proving existence and uniqueness for global solutions is fundamentally different from the usual procedure for problems such as these. One normally obtains estimates on local classical solutions and then uses these estimates to extend these local solutions to global ones. Here, we modify the constitutive functions and global weak solutions are obtained under these modifications which are realistic exactly when suitable pointwise bounds on the strain or specific volume can be obtained. These estimates are then established and an "approximate" problem is determined whose solutions have the property that on the global time interval of interest, the specific volume or strain remains in the unmodified region of the constitutive functions. The resulting solutions are weak, not classical, but the procedure has the advantage of allowing the consideration of problems of extreme generality. For example, in [**9**] it was not even necessary to require $\alpha(V)$, the coefficient of viscosity, to be a continuous function of V. Neither is it necessary to have any smoothness on the body force or initial data. There is some interest in such generalization [**17**]. Also, this approach adapts well to the inclusion of memory dependent terms and can be used as a basis for developing numerical methods which can be shown to converge to the solution of the boundary value problem of interest. If more smoothness is assumed on α, P, the initial specific volume, and the initial velocity, one can obtain regularity theorems for these weak solutions duplicating, and generalizing the boundary conditions of, more standard results [**10**]. These regularity theorems involve some fairly subtle arguments because of the nonlinearity in the viscosity. Everything is much easier if $\alpha(V)$ is a constant. For example, in [**15**] a clever transformation is used to reduce some initial boundary value problems to the form $u' + Au = f(u)$ where A generates an analytic semigroup. This makes possible the use of a very well developed theory, but attempts to adapt this method to (0.5) have been unsuccessful. The problem is not semilinear and it seems impossible to transform it into a semilinear problem. Furthermore, the boundary conditions (0.6) in which k_0 and k_1 are allowed to be functions of velocity, while they are physically reasonable, introduce additional nonlinearities which complicate the mathematics. It would be interesting to determine the extent to which the regularity theory of [**10**] can be generalized to the velocity dependent boundary conditions considered in this paper.

Although no one has dealt with prescribed velocity dependent traction forces before this, partial differential equations having a formal resemblance to (0.5) have been extensively studied. In [**4**], a system of nonlinear wave equations with nonlinear viscosity is considered. If this system is specialized to one dimension, it does not contain (0.5) as a special case because we do not assume that P is monotone and the viscosity term here would only be a special case if $\alpha(V)$ is a constant. We also make fewer demands on the regularity of the constitutive functions. The isentropic gas model considered by Kanel and discussed in Smoller [**19**] is formally similar to our (0.5) but does not include the nonmonotone function of state. Furthermore, there are no boundary conditions at all, the treatment being for the initial value problem on the real line. Probably the paper most closely related to what we are discussing here is [**5**]. This is the first paper to deal with a nonmonotone equation of state along with the Navier Stokes viscosity, $V^{-1}(\partial v/\partial x)$ for v the velocity. Like [**4**], the boundary condition is of zero displacement on the boundary. The paper by MacCamey [**14**] involves a viscosity term formally similar to ours, but the coefficient of viscosity is assumed to be bounded away from zero while the function of state corresponding to our P is assumed monotone. Because of the form of the viscosity, this paper by MacCamey is not a special case of [**1**].

Sections 2–4 of the paper are devoted to developing existence and uniqueness of global solutions to the initial boundary value problems for (0.5) in which the constitutive functions have been modified. In these sections, we assume a global Lipschitz condition on P and we assume the coefficient of viscosity $\alpha(V)$ is bounded away from zero and infinity contrary to reasonable physical assumptions. We also make no assumptions on the sign of the given traction forces k_i because there is no need to do so. It is not until Section 5 that we include physically reasonable assumptions on α, P, and the traction forces and show that these physically reasonable assumptions can replace the unrealistic Lipschitz assumptions and bounds on the viscosity coefficient.

Let E be a subspace of $H^1(0,1)$ closed in $H^1(0,1)$, which contains the test functions. For example, E could be $\{u \in H^1(0,1) : u(1) = 0\}$ or E could be $H^1(0,1)$. We do not specify E because each choice of E will determine "stable" boundary conditions and we want to retain as much generality as possible. We will eventually insist that E contain $\{u \in H^1(0,1) : u(1) = 0\}$ which eliminates $H_0^1(0,1)$, the space appropriate for studying zero displacement boundary conditions. This case has been dealt with elsewhere [**9**]. Let $\varphi \in C_0^\infty(0,T;E)$. This means that φ is an infinitely differentiable map from $(0,T)$ to the space E which vanishes at 0 and at T, $\varphi(t)$ being an element of E for each t.

Multiply (0.5) by φ, integrate by parts and use (0.6). This yields for all $\varphi \in C_0^\infty(0,T;E)$ the equation

(0.7a)
$$-\int_0^T \int_0^1 u_t \varphi_t \, dx \, dt + \int_0^T \int_0^1 \alpha(V) u_{tx} \varphi_x \, dx \, dt - \int_0^T \int_0^1 P(V) \varphi_x \, dx \, dt$$
$$= \int_0^T \left[k_1\left(t, u_t(t,1), u(t,1)\right)\varphi(t,1) - k_0\left(t, u_t(t,1), u(t,0)\right)\varphi(t,0)\right] dt$$

to which we add the initial conditions

$$u_t(0,x) = v_1(x), \qquad u(0,x) = 0. \tag{0.7b}$$

Stable boundary conditions are obtained by choosing E while variational conditions are determined by the above variational form of our problem. For example, if $E = H^1(0,1)$, we obtain both boundary conditions (0.6). If $E = [u \in H^1(0,1) : u(1) = 0]$, we obtain a zero displacement condition at 1 and the condition (0.6a) at 0.

We will always study (0.7) in the abstract form described in the following sections. We let $H = L^2(0,1)$ and identify H and H', the dual space of H. Thus we may write $E \subseteq H = H' \subseteq E'$. The symbol $\langle f, u\rangle$ will denote the value $f(u)$ where $f \in E'$, the dual space of E, and $u \in E$, while the symbol (u, v) will denote the value of an inner product in a Hilbert space. Since E is a Hilbert space, we can define the Riesz map, $R : E \to E'$ by $\langle Ru, v\rangle = (u, v)$. When we write "$'$" we mean d/dt. More precisely, the "$'$" will denote differentiation in the sense of E' valued distributions. Thus for $\varphi \in C_0^\infty(0,T)$ and $f \in L^1(0,T;E')$, we consider f as an E' valued distribution according to the rule

$$f'\varphi = -\int_0^T f(t)\varphi'(t)dt,$$

and we say $f' \in L^1(0,T;E')$ if there exists $g \in L^1(0,T;E')$, necessarily unique, such that

$$f'(\varphi) = \int_0^T g(t)\varphi(t)dt$$

for all $\varphi \in C_0^\infty(0,T)$. It is well known [**12, 6**] that if $f \in L^2(0,T;E)$ and $f' \in L^2(0,T;E')$, then $f \in C(0,T;H)$, the space of continuous H valued functions.

The symbol $L^2(0,T;X)$ will denote the space of measurable X valued functions which are square integrable. The symbol $W_1^1(0,1)$ is the space of functions in $L^1(0,1)$ whose weak derivatives are also in $L^1(0,1)$.

Supporting theorems. Let V and W be reflexive Banach spaces with V dense in W ($\overline{V} = W$ where closure is taken in the topology of W) and for $u \in V$, $\|u\|_V \geq \|u\|_W$. Thus we may consider $W' \subseteq V'$. Let $B : W \to W'$ be a linear operator satisfying

(1.1) $$\langle Bu, u\rangle \geq 0, \qquad \overline{\langle Bu, v\rangle} = \langle Bv, u\rangle$$

for all $u, v \in W$. Let $A : L^2(0,T;V) \to L^2(0,T;V')$ be an operator satisfying

(1.2a) A is hemicontinuous $(\lim_{t\to 0}\langle A(u+tv), w\rangle = \langle Au, w\rangle)$

For some $\lambda \geq 0$,

(1.2b) $$A + \lambda B \text{ is strictly monotone,}$$

(1.2c) $$\lim_{\|u\| \to \infty} \frac{\operatorname{Re}\langle Au, u\rangle + \lambda\langle Bu, u\rangle}{\|u\|} = \infty,$$

where B is considered as a map from $L^2(0,T;V)$ to $L^2(0,T;V')$ according to the rule $(Bu)(t) = B(u(t))$ and $\|u\|$ in (1.2c) signifies the norm in $L^2(0,T;V)$, the space of measurable, square integrable, V valued functions [**3**]. Let

(1.3) $$X = \{u \in L^2(0,T;V) \text{ s.t. } (Bu)' \in L^2(0,T;V')\}$$

where the $'$ denotes d/dt in the sense of V' valued distributions.

Under these conditions, the following Theorem is a special case of Theorem 1 of [**6**].

Theorem 1.1. *If $u \in X$, $t \to B(u(t))$ is equal to a function in $C(0,T;W')$ almost everywhere.*

We will denote the continuous function whose existence is the conclusion of Theorem 1.1 by Bu. Then we have the following theorem which is a special case of Theorem 5 of [**6**].

Theorem 1.2. *If $f \in L^2(0,T;V')$ and $u_0 \in W$, there exists a unique $u \in X$ satisfying*

(1.4a) $$(Bu)' + Au = f \qquad \text{in } L^2(0,T;V'),$$

(1.4b) $$Bu(0) = Bu_0,$$

Now let E be a reflexive Banach space, and let H be a Hilbert space (these will be the spaces mentioned in the introduction) with

(1.5a)
$$E \subseteq H. \qquad E \text{ dense in } H \text{ (in topology of } H), \qquad \|\ \|_E \geq |\ |_H$$

Then, identifying H and H', we write

$$E \subseteq H = H' \subseteq E'. \tag{1.5b}$$

Let

$$g, g' \in L^2(0, T; E') \tag{1.5c}$$

where the $'$ on g signifies E' valued distributions, and suppose $M : E \to E'$ satisfies

$$M \text{ is bounded} \tag{1.6a}$$

$$\lim_{t \to o} M(u + tw) = Mu \tag{1.6b}$$

$$\langle Mu - Mv, u - v \rangle \geq \delta^2 \rho(u - v)^2 \tag{1.6c}$$

where ρ is a seminorm satisfying

$$\|u\|_E^2 = \rho(u)^2 + |u|_H^2. \tag{1.6d}$$

Let

$$u_0 \in E, \;\; w_1 \in H \tag{1.7a}$$

and suppose

$$g(0) - Mu_0 = w_1 \tag{1.7b}$$

Then with (2.5)–(2.7) we obtain the regularity theorem of [**9**]

Theorem 1.3. *There exists a unique solution to*

$$u' + Mu = g \tag{1.8a}$$

$$u(0) = u_0, \qquad u \in L^2(0, T; E), \;\; u' \in L^2(0, T; E'). \tag{1.8b}$$

This solution has the regularity properties

$$u, \qquad u' \in L^2(0, T; E), \;\; u' \in L^\infty(0, T; H) \tag{1.8c}$$

Preliminary considerations. In this section, assume

$$|P(V_1) - P(V_2)| \leq K_0|V_1 - V_2|, \tag{2.1a}$$

$$|P(V)| \leq C_0 \tag{2.1b}$$

and

$$K_0 > \alpha(V) \geq \delta > 0 \tag{2.1c}$$

for some constant K_0. Also let α be continuous and let β be defined by

$$\beta(V) = \int_1^V \alpha(s)ds. \tag{2.2}$$

Letting E be a closed subspace of $H^1(0,1)$ containing $C_0^\infty(0,1)$, define mappings $Q(u)$, M, and N from E to E' by

$$\langle Q(u)w, v\rangle = (\alpha(u_x + V_0)w_x, v_x)_H, \tag{2.3a}$$

$$\langle Mw, v\rangle = (\beta(w_x + V_0), v_x)_H, \tag{2.3b}$$

$$\langle Nw, v\rangle = -(P(w_x + V_0), v_x)_H. \tag{2.3c}$$

Let $\mathcal{V} = L^2(0,T;E)$, $\mathcal{V}'$ be it's dual space, and define $\mathcal{Y}$ by

$$\mathcal{Y} = \{u \in \mathcal{V} : u' \in \mathcal{V} \text{ and } u'' \in \mathcal{V}' = L^2(0,T;E')\}. \tag{2.4a}$$

(2.4b)
$$\|u\|_{\mathcal{Y}} = \| u \|_{\mathcal{V}} + \| u' \|_{\mathcal{V}} + \| u'' \|_{\mathcal{V}'}$$

We want to consider the problem

$$u'' + (Mu)' + Nu = f \tag{2.5a}$$

$$u(0) = 0, \qquad u'(0) = v_1 \tag{2.5b}$$

$$u \in \mathcal{Y} \tag{2.5c}$$

where $f \in \mathcal{V}' = L^2(0, T : E')$ and $v_1 \in H = L^2(0,1)$. In particular, we want to consider the dependence of solutions to (2.5) on f.

Theorem 2.1. *There exists a unique solution to* (2.5). *Furthermore, if u_n is the solution to* (2.5) *with f_n replacing f and if $f_n \to f$ in $\mathcal{V}'$ then $u_n \to u$ in $\mathcal{V}$ and $u_n \rightharpoonup u$ in $\mathcal{Y}$.*

Proof. Let R be the Riesz map of E onto E' and let $w_1 \in E$ be such that

$$v_1 + M(0) = Rw_1.$$

Consider the system

$$\left(\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix}\right)' + \begin{pmatrix} Nu \\ Mu - Rw \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \tag{2.6a}$$

and initial condition

$$\begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix}(0) = \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} w_1 \\ 0 \end{pmatrix} \tag{2.6b}$$

satisfying

$$u, w \in \mathcal{V}, (Rw)', u' \in \mathcal{V}' \tag{2.6c}$$

It follows from Theorem 1.1 and 1.2 that there exists a unique solution to (2.6). In using these theorems, $V = E \times E$, $W = E \times H$, $H = L^2(0,1)$, and B is given by

$$B = \begin{pmatrix} R & O \\ O & I \end{pmatrix}$$

while A is given by

$$A\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} Nu \\ Mu - Rw \end{pmatrix}$$

It is clear that if u solves (2.5), then if $u' + Mu = Rw$, $(w,u)^T$ solves (2.6). Conversely, suppose $(w,u)^T$ solves (2.6). Then since $(Rw)' \in \mathcal{V}'$,

$$w' = R^{-1}(Rw)' = R^{-1}(f - Nu) \in \mathcal{V}. \tag{2.7}$$

Furthermore it follows from Theorem 1.3 applied to bottom line of (2.6a) that $u' \in \mathcal{V}$.

Therefore,

$$(Mu)' = Q(u)u' \in \mathcal{V}', \tag{2.8}$$

and from the above,

$$\begin{aligned} u' + Mu &= \int_0^t (f - Nu)\, ds + v_1 + M(0), \\ u'(0) &= v_1, \;\; u(0) = 0. \end{aligned} \tag{2.9}$$

Using (2.8) in (2.9), we see that u solves (2.5). Multiply (2.5a) by u' and do $\int_0^t$. This yields

$$\begin{aligned} \frac{1}{2}|u'(t)|_H^2 - \frac{1}{2}|v_1|_H^2 &- \int_0^t (P(u_x(s) + V_0), u_x'(s))\, ds \\ &+ \int_0^t (\alpha(u_x(s) + V_0)u_x'(s), u_x'(s))_H\, ds \\ &= \int_0^t \langle f(s), u'(s)\rangle ds \end{aligned} \tag{2.10}$$

Using (2.1) we obtain

$$\begin{aligned} \frac{1}{2}|u'(t)|_H^2 - \frac{1}{2}|v_1|_H^2 + \delta \int_0^t |u_x'(s)|^2\, ds \le \|f\|_{\mathcal{V}'} \left(\int_0^t \|u'(s)\|_E^2\, ds \right)^{1/2} \\ + C_0\sqrt{T}\left(\int_0^t |u_x'(s)|_H^2\, ds\right)^{1/2} \end{aligned}$$

Adding $\delta \int_0^t |u'(s)|_H^2\, ds$ to both sides we can get

$$\begin{aligned} |u'(t)|_H^2 - |v_1|_H^2 + \delta \int_0^t \|u'(s)\|_E^2\, ds \\ \le K_\delta(\|f\|_{\mathcal{V}}^2 + C_0^2 T) + \frac{\delta}{2}\int_0^t |u'(s)|_H^2\, ds. \end{aligned} \tag{2.11}$$

where K_δ is a constant depending on δ. By Gronwall's inequality, there exists a constant $M = M(T, \delta, \|f\|_{\mathcal{V}}, C_0, |v_1|_H)$ such that

$$\|u'\|_{C(0,T;H)} + \|u'\|_{\mathcal{V}} + \|u\|_{L^\infty(0,T;E)} + \|u''\|_{\mathcal{V}'} \le M \tag{2.12}$$

Now let u_n be the solution to (2.5) with f replaced with f_n. Using (2.9) we can write:

$$\begin{aligned}\frac{1}{2}|u(t) - u_n(t)|_H^2 &+ \delta \int_0^t |u_x(s) - u_{nx}(s)|_H^2 \, ds \\ &\le \int_0^t \left(\int_0^s \|f(r) - f_n(r)\|_{E'} \, dr \right) \|u(s) - u_n(s)\|_E \, ds \\ &\quad + K_0 \int_0^t \left(\int_0^s |u_x(r) - u_{nx}(r)|_H \, dr \right) |u_x(s) - u_{nx}(s)|_H \, ds\end{aligned}$$

It is routine to obtain the inequality

$$\begin{aligned}|u(t) - u_n(t)|_H^2 &+ \int_0^t \|u(s) - u_n(s)\|_E^2 \, ds \\ &\le K\|f - f_n\|_{\mathcal{V}'}^2 + K \int_0^t (1+s) \int_0^s \|u(r) - u_n(r)\|_E^2 \, dr \, ds\end{aligned}$$

where K is some constant depending on δ, K_0 and T. Thus, an application of Gronwall's inequality yields

$$\begin{aligned}|u(t) - u_n(t)|_H^2 &+ \int_0^t \|u(s) - u_n(s)\|_E^2 \, ds \\ &\le K\|f - f_n\|_{\mathcal{V}'}^2 e^{t+t^2/2}.\end{aligned} \tag{2.13}$$

Therefore, $u_n \to u$ in $\mathcal{V}$ if $f_n \to f$ in $\mathcal{V}'$.

To get the weak convergence in $\mathcal{Y}$, it remains to show that $u_n' \rightharpoonup u'$ in $\mathcal{V}$ and that $u_n'' \rightharpoonup u''$ in $\mathcal{V}'$. But this is routine from the estimate (2.12) along with the strong convergence of u_n to u. This proves the Theorem. □

3. Existence. For $u \in \mathcal{Y}$, let $f(u, u') \in \mathcal{V}'$ be given by

$$\begin{aligned}\langle f(u, u'), v \rangle = &\int_0^T k_1(t, u'(t)(1), u(t)(1)) v(t)(1) \, dt \\ &- \int_0^T k_0(t, u'(t)(0), u(t)(0)) v(t)(0) \, dt\end{aligned} \tag{3.1}$$

where for $i = 0, 1$,

(3.2a)
$$|k_i(t, z_1, z_2) - k_i(t, w_1, w_2)| \le K_1(|z_1 - w_1| + |z_2 - w_2|),$$

(3.2b) $$|k_i(t, z_1, z_2)| \le K_1,$$

for some constant K_1.

Lemma 3.1. *If $u_n \rightharpoonup u$ in $\mathcal{Y}$, then*

(3.3a)
$$u_n'(\cdot)(1) \to u'(\cdot)(1), u_n'(\cdot)(0) \to u'(\cdot)(0) \qquad \text{in } L^2(0,T)$$

(3.3b)
$$u_n(\cdot)(1) \to u(\cdot)(1), u_n(\cdot)(0) \to u(\cdot)(0) \qquad \text{in } L^2(0,T)$$

(3.3c)
$$f(u_n, u_n') \to f(u', u) \qquad \text{in } \mathcal{V}'$$

Proof. Let W be a reflexive Banach space satisfying:

$$E \subseteq W \subseteq H$$

with the imbedding of E into W compact and also let $W \subseteq C([0,1])$ with continuous injection. For example, let $W = H^{3/4}(0,1)$. By a well-known result in Lions [**11**, p. 58], $u_n \rightharpoonup u$ in $\mathcal{Y}$ implies that $u_n' \to u'$ in $L^2(0,T;W)$. Therefore, $u_n'(\cdot)(e) \to u'(\cdot)(e)$ for $e \in \{0,1\}$. This proves (3.3a),(3.3b) is similar. Finally (3.3c) follows from this and (3.2a).

Theorem 3.1. *There exists a solution to the problem*

(3.4a) $$u'' + (Mu)' + Nu = f(u, u')$$

(3.4b) $$u(0) = 0, \quad u'(0) = v_1 \in H$$

(3.4c) $$u \in \mathcal{Y}.$$

Proof. By (3.2b). $\|f(w,w')\|_{\mathcal{V}'}$ is bounded independently of the choice of $w \in \mathcal{Y}$. For $w \in \mathcal{Y}$, let $\theta(w)$ be the solution of the problem.

$$u'' + (Mu)' + Nu = f(w, w'),$$

$$u(0) = 0, \quad u'(0) = v_1 \in H,$$
$$u \in \mathcal{Y}.$$

Then by (2.12) θ maps $\mathcal{Y}$ into a bounded subset of $\mathcal{Y}$. Lemma 3.1 implies that θ is weakly continuous on this bounded subset of $\mathcal{Y}$. By the Tikhanov fixed point theorem [**2**], θ has a fixed point in $\mathcal{Y}$. This proves Theorem (3.1). □

4. Uniqueness. Establishing uniqueness of problem (3.4) is not so easy and we will only do so under additional assumptions. In addition to (2.1) assume

$$|P'(V_1) - P'(V_2)| \leq K_0|V_1 - V_2| \tag{4.1a}$$

and

$$|\alpha(V_1) - \alpha(V_2)| \leq K_0|V_1 - V_2| \tag{4.1b}$$

Also assume $V_0 \in H^1(0,1)$ in addition to being in $L^\infty(0,1)$.

We will use the following lemma.

Lemma 4.1. *Let u, $u' \in \mathcal{V}$, and let $u_t(\cdot,\cdot)$ be a Borel measurable representative for $u'(\cdot)(\cdot)$ and $u_{tx}(\cdot,\cdot)$ a Borel measurable representative for $\partial u'(\cdot)(\cdot)/\partial x$. If $u_0 = u(0)$, define*

$$u_x(t,x) = u_{0x}(x) + \int_0^t u_{tx}(s,x)\,ds \tag{4.2a}$$

$$u(t,x) = u_0(x) + \int_0^t u_t(s,x)\,ds. \tag{4.2b}$$

Then

$$u(t,\cdot) = u(t) \text{ a.e.}, \quad u_t(t,\cdot) = u'(t) \text{ a.e.},$$
$$u_x(t,\cdot) = \frac{\partial u}{\partial x}(t) \text{ a.e.}, \quad u_{tx}(t,\cdot) = \frac{\partial u'(t)}{\partial x} \text{ a.e.},$$
$$u_x(\cdot,\cdot) = \frac{\partial u}{\partial x}(\cdot,\cdot) \text{ in the sense of distributions},$$
$$u_{tx}(\cdot,\cdot) = \frac{\partial u_t}{\partial x}(\cdot,\cdot) \text{ in the sense of distributions.}$$

Because of this lemma, we adopt the convention that if u, $u' \in \mathcal{V}$, $u(\cdot,\cdot)$ will be the above measurable representative. Also we let $V(t,x) = V_0(x) + u_x(t,x)$.

Another lemma which will be of use is

Lemma 4.2. *Let X, Y and Z be three Banach spaces with X reflexive, $X \subseteq Y \subseteq Z$, $\| \ \|_Y \leq \| \ \|_X$ and the injection map of X into Y is compact. Then for all $\varepsilon > 0$ there exists a constant K_ε such that*

$$\|u\|_Y \leq \varepsilon \|u\|_X + K_\varepsilon \|u\|_Z$$

for all $u \in X$.

This is a well-known lemma due to Lions [**11**, pp. 58, 59].

Now define, for u a solution to problem (3.4),

$$q(t,x) = \int_0^x u'(t)(z)dz + \int_0^t P(V(s,x))ds - \beta(V(t,x)). \tag{4.3}$$

It is routine to show that in the sense of distributions

$$q_x = v_1 - (\beta(V_0))_x \in L^2((0,T)\times(0,1)). \tag{4.4}$$

It follows that there exists a set of measure zero, $D \subseteq [0,T]$ such that for all $t \notin D$, $x \to q(t,x)$ is absolutely continuous a.e. and for $t \notin D$, the following holds for a.e. $x \in [0,1]$:

(4.5a)

$$q(t,x) = q(t,0) + \int_0^x q_x(t,z)\,dz$$

$$= q(t,0) + \int_0^x v_1(z)dz - \beta(V_0(x)) + \beta(V_0(0)). \tag{4.5b}$$

Using (4.5) and (2.9), a little computation shows that for $\varphi \in$

$C_0^\infty(0,T;E)$,

$$(4.6)\quad -\int_0^T q(t,0)(\varphi(t,1)-\varphi(t,0))\,dt$$
$$= \int_0^T \varphi(t,1)\left[\int_0^1 v_1(z)\,dz - \int_0^1 u'(t)(z)\,dz\right]dt$$
$$+ \int_0^T \beta(V_0(0))[\varphi(t,1)-\varphi(t,0)]\,dt$$
$$+ \int_0^T \int_0^t k_1(s,u'(s)(1),u(s)(1))\varphi(t,1)\,ds\,dt$$
$$- \int_0^T \int_0^t k_0(s,u'(s)(0),u(s)(0))\varphi(t,0)\,ds\,dt$$

If $E = H_0^1(0,1)$, (4.6) gives no information about $q(t,0)$ so from now on we assume

$$(4.7)\qquad E \supseteq \{u \in H^1(0,1) : u(1) = 0\}$$

Therefore, we may take $\varphi(t,1) = 0$ and $\varphi(t,0) \in C_0^\infty(0,T)$. Therefore (4.6) implies

$$(4.8)\qquad q(t,0) = -\beta(V_0(0)) - \int_0^t k_0(s,u'(s)(0),u(s)(0))\,ds \quad \text{a.e.t.}$$

Therefore, enlarging D, if necessary, (4.5) implies that for $t \notin D$, $m(D) = 0$,

(4.9)
$$q(t,x) = -\int_0^t k_0(s,u'(s)(0),u(s)(0))ds - \beta(V_0(x)) + \int_0^x v_1(z)\,dz,$$

for a.e.x. Letting

$$(4.10a)\qquad b(t) = -\int_0^t k_0(s,u'(s)(0),u(s)(0))\,ds,$$

$$(4.10b)\qquad a(x) = -\beta(V_0(x)) + \int_0^x v_1(z)\,dz,$$

We obtain

Lemma 4.3. *If q is given by (4.3), there exists a set of measure zero $D \subseteq [0,T]$ such that for $t \notin D$*

$$q(t,x) = b(t) + a(x) \text{ a.e. } x \tag{4.11}$$

where $a(\cdot) \in H^1(0,1)$.

Now suppose u_1 and u_2 are two solutions to (3.4). Let the corresponding specific volumes be denoted by V_1 and V_2, respectively, and let the exceptional subsets of $[0,T]$ be D_1 and D_2. Let $D = D_1 \cup D_2$. Also let $b_i(t) = -\int_0^t k_0(s, u_i'(s)(0), u_i(s)(0))\, ds$.

Lemma 4.4. *For each $\varepsilon > 0$ there is a constant K_ε such that*

$$\begin{aligned} &|b_1(t) - b_2(t)| \\ &\quad \le \int_0^t \epsilon |u_{1x}'(s) - u_{2x}'(s)|_H + K_\varepsilon |u_1'(s) - u_2'(s)|_H \, ds \end{aligned} \tag{4.12}$$

Proof. Let $\eta > 0$ be given and use Lemma 4.2 with $X = H^1(0,1)$, $Y = C(0,1)$, $Z = L^2(0,1)$ to obtain

$$\begin{aligned} |u_1'(s)(0) - u_2'(s)(0)| &\le \|u_1'(s) - u_2(s)\|_{C(0,1)} \\ &\le \eta \|u_1'(s) - u_2'(s)\|_{H^1} + K_\eta |u_1'(s) - u_2'(s)|_H \end{aligned}$$

Then using (3.2a) and the inequality

$$|u_1(s)(0) - u_2(s)(0)| \le \int_0^s \|u_1'(r) - u_2'(r)\|_{C(0,1)} \, dr,$$

one obtains

$$\begin{aligned} &|b_1(t) - b_2(t)| \\ &\quad \le K_1(1+T) \int_0^t (\eta \|u_1'(s) - u_2'(s)\|_{H^1} + K_\eta |u_1'(s) - u_2'(s)|_H)\, ds. \end{aligned}$$

which implies (4.12) since η is arbitrary. □

From Lemma 4.3, we can obtain for $t \notin D$, the equations

$$V_i(t,x) = \beta^{-1}\biggl(\int_0^x u_i'(t)(z)dz + \int_0^t P(V_i(s,x)\biggr)\, ds - a(x) - b_i(t)) \quad \text{a.e.x} \tag{4.13}$$

for $i = 1, 2$. It follows from this that for $t \notin D$,

$$|V_1(t,x) - V_2(t,x)| \le K\biggl\{|u_1'(t) - u_2'(t)|_H + \int_0^t |V_1(s,x) - V_2(s,x)|\, ds + |b_1(t) - b_2(t)|\biggr\}, \tag{4.14}$$

where K is a constant. From now on, K will be used to denote some constant. K may depend on T. From (4.12),

$$|V_1(t,x) - V_2(t,x)| \le K\biggl\{|u_1'(t) - u_2'(t)|_{_H} + \int_0^t |V_1(s,x) - V_2(s,x)|\, ds + \int_0^t (\varepsilon|u_{1x}'(s) - u_{2x}'(s)|_{_H} + K_\varepsilon |u_1'(s) - u_2'(s)|_{_H})\, ds\biggr\}. \tag{4.15}$$

Taking $\int_0^1$ of both sides of (4.15), we have that for a.e.t,

$$|V_1(t) - V_2(t)|_{_{L^1}} \le K\biggl\{|u_1'(t) - u_2'(t)|_{_H} + \int_0^t |V_1(s) - V_2(s)|_{_{L^1}}\, ds + \int_0^t (\varepsilon|u_{1x}'(s) - u_{2x}'(s)|_{_H} + K_\varepsilon |u_1'(s) - u_2'(s)|_{_H})\, ds\biggr\} \tag{4.16}$$

Using (4.13) we may write that for a.e.x

$$V_{ix}(t,x) = (\beta^{-1})'\biggl(\int_0^x u_i'(t)(z)\, dz + \int_0^t P(V_i(s,x))\, ds - a(x) - b_i(t)\biggr) \cdot \biggl(u_t'(t)(x) + \int_0^t P'(V_i(s,x))V_{ix}(s,x)\, ds - a'(x)\biggr)$$

for $i = 1, 2$. The formula is clear formally. To verify that V_{ix} exists, see the discussion in [**10**, pp. 99–100]. This formula also shows that

$$\|V_{ix}\|_{L^\infty(0,T;H)} \leq K, \qquad i = 1, 2, \tag{4.17}$$

since $(\beta^{-1})'$ is bounded. Some computations using (4.1) yield

$$\begin{aligned} |V_{1x}(t,x) - V_{2x}(t,x)| \leq K\bigg\{ & |u_1'(t)(x) - u_2'(t)(x)| \\ & + \int_0^t |V_1(s,x) - V_2(s,x)|\, ds \\ & + \int_0^t |V_{1x}||V_1(s,x) - V_2(s,x)| ds \bigg\} \\ & + K\bigg[|u_2'(t)(x)| + \int_0^t |V_{2x}| ds + |a'(x)|\bigg] \\ & \cdot \bigg\{ |u_2'(t) - u_1'(t)|_H \\ & + \int_0^t |V_1(s,x) - V_2(s,x)|\, ds + |b_1(t) - b_2(t)| \bigg\} \end{aligned} \tag{4.18}$$

When the righthand side of (4.18) is multiplied and $\int_0^1$ is applied to both sides, one obtains, after some simplification, the inequality

$$\begin{aligned} |V_{1x}(t) - V_{2x}(t)|_{L^1} \leq K\bigg\{ & |u_1'(t) - u_2'(t)|_H \\ & + \int_0^t (|V_1(s) - V_2(s)|_{L^1} + |V_{1x}(s) - V_{2x}(s)|_{L^1})\, ds \\ & + |b_1(t) - b_2(t)| \bigg\} \end{aligned} \tag{4.19}$$

As an example of the type of argument used in the simplification,

$$\begin{aligned}
\int_0^1 \left(\int_0^t |V_{2x}|\, ds \int_0^t |V_1 - V_2|\, ds \right) dx \\
&= \int_0^1 \left(\int_0^t |V_{2x}|\, dr \int_0^t |V_1 - V_2|\, ds \right) dx \\
&= \int_0^1 \int_0^t \int_0^t |V_{2x}(r,x)||V_1(s,x) - V_2(s,x)|\, dr\, ds\, dx \\
&= \int_0^t \int_0^1 \int_0^t |V_{2x}(r,x)||V_1(s,x) - V_2(s,x)|\, dr\, dx\, ds \\
&\le K \int_0^t (|V_1(s) - V_2(s)|_{L^1} + |V_{1x}(s) - V_{2x}(s)|_{L^1}) \\
&\qquad \cdot \int_0^1 \int_0^t |V_{2x}(r,x)|\, dr\, dx \\
&\le K \int_0^t (|V_1(s) - V_2(s)|_{L^1} + |V_{1x}(s) - V_{2x}(s)|_{L^1})\, ds
\end{aligned}$$

because of (4.17).

Returning to (4.12), (4.19) implies

$$\begin{aligned}
(4.20) \quad |V_{1x}(t) - V_{2x}(t)|_{L^1} \le K \Big\{ |u_1'(t) - u_2'(t)|_H \\
+ \int_0^t (|V_1(s) - V_2(s)|_{L^1} + |V_{1x}(s) - V_{2x}(s)|_{L^1})\, ds \\
+ \int_0^t \varepsilon |u_{1x}'(s) - u_{2x}'(s)|_H + K_\varepsilon |u_1'(s) - u_2'(s)|_H\, ds \Big\}
\end{aligned}$$

Therefore, from (4.20) and (4.15), we obtain for all $t \notin D$,

$$\begin{aligned}
(4.21) \quad \|V_1(t) - V_2(t)\|_{W_1^1(0,1)} \\
\le K \Big\{ |u_1'(t) - u_2'(t)|_H + \int_0^t (\|V_1(s) - V_2(s)\|_{W_1^1(0,1)}\, ds \\
+ \int_0^t \varepsilon |u_{1x}'(s) - u_{2x}'(s)|_H + K_\varepsilon |u_1'(s) - u_2'(s)|_H\, ds \Big\}
\end{aligned}$$

Lemma 4.5. *If $f(t) \leq a(t) + b(t) + K\int_0^t f(s)\,ds$ almost everywhere and b is increasing, then $f(t) \leq a(t) + b(t)e^{Kt} + \int_0^t a(s)e^{K(t-s)}\,ds$ almost everywhere.*

Using Lemma 4.5 in (4.21) with $a(t) = K|u_1'(t) - u_2'(t)|_H$ and $b(t)$, the last integral in $\{\ \}$, we obtain for a.e.t.,

$$\|V_1(t) - V_2(t)\|_{W_1^1(0,1)} \leq K|u_1'(t) - u_2'(t)|_{H} + K\int_0^t |u_1'(s) - u_2'(s)|_{H} e^{K(t-s)}\,ds + Ke^{Kt}\int_0^t \varepsilon|u_{1x}'(s) - u_{2x}'(s)|_{H} + K_\varepsilon|u_1'(s) - u_2'(s)|_{H}\,ds \tag{4.22}$$

for K independent of ε.

This implies

Theorem 4.1. *Let u_1 and u_2 be two solutions to* (3.4)*. Then for all $\varepsilon > 0$ there exists a constant, K, depending on ε and T such that*
(4.23)

$$\|V_1(t) - V_2(t)\|_{W_1^1(0,1)} \leq \int_0^t (\varepsilon|u_{1x}'(s) - u_{2x}'(s)|_{H} + K|u_1'(s) - u_2'(s)|_{H})ds + K|u_1'(t) - u_2'(t)|_{H}$$

With estimate (4.23), we proceed with the uniqueness proof. Since u_1 and u_2 both solve (3.4),

$$\begin{aligned} &\frac{1}{2}|u_1(t) - u_2(t)|_{H}^2 + \int_0^t \langle Nu_1 - Nu_2, u_1' - u_2'\rangle\,ds \\ &\quad + \int_0^t \langle Q(u_1)u_1' - Q(u_2)u_2', u_1' - u_2'\rangle\,ds \\ &= \int_0^t \langle f(u_1', u_1) - f(u_2', u_2), u_1' - u_2'\rangle\,ds \end{aligned} \tag{4.24}$$

The third term equals

$$(4.25)\quad \int_0^t \langle Q(u_1)(u_1' - u_2'), u_1' - u_2'\rangle\, ds + \int_0^t \langle (Q(u_1) - Q(u_2))u_2', u_1' - u_2'\rangle\, ds$$

and we estimate the second term in (4.25). This term is bounded by

$$\begin{aligned}(4.26)\quad &\int_0^t \int_0^1 (\alpha(V_1) - \alpha(V_2)) u_{2x}'(s)(x)(u_{1x}'(s)(x) - u_{2x}'(s)(x))\, dx\, ds \\ &\leq K_0 \int_0^t \int_0^1 |V_1(s,x) - V_2(s,x)|\, |u_{2x}'(s)(x)||u_{1x}'(s)(x) - u_{2x}'(s)(x)|\, dx\, ds \\ &\leq K_0 \int_0^t \|V_1(s) - V_2(s)\|_{L\infty} \int_0^1 |u_{2x}'(s)(x)||u_{1x}'(s)(x) - u_{2x}'(s)(x)|\, dx\, ds \\ &\leq K_0 \int_0^t \|V_1(s) - V_2(s)\|_{W_1^1(0,1)} \int_0^1 |u_{2x}'(s)(x)|\, |u_{1x}'(s)(x) - u_{2x}'(s)(x)|\, dx\, ds.\end{aligned}$$

By Theorem 4.1, this is less than

$$(4.27)\quad K_0 \int_0^t \left[\int_0^s (\varepsilon |u_{1x}'(r) - u_{2x}'(r)|_{_H} + K|u_1'(r) - u_2'(r)|_{_H})\, dr + K|u_1'(s) - u_2'(s)|_{_H}\right](|u_{2x}'(s)|_{_H} |u_{1x}'(s) - u_{2x}'(s)|_{_H})\, ds.$$

Splitting this into pieces we consider

$$\begin{aligned}\mathcal{A} &= \varepsilon K_0 \int_0^t \int_0^s |u_{1x}'(r) - u_{2x}'(r)|_{_H}\, dr |u_{2x}'(s)|_{_H} |u_{1x}'(s) - u_{2x}'(s)|_{_H}\, ds, \\ \mathcal{B} &= K_0 K \int_0^t \int_0^s |u_1'(r) - u_2'(r)|_{_H}\, dr |u_{2x}'(s)|_{_H} |u_{1x}'(s) - u_{2x}'(s)|_{_H}\, ds, \\ \mathcal{C} &= K_0 K \int_0^t |u_1'(s) - u_2'(s)|_{_H} |u_{2x}'(s)||u_{1x}'(s) - u_{2x}'(s)|_{_H}\, ds,\end{aligned}$$

and estimate these terms, the second term of (4.25) being dominated by their sum. An application of Fubini's theorem on $\mathcal{A}$ followed by Holder's inequality yields

$$\mathcal{A} \leq \varepsilon K_0 \|u_2'\|_{\mathcal{V}} T^{1/2} \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2 \, ds \tag{4.28}$$

A similar procedure applied to $\mathcal{B}$ yields

$$\mathcal{B} \leq K_0 K \|u_2'\|_{\mathcal{V}} T^{1/2} \left(\int_0^t |u_{1x}' - u_{2x}'|_H^2 \, ds \right)^{\frac{1}{2}} \left(\int_0^t |u_1'(r) - u_2'(r)|^2 \, dr \right)^{1/2}$$

which implies

$$\begin{aligned} \mathcal{B} \leq{} & K_0 \|u_2'\|_{\mathcal{V}} T^{1/2} \varepsilon \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2 \, ds \\ & + K_0 K \|u_{2x}'\|_{\mathcal{V}} T^{1/2} \int_0^t |u_1'(r) - u_2'(r)|_H^2 \, dr, \end{aligned} \tag{4.29}$$

where K depends on ε. Note that the K is just a name for a constant and the K in (4.29) is not the same as the K in the preceding inequality.

$$\begin{aligned} \mathcal{C} &\leq K_0 K \left(\int_0^t |u_1'(s) - u_2'(s)|_H^2 |u_{2x}'(s)|_H^2 \, ds \right)^{1/2} \\ &\quad \cdot \left(\int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2 \, ds \right)^{1/2} \\ &\leq K_0 K \left[\frac{\varepsilon}{K} \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2 \, ds \right. \\ &\quad \left. + \frac{K}{\varepsilon} \int_0^t |u_1'(s) - u_2'(s)|_H^2 |u_{2x}'(s)|_H^2 \, ds \right] \\ &\leq \varepsilon K_0 \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2 \, ds \\ &\quad + K_0 K \left(\int_0^t |u_1'(s) - u_2'(s)|_H^2 |u_{2x}'(s)|_H^2 \, ds \right. \end{aligned} \tag{4.30}$$

It follows from (4.30), (4.29), and (4.28) that if ε is chosen small enough, then $\mathcal{A}+\mathcal{B}+\mathcal{C}$ is dominated by an expression of the form

$$\frac{\delta}{4}\int_0^t |u'_{1x}(s)-u'_{2x}|_H^2\,ds + K\int_0^t |u'_1(s)-u'_2(s)|_H^2\,(1+|u'_{2x}(s)|_H^2)\,ds \tag{4.31}$$

(Note that (2.12) gives an estimate for $\|u'_2\|_{\mathcal{V}}$ that depends only on T, δ, $|v_1|_H$, K_1, and C_0.) With (4.31), we return to (4.24) using (3.2) and (2.1) we can obtain

$$\begin{aligned}
\frac{1}{2}|u'_1(t)-u'_2(t)|_H^2 &+ \delta\int_0^t |u'_{1x}(s)-u'_{2x}(s)|_H^2 \\
&\le \frac{\delta}{4}\int_0^t |u'_{1x}(s)-u'_{2x}(s)|_H^2\,ds \\
&\quad + K\int_0^t (1+|u'_{2x}(s)|_H^2)|u'_1(s)-u'_2(s)|_H^2\,ds \\
&\quad + K_0\int_0^t |V_1-V_2|_H|u'_{1x}-u'_{2x}|_H\,ds \\
&\quad + 2K_1\int_0^t \Big(\|u'_1(s)-u'_2(s)\|_C^2 \\
&\qquad\qquad + \int_0^s \|u'_1(r)-u'_2(r)\|_C^2\,dr\Big)\,ds
\end{aligned} \tag{4.32}$$

where $C=C([0,1])$. The last term in (4.32) is dominated by

$$2K_1(1+T)\int_0^t \|u'_1(s)-u'_2(s)\|_C^2\,ds,$$

and therefore, using Lemma 4.2 on this along with some elementary inequalities on the term involving $|V_1 - V_2|_H$, we can obtain

$$\frac{1}{2}|u_1'(t) - u_2'(t)|_H^2 + \frac{3\delta}{4}\int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2\, ds$$
$$\leq K\int_0^t (1 + |u_{2x}'(s)|_H^2)|u_1'(s) - u_2'(s)|_H^2\, ds$$
$$+ K\int_0^t |V_1 - V_2|^2\, ds + \frac{\delta}{4}\int_0^t |u_{1x}' - u_{2x}'|_H^2\, ds$$
$$+ K\int_0^t |u_1'(s) - u_2'(s)|_H^2\, ds,$$

from which we obtain, after changing K,

$$|u_1'(t) - u_2'(t)|_H^2 + \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2\, ds$$
$$\leq K\int_0^t (1 + |u_{2x}'(s)|_H^2)|u_1'(s) - u_2'(s)|_H^2\, ds$$
$$+ K\int_0^t |V_1 - V_2|_H^2\, ds.$$

The last term is dominated by a term of the form

$$K\int_0^t \int_0^s |u_{1x}'(r) - u_{2x}'(r)|_H^2\, dr\, ds$$

where K may depend on T. It follows that

$$|u_1'(t) - u_2'(t)|_H^2 + \int_0^t |u_{1x}'(s) - u_{2x}'(s)|_H^2\, ds \tag{4.33}$$
$$\leq K\int_0^t (1 + |u_{2x}'(s)|_H^2)\Big[|u_1'(s) - u_2'(s)|_H^2 + \int_0^s |u_{1x}'(r) - u_{2x}'(r)|_H^2\, dr\Big]\, ds$$

An application of Gronwall's inequality yields the desired theorem.

Theorem 4.2. *If in addition to* (2.1), (4.1) *holds, and* $V_0 \in H^1(0,1)$, *and if* $E \supseteq \{u \in H^1 : u(1) = 0\}$, *then the solution to problem (3.4) is unique.*

Different assumptions. The assumptions on P and α given in (2.1) and (4.1) are not reasonable for many examples. In particular, one would expect P and α to have an asymptote at $V = 0$, and one would also expect that $\lim_{V\to\infty} \alpha(V) = 0$. However, we show in this section that the study of (3.4) under more reasonable assumptions for P and α can be reduced to the problem discussed in Sections 2–4. Here we assume

$$\alpha, P, P' \text{ are locally Lipschitz on } (0,\infty), \tag{5.1a}$$

If $V \in [a,b]$, $0 < a < b < \infty$, then there exist constants K_{ab} and δ_{ab} such that

$$0 < \delta_{ab} \leq \alpha(V) \leq K_{ab} \tag{5.1b}$$

$$-\infty = \lim_{V\to 0+} \beta(V), \qquad \infty = \lim_{V\to\infty} \beta(V), \tag{5.1c}$$

$$P(V) > 0, \qquad \lim_{V\to\infty} P(V) = 0, \qquad \lim_{V\to 0} P(V) = \infty. \tag{5.1d}$$

$$\limsup_{V\to 0+} P'(V) < 0 \tag{5.1e}$$

Now we will examine the prescribed stress k_1 and k_0 more carefully. To begin with we assume that the material does not lie in a vacuum and so

$$k_i(t,v,u) = l_i(t,v,u) + s_i(v) \tag{5.2}$$

where l_i is an applied stress from some sort of machinery and s_i is a stress that results from air pressure and air resistance.

We assume that whatever produces the applied stress, l_i, has finite power. Thus

$$|l_i(t,v,u)v| \leq M/2 \tag{5.3}$$

for some constant M. The term s_i has the form

$$s_1(v) = -\eta_1 - h_1(v), \qquad s_0(v) = -\eta_0 + h_0(v), \tag{5.4a}$$

where

$$h_i(v)(v) \geq 0, \qquad h_i(0) = 0, \qquad \eta_i > 0. \tag{5.4b}$$

The η_i are forces which result from air pressure when the ith face of the material is at rest. These forces are modified when $v \neq 0$ by adding the forces $\pm h_i(v)$. From (5.2)–(5.4) we obtain

$$\begin{aligned} k_1(t,v,u) + \eta_1 &= l_1(t,v,u) - \eta_1 - h_1(v) + \eta_1 \\ &= l_1(t,v,u) - h_1(v), \end{aligned} \tag{5.5a}$$

and similarly,

$$k_0(t,v,u) + \eta_0 = l_0(t,v,u) + h_0(v). \tag{5.5b}$$

Therefore from (5.4b) and (5.3),

$$(k_1(t,v_1,u_1) + \eta_1)v_1 - (k_0(t,v_0,u_0) + \eta_0)v_0 \leq M, \tag{5.6}$$

and we shall assume that (5.6) holds for some positive constants η_0 and η_1 if the material is not in a vacuum.

If the material is in a vacuum, $l_i = k_i$ and we assume (5.3) along with

$$\int_1^\infty P(V)dV < \infty. \tag{5.7}$$

Let $0 < a < 1 < b < \infty$ and let α_{ab} and P_{ab} equal α and P respectively for $V \in [a,b]$, and let α_{ab} and P_{ab} satisfy (2.1) and (4.1). Also let

$$\beta_{ab}(V) = \int_1^V \alpha_{ab}(y)\,dy \tag{5.8a}$$

$$W(V) = \int_1^V P(y)\,dy, \qquad W_{ab}(V) = \int_1^V P_{ab}(y)\,dy, \tag{5.8b}$$

and choose P_{ab} in such a way that for $V > 1$, $W_{ab}(V) \leq W(V)$ and $P_{ab}(V) \leq P(V)$.

Let u be a solution of (3.4) in which α and P have been replaced by α_{ab} and P_{ab}. Then, multiplying (3.4a) by u' and doing $\int_0^t$, we obtain

$$\begin{aligned}\frac{1}{2}|u'(t)|_H^2 - \frac{1}{2}|v_1|_H^2 - \int_0^1 W_{ab}(V(t,x))\,dx \\ + \int_0^1 W(V_0)\,dx + \int_0^t (\alpha(V)u'_x(s), u'_x(s))_H\,ds \\ = \int_0^t \langle f(u,u'), u'\rangle\,ds,\end{aligned} \tag{5.9}$$

because we take $a < a_0 < 1 < b_0 < b$ where $V_0(x) \in [a_0, b_0]$ a.e.

Now suppose (5.6) holds. The last integral on the right in (5.9) equals

$$\begin{aligned}\int_0^t (k_1(s,u'(s)(1),u(s)(1)) + \eta_1)u'(s)(1)\,ds \\ - \int_0^t (k_0(s,u'(s)(0),u(s)(0)) + \eta_0)u'(s)(0)\,ds \\ - \int_0^1 ((x\eta_1 + (1-x)\eta_0)u(t)(x))_x\,dx,\end{aligned} \tag{5.10}$$

which is bounded above by

$$Mt + (\eta_1+\eta_0)\int_0^1 |u(t)(x)|dx - \int_0^1 (x\eta_1 + (1-x)\eta_0)u_x(t)(x)\,dx \tag{5.11}$$

The second term in (5.11) is bounded above by

$$(\eta_1+\eta_0)\int_0^t |u'(s)|_H\,ds$$

which is no larger than

$$(\eta_1+\eta_0)^2 + T\int_0^t |u'(s)|_H^2\,ds. \tag{5.12}$$

Therefore, (5.9) implies

$$\begin{aligned}\frac{1}{2}|u'(t)|_H^2 \le -\int_0^1 W(V_0)\,dx + \int_0^1 W_{ab}(V)^+dx \\ + (\eta_1+\eta_0)^2 + T\int_0^t |u'(s)|_H^2\,ds \\ + MT - \int_0^1 (x\eta_1 + (1-x)\eta_0)V\,dx + C,\end{aligned} \tag{5.13}$$

where C is a constant depending on V_0. Since $W_{ab}(V) \leq W(V)$ for $V > 1$, we obtain

$$(5.14) \qquad \begin{aligned} |u'(t)|_H^2 &\leq C + \int_0^t |u'(s)|_H^2 \, ds \\ &\quad + \int_0^1 [W(V)^+ - (x\eta_1 + (1-x)\eta_0)V] \, dx. \end{aligned}$$

where C is a constant depending on V_0, T, η_0, and η_1. The integrand of the last integral in (5.14) is bounded above since η_0 and η_1 are positive and $\lim_{V\to\infty} P(V) = 0$. Therefore, Gronwall's inequality implies

$$(5.15) \qquad |u'(t)|_H^2 \leq C,$$

for some C depending on V_0, T, η_0 and η_1. If (5.3) and (5.7) hold it is also routine to obtain (5.15). This proves:

Lemma 5.1. *Let u be a solution of* (3.4) *in which α and P are replaced with α_{ab} and P_{ab}, respectively. Then there exist a_0 and b_0 with $0 < a_0 < 1 < b_0 < \infty$ such that $|u'(t)|_H$ is bounded independently of the choice of $a < a_0$ and $b > b_0$.*

Let $E \supseteq \{u \in H^1(0,1) : u(1) = 0\}$, and let

$$(5.15) \qquad \int_0^x u'(t)(z) \, dz - \beta_{ab}(V(t,x)) = r(t,x).$$

Choose J large enough that

$$(5.16a) \qquad C - J < 0, \qquad r(0,x) = \int_0^x v_1(z) dz - \beta(V_0(x)) < J,$$

and if $\beta(V) \leq C - J$, then

$$(5.16b) \qquad -(P(V) + k_0(t,v,u)) < 0$$

for all $t \in [0,T]$ and u, v. In (5.16), C is the constant of Lemma 5.1, bounding $|u'(t)|_H$.

By (5.1e), there exists $\gamma > 0$ such that for $V \in (0, \gamma)$, $P'(V) < 0$. Letting J also be large enough that $\beta^{-1}(C - J) < \gamma$, we may assume that P_{ab} was chosen such that (5.16b) holds for P_{ab} replacing P, whenever $\beta(V) \leq C - J$. Now make a_0 smaller if necessary such that

$$0 < a_0 < \beta^{-1}(C - J). \tag{5.17}$$

By the choice of E, one can show that in the sense of distributions,

$$r_t(t, x) = -P_{ab}(V(t, x)) - k_0(t, u'(t)(0), u(t)(0)) \text{ a.e.}$$

It follows that there exists a set of measure zero, B, such that for $x \notin B$, $t \to r(t, x)$ equals an absolutely continuous function a.e.t and

(5.18)

$$r(t, x) = r(0, x) + \int_0^t -(P_{ab}(V(s, x)) + k_0(s, u'(s)(0), u(s)(0)))\, ds.$$

From Lemma 4.1, $t \to r(t, x)$ is continuous for a.e.x. and so we may conclude that the above formula holds for all $t \in [0, T]$.

Lemma 5.2. *If $x \notin B$, then $r(t, x) < J + 1$ for all $t \in [0, T]$.*

Proof. If the lemma is not true, there exists $x \notin B$ and $t \in [0, T]$ such that $r(t, x) = J + 1$ but $r(s, x) < J + 1$ for all $s < t$. Hence

(5.19)

$$r(t, x) - r(s, x) = \int_s^t -(P_{ab}(V(r, x)) + k_0(r, u'(r)(0), u(r)(0)))\, dr$$

By continuity of $s \to r(s, x)$, we can conclude that for all s close enough to t, $r(s, x) > J$. Pick such an s. Then if $r \in (s, t)$, $\beta_{ab}(V(r, x)) < C - J < 0$. This implies $\beta(V(r, x)) < C - J$ also (if $\beta(V(r, x)) \geq C - J$, then $V(r, x) \geq \beta^{-1}(C - J) > a_0$ and so $\beta_{ab}(V(r, x)) = \beta(V(r, x)) \geq C - J$ contradicting $\beta_{ab}(V(r, x)) < C - J$). Therefore, by the validity of (5.16b) with P_{ab} replacing P, the integrand in (5.19) is negative. Thus $J + 1 = r(t, x) < r(s, x) < J + 1$, a contradiction. □

We conclude from Lemma 5.2 that $r(t, x) < J + 1$ for all $x \notin B$ and $t \in [0, T]$. Consequently, for all $a < a_0$ and $b > b_0$,

$$C - J - 1 < \beta_{ab}(V). \tag{5.20}$$

Now let $a_0 < \beta^{-1}(C - J - 1)$. If $V(t,x) = a_0$, then $\beta(V(t,x)) = \beta_{ab}(V(t,x)) = \beta(a_0) < C - J - 1$ contradicting (5.20). Thus for a.e.x, $V(t,x) > a_0$ for all $t \in [0,T]$.

Letting $N > \max\{P(V) : V \geq a_0\} + \sup\{|k_0(t,v,u)| : t \in [0,T], (v,u) \in \Re^2\}$, (5.18) shows that

$$|r(t,x)| < NT + |r(0,x)|,$$

and so there exists a constant, $-R$, independent of a and b such that $-R < r(t,x)$, and so

$$\beta_{ab}(V(t,x)) < C + R. \tag{5.21}$$

Let $b_0 > \beta^{-1}(C + R)$. If $V(t,x) = b_0$, then $\beta(V(t,x)) = \beta_{ab}(V(t,x)) = \beta(b_0) > C + R$. This contradicts (5.21) and so $V(t,x) \neq b_0$ for all $t \in [0,T]$. Therefore, for a.e.x, $V(t,x) < b_0$ for all $t \in [0,T]$. This proves the following

Theorem 5.1. *Let $E \supseteq \{u \in H^1(0,1) : u(1) = 0\}$, let $V_0(x)$ be bounded away from 0 and ∞, and let (5.1) and either (5.6) or (5.7) and (5.3) hold in addition to (3.2). Then there exists a solution to problem (3.4) satisfying $0 < a_0 \leq V(t,x) \leq b_0 < \infty$ for a.e.x. If $V_0 \in H^1(0,1)$, this solution is unique.*

Proof. This follows by picking $a < a_0$ and $b > b_0$ for the a_0 and b_0 described above. If u solves (3.4) with α_{ab} and P_{ab} replacing α and P, the above argument has shown that $a_0 < V(t,x) < b_0$ and so $\alpha_{ab}(V) = \alpha(V)$ and $P_{ab}(V) = P(V)$. ◻

In this section we have considered prescribed stress satisfying (5.2), (5.3), and (5.4). It seems reasonable to assume that $|l_i(t,v,u)|$ should satisfy (3.2a) and (3.2b), but then we would have to conclude $h_i(v)$ is bounded. It might be more desirable to not make this assumption. To remove it, let

$$h_{in}(v) = \begin{cases} h_i(v) & \text{if } |v| \leq n \\ n & \text{if } v > n \\ -n & \text{if } v < -n \end{cases} \tag{5.22}$$

and denote by $\mathcal{P}(n,a,b)$ the problem (3.4) in which P is replaced with P_{ab}, β is replaced with β_{ab} and h_i is replaced with h_{in}. Also let $\mathcal{P}(a,b)$ denote problem (3.4) in which h_i has not been replaced. Then (5.9) and (5.10)–(5.11) imply that, for u_n a solution to $\mathcal{P}(n,a,b)$,

$$
\begin{aligned}
\frac{1}{2}|u_n'(t)|_H^2 - \frac{1}{2}|v_1|_H^2 & \\
&+ \int_0^1 (x\eta_1 + (1-x)\eta_0)V_n(t,x) - W_{ab}(V_n(t,x))dx \\
&+ \int_0^1 W(V_0(x))\,dx + \delta_{ab}\int_0^t |u_{nx}'(s)|_H^2\,ds \\
&\qquad \le MT + (\eta_0+\eta_1)^2 + T\int_0^t |u_n'(s)|_H^2\,ds.
\end{aligned} \tag{5.23}
$$

Since $P(V) > 0$, $\lim_{V\to 0+} P(V) = \infty$, and $\lim_{V\to\infty} P(V) = 0$, we can assume that a_0 is small enough and b_0 is large enough that for $a < a_0$ and $b > b_0$ there is a lower bound, $-J$, for the function $V \to (x\eta_1 + (1-x)\eta_0)V - W_{ab}(V)$ which is independent of such a, b, and $x \in [0,1]$. Therefore, an application of Gronwall's inequality in (5.23) yields a constant, C, independent of n such that

$$\|u_n'\|_{L^\infty(0,T;H)} + \|u_n'\|_{\mathcal{V}} + \|u_n\|_{L^\infty(0,T;E)} \le C. \tag{5.24}$$

It then follows from (3.2a) that $f_n(u_n', u_n)$ is bounded in $\mathcal{V}'$ from which we may conclude that $\|u_n\|_{\mathcal{Y}}$ is bounded independently of n. Therefore, there exists a subsequence, still denoted by n, such that

$$u_n \rightharpoonup u \text{ in } \mathcal{Y}. \tag{5.25}$$

Applying (3.3a) and (3.3b), we may take a further subsequence and assume that for a.e.t, and $e = 0$ or 1,

$$u_n'(t)(e) \to u'(t)(e), \quad u_n(t)(e) \to u(t)(e). \tag{5.26}$$

Therefore, for a.e.t,

$$h_{n1}(u_n'(t)(1)) \to h_1(u'(t)(1)), \quad h_{n0}(u_n'(t)(0) \to h_0(u'(t)(0)), \tag{5.27}$$

assuming h_0 and h_1 are Lipschitz continuous. Thus

$$f_n(u_n, u_n') \to f(u, u') \quad \text{in } \mathcal{V}'. \tag{5.28}$$

Let w be the solution to (2.5) where $f = f(u, u')$. By Theorem 2.1 $u_n \to w$ in $\mathcal{V}$. It follows from (5.25) that $u = w$. Therefore u is a solution to $\mathcal{P}(a, b)$.

Since (3.2b) is no longer assumed to hold, some changes are necessary in estimating the specific volume. We assume that, in addition to (5.1), there exist constants a_1 and b_1 with $0 < a_1 < b_1$ such that

$$P \text{ is decreasing on } (0, a_1] \cup [b_1, \infty), \tag{5.29a}$$

and

$$\int_0^1 P(V)dv = \infty \tag{5.29b}$$

One defines, for x, $y \in [0, 1]$ and u the solution of $\mathcal{P}(a, b)$,

$$q(t, x, y) = \int_x^y u'(t)(z)dz - [\beta_{ab}(V(t, y)) - \beta_{ab}(V(t, x))].$$

Then the procedure of [**9**, Section 4] is used to estimate the specific volume independent of a and b for all a small enough and b large enough. Since (3.2b) was not used in Section 4 of this paper, we obtain:

Theorem 5.2. *Delete assumption* (3.2b) *but assume* (3.2a) *holds for* l_i *and* k_i *given in* (5.2). *Also assume* (5.29) *in addition to* (5.1). *Then there exists a solution to* (3.4) *and constants* a_0 *and* b_0, $a_0 > 0$, *such that* $V(t, x) \in [a_0, b_0]$ *a.e.x. If the initial specific volume* V_0 *is in* $H^1(0, 1)$ *and if* $E \supseteq \{u \in H^1(0, 1) : u(1) = 0\}$, *then the solution is unique.*

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Michigan Technological University, Houghton, MI 49931