ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

ISOTOPIC HOMEOMORPHISMS AND NIELSEN FIXED POINT THEORY

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ABSTRACT. Given a topological manifold, M, and an embedding $h : M \to M$ we define a certain class of balls in M called isotopy-standard-for h. The existence of such balls is established and also the following fixed point removability criterion is given: if B is isotopy-standard-for h, and $h|B$ has index zero, then there is an isotopy with support on B taking h to a map which has no fixed points in B.

1. Introduction. For a large class of compact topological spaces, Nielsen theory provides a way to estimate the number of fixed points for any given self-map. If X is a compact polyhedron and $f : X \to X$ a self-map, then the *Nielsen number*, $N(f)$, has the two important properties given below:

(1) $N(f)$ is a lower bound for the number of fixed points of $f : X \to X$,

(2) $N(f)$ is a homotopy invariant (i.e., homotopic maps have the same Nielsen number).

The definition of $N(f)$ and some of its properties can be found in either [**2**] or [**5**]. Loosely, it gives the number of fixed point classes of f which have nonzero index.

A very natural question–and one of general interest–concerns the realizability of the Nielsen number as a lower bound. Specifically, given $f : X \to X$ does there exist a map g homotopic to f having exactly $N(f)$ fixed points? The first affirmative results were due to Nielsen (maps on the torus) and Wecken (maps on PL manifolds of dimension greater than two). Results by a number of authors have since culminated in the following theorem due to Jiang [**6**].

Received by the editors on September 2, 1991, and in revised form on November 16, 1992.
1980 *Mathematics Subject Classification.* Primary 55M20, Secondary 57R52, 58C30.
Key words and phrases. Fixed point, Nielsen number, manifold, isotopy.

Theorem 1.1. *Suppose that X is a compact polyhedron which is not either* (1) *a polyhedron with local separating points or* (2) *a surface with negative Euler characteristic. For $f : X \to X$ define*

$$MF[f] = \min_g \{\# \operatorname{Fix}(g) | g \text{ is homotopic to } f\}.$$

Then $MF[f] = N(f)$.

This theorem is best possible in the sense that counterexamples exist under either condition (1) or (2). (See [**2, 8, 10**].) On the other hand, if we restrict our attention to homeomorphisms in the case (2), as a consequence of the Nielsen-Thurston classification of surface automorphisms the following theorem holds.

Theorem 1.2 [**4**]. *If F is a compact, orientable, surface and $h : F \to F$ is a homeomorphism, then $MF[h] = N(h)$.*

Remarks. (1) It cannot always be arranged that the minimizing map in Theorem 2 is also a homeomorphism. Jiang [**7**] pointed out that for an orientation reversing involution of the pair of pants no homeomorphism will work. However, the minimum can always be achieved by an embedding into the interior of M.

(2) If in place of $N(h)$ we were to use the relative Nielsen number, $N(h, h|\partial F)$, as defined by H. Schirmer [**12**], then this new minimum can be achieved by an actual homeomorphism. Details appear in [**9**].

Given a topological manifold M, the results of Theorems 1.1 and 1.2 can be combined to give the following theorem.

Theorem 1.3. *If M is an orientable topological manifold, then for any homeomorphism $h : M \to M$, we have $MF[h] = N(h)$.*

2. Homotopy vs. isotopy in Nielsen theory. Theorem 1.3 is a result about a homotopy invariant (the Nielsen number) in the category of topological manifolds and homeomorphisms. But in this category isotopy is perhaps more natural to consider. Certainly, $N(h)$ is an isotopy invariant and a lower bound for the number of fixed points. In the paper [**7**], Jiang raised the question of the realizability of this lower bound. When $\dim(M) = 2$, this follows immediately from Theorem 1.3 and the fact that homotopy implies isotopy in dimension two. Here, and in what follows, we take isotopy to mean *isotopy through embeddings* when necessary.

In higher dimensions, standard techniques (for example, the use of the Whitney trick in [**7**]) don't seem to apply. Also, problems of a local nature may occur. For example, if N is a topological ball in M which contains exactly one fixed point, a, in its interior there is the following removability criterion,

Proposition 2.1 [**5**, p. 13]. *If* $\operatorname{index}(h, a) = 0$*, then h is homotopic to a map g which agrees with h on $M \backslash N$ and* $\operatorname{Fix}(g) \cap N = \varnothing$*.*

The analogous result using isotopy in place of homotopy is unknown. The purpose of this paper is to obtain a partial analog to the above proposition as well as some other corresponding results. In the next section we define a certain class of balls in a manifold called *isotopy-standard*. Various properties are discussed including the following result.

Theorem 2.2. *Let M be a topological manifold and $h : M \to M$ an embedding into the interior of M. If B is an n-ball in M which is isotopy-standard-for h and with* $\operatorname{index}(h, B) = 0$*, then h is isotopic to g with support on B so that* $\operatorname{Fix}(g) \cap B = \varnothing$*. If M and h are smooth, then so is g.*

The above theorem can be used to reduce the realizability problem (of achieving the Nielsen number as a minimum) to the problem of finding certain isotopy-standard balls in M. This is indicated in the following consequence of Theorem 2.2.

Corollary 2.3. *Suppose $h : M \to M$ is an embedding and $B_1, \dots, B_k$ are isotopy-standard balls in M satisfying*

(1) $B_i \cap B_j = \varnothing$ *if* $i \neq j$

(2) $\operatorname{Fix}(h) \subset \cup_{i=1}^{k} B_i$

(3) *Suppose γ is a path in M with* $\partial\gamma \subset \operatorname{Fix}(h)$*. If $\gamma \cup h(\gamma)$ is*

null-homotopic then $\partial\gamma \subset B_j$ for some j.

Then h is isotopic to an embedding having exactly $N(h)$ fixed points.

Proof. Conditions (2) and (3) in the hypothesis imply that each fixed point class is contained in some B_j. Proposition 3.1 reduces to one fixed point in each class and Theorem 2.2 can be applied to remove those which have zero index. As the Nielsen number gives the number of fixed point classes with nonzero index, this is the desired embedding. $\square$

3. Isotopy-standard balls in manifolds. We first give some notation which will be used throughout this section. For $n \geq 2$, write $\mathbf{R}^n = \mathbf{R}^2 \times \mathbf{R}^{n-2}$ and identify $\mathbf{R}^2 \times \{0\}$ with $\mathcal{C}$. Let $\mathbf{O}$ denote the origin in the complex plane, $\mathcal{C}$ (or in $\mathbf{R}^n$). Let

$$V_1 = \{z \in \mathcal{C} | |z| \leq 1\} \quad \text{and} \quad V_2 = \{z \in \mathcal{C} | |z| \leq 2\}.$$

Let $E_0 = \{\mathbf{O}\} \subset \mathcal{C}$, and for each integer $k > 0$, let

$$E_k = \{t \cdot z | z^k = 1, 0 \leq t \leq 1, z \in \mathcal{C}\}.$$

With J denoting the interval $[-1, 1]$ in $\mathbf{R}$, let

$$V = V_1 \times J^{n-2} \subset \mathbf{R}^n \quad \text{and for } k \geq 0,$$
$$Q_k = \left\{x \in \mathbf{R}^n | \text{dist}\,(x, E_k) \leq \frac{1}{k+2}\right\}.$$

Notice that $Q_k \cap (\mathbf{R}^2 \times \{0\})$ is a disk contained in V_2 and that $V \cap Q_k$ is an n-ball as is each component of $Q_k \backslash \text{int}\,(V)$. These components each meet $V \cap Q_k$ in an $(n-1)$-ball.

Let M^n be a compact, connected, topological manifold of dimension n. As techniques from differential topology will be applied to certain subsets of M, we will assume, for convenience, that M is smooth away from a finite set of points. If $\partial M = \varnothing$, let $h : M \to M$ be a homeomorphism, otherwise let h be an embedding of M into the interior of M. Let B be a smooth n-ball in M, and suppose that h restricted to B is a smooth map onto its image.

Definition. We say that B is *isotopy-standard-for* h if there exists a smooth, locally flat disk D in M, a nonnegative integer k, and a homeomorphism of triples

$$\phi : (B, h(B), D) \to (V, Q_k, V_2)$$

such that the following conditions are satisfied:

(c1) $\mathrm{Fix}\,(h) \cap \partial B = \varnothing$,

(c2) $\phi h \phi^{-1}(V_1) \subset V_2$, and it is orientation preserving; $\phi h \phi^{-1}(V \backslash V_1) \cap V_2 = \varnothing$,

(c3) for each $x \in B \backslash D$, $\mathrm{dist}\,(\phi(h(x)), V_2) < \mathrm{dist}\,(\phi(x), V_2)$.

Remark. In what follows, the dependence on the homeomorphism is usually suppressed, in which case we just write isotopy-standard.

Example. Suppose x is a hyperbolic fixed point of h such that the dimension of the unstable manifold, $W^u(x)$, is one (see [**3**]). Suppose further that the stable manifold contains a one-dimensional invariant subspace, L. If h is orientation preserving on each of these subspaces, then small regular neighborhoods of x form isotopy-standard balls. The locally flat disk D is contained in the product $L \times W^u(x)$. This idea is developed further in Proposition 3.3.

Example. Suppose $\dim(M) = 2$ and N is a disk in M which satisfies (i) $h(\partial N)$ is transverse to ∂N with no fixed points on ∂N, (ii) $N \backslash h(N) \neq \varnothing$ and (iii) $N \cap h(N)$ is a disk. Under these conditions we may assume, without loss of generality, that both N and $h(N)$ are contained in $\mathbf{R}^2$. Now if h is orientation preserving, then N is an isotopy-standard ball.

If B is isotopy-standard, then it is easy to see that all the fixed points in B are Nielsen equivalent. By condition (c3) they must all lie on the disk D. Our first result says that they can all be merged together to form a single fixed point.

Proposition 3.1. *Suppose that B is an isotopy-standard ball. Then h is isotopic to h' with support on B such that B is an isotopy-standard* *ball and* $\mathrm{Fix}\,(h') \cap B$ *consists of exactly one point.*

Proof. For each $p \in \mathbf{bd}(V) \cup \mathbf{bd}(Q_k)$ let l_p denote the segment in $\mathbf{R}^n$ from p to $\mathbf{O}$. Define a map $g : V \to Q_k$ by sending l_p linearly onto $l_{\phi h \phi^{-1}(p)}$ with $\mathbf{O}$ being sent to $\mathbf{O}$. Since V is convex and Q_k is star-shaped, g is well-defined; and from (c1) it follows that $\mathrm{Fix}\,(g) = \mathbf{O}$. Let $h'(p) = \phi^{-1} \circ g \circ \phi(p)$ for $p \in B$ and $h'(p) = h(p)$ for $p \notin B$. Then by the disk theorem [**3**; p. 185] h' is isotopic to h. Clearly, $\mathrm{Fix}\,(h') \cap B = \phi^{-1}(\mathbf{O})$. Finally, B is isotopy-standard. For given $x \in B \backslash D$, then $\phi(x) \in l_p$ for a unique point $p \in \phi(\partial B)$ and

$$\begin{aligned} &\mathrm{dist}\,(\phi(h'(x)), V_2) < \mathrm{dist}\,(\phi(x), V_2) \\ \text{iff} \quad &\mathrm{dist}\,(g(\phi(x)), V_2) < \mathrm{dist}\,(\phi(x), V_2) \\ \text{iff} \quad &\mathrm{dist}\,(g(p), V_2) < \mathrm{dist}\,(p, V_2). \end{aligned}$$

This last inequality holds because $g = \phi h \phi^{-1}$ on $\phi(\partial B)$ and the ball B is isotopy-standard. □

Given $h : M \to M$ and a set X which denotes either a compact set of fixed points, an open set in M, or the closure of an open set having no fixed points on its boundary, let $\mathrm{index}\,(h, X)$ denote the topological index for the fixed point set of h on X. See [**2**] or [**5**] for the definition. From the proof of Proposition 3.1, if B is any isotopy-standard ball we know that

$$\mathrm{index}\,(h, B) = \mathrm{index}\,(h', \phi^{-1}(\mathbf{O})).$$

If U and V are subspaces of M with U open in V and $h(U) \subset V$, then we may consider the restricted index for the map $h|U : U \to V$ denoted $\mathrm{index}\,(h|U, Y)$ where $Y \subseteq U$ is similar to X above. In general, for $p \in \mathrm{Fix}\,(h)$ and $p \in U \subset M$, $\mathrm{index}\,(f, p)$ may not be equal to $\mathrm{index}\,(f|U, p)$ but for an isotopy-standard ball B we have

Lemma 3.2. $\mathrm{index}\,(h', \phi^{-1}(\mathbf{O})) = \mathrm{index}\,(h'|D \cap B, \phi^{-1}(\mathbf{O}))$.

Proof. We show that $\mathrm{index}\,(\phi h' \phi^{-1}, \mathbf{O}) = \mathrm{index}\,(\phi h' \phi^{-1}|V_1, \mathbf{O})$. Define a map $g : V \to \mathbf{R}^n$ as follows. For $p \in V \cup Q_k$ let $q(p)$ denote the point on V_2 such that $\mathrm{dist}\,(p, q(p))$ is minimal, and let s_p denote the segment joining p to $q(p)$. Let $\lambda : V \to [0, 1]$ be any continuous function satisfying (1) $\lambda(\partial V) = 0$ and (2) if $x \in N = \{y \in V | \text{dist}\,(y, \mathbf{O}) \leq 1/2\}$ then $\lambda(x) = 1$. Now, for $x \in V$ let $p = \phi h' \phi^{-1}(x)$ and define $g(x) = (1 - \lambda(x))p + (\lambda(x)) \cdot q(p) \in s_p$. Clearly, g is homotopic to $\phi h' \phi^{-1} \text{rel}\, \partial V$. Also, condition (c3) of the definition of isotopy-standard ensures that $\text{Fix}\,(g) = \mathbf{O}$. Thus,

$$\text{index}\,(\phi h' \phi^{-1}, \mathbf{O}) = \text{index}\,(g, \mathbf{O}) = \text{index}\,(g, N).$$

Now apply I.3.2(3) in [**5**] to get

$$\text{index}\,(g, N) = \text{index}\,(g|V_1, N \cap V_1) = \text{index}\,(g|V_1, \mathbf{O}).$$

The result now follows as $g|V_1 = \phi h' \phi^{-1}|V_1$. □

This section is concluded by showing that, in the smooth category, isotopy-standard balls which cover the entire fixed point set do exist. This result doesn't directly apply to Corollary 2.3 as the construction will not address the third condition in the hypothesis of 2.3. For the proof of existence, we will use the notion of a hyperbolic fixed point.

Definition. Given a smooth embedding $f : M \to M$, a point $x \in \text{Fix}\,(f)$ is said to be *hyperbolic* if Df_x is a linear isomorphism of TM_x for which all of its eigenvalues have modulus different from 1. In this case there is a splitting of $TM_x = E_f^s \oplus E_f^u$ where both E_f^s and E_f^u are Df_x-invariant,

$$E_f^s = \{\nu \in TM_x \mid ||Df_x(\nu)|| < ||\nu||\}$$

and

$$E_f^u = \{\nu \in TM_x \mid ||Df_x(\nu)|| > ||\nu||\}.$$

Proposition 3.3 (Existence of isotopy-standard neighborhoods). *Let $f : M \to M$ be either a diffeomorphism (if $\partial M = \varnothing$) or a smooth embedding into* $\text{int}(M)$. *If* $\dim\,(M) = 2$, *we assume that f is orientation preserving. Then f is isotopic to $h : M \to M$ such that* $\text{Fix}\,(h)$ *is finite and each fixed point is contained in an isotopy-standard* *ball. Moreover, these balls can be chosen inside any open set containing* $\mathrm{Fix}\,(h)$.

Proof. First, by standard transversality techniques (see [**1**]) arrange that $\mathrm{Fix}\,(f)$ is a 0-dimensional submanifold and, hence, a finite set of points. Also, for each $x \in \mathrm{Fix}\,(f)$, $Df_x : TM_x \to TM_x$ is such that Df_x has no eigenvalues of modulus one. Thus, all fixed points are hyperbolic.

First consider the case $\dim(M) = 2$. If $x \in \mathrm{Fix}\,(f)$ is such that either of the eigenvalues of Df_x has modulus less than one, then small regular neighborhoods of x are isotopy-standard. Otherwise, f is locally expanding at x and, hence, has a regular neighborhood, N, such that $N \subset f(N)$ with no other fixed points in N. Let U be an open disk in M with $f(N) \subset U$. Choose a diffeomorphism $\Theta : U \to \mathbf{R}^2$ such that $\Theta(x) = \mathbf{O}$, $\Theta(N) = \{x \in \mathbf{R}^2 \mid |x| \leq 2\}$, and for $x \in \partial N$, $\Theta(f(x)) = 2 \cdot \Theta(x)$ ($\mathbf{R}^2$ as a vector space).

Let

$$\lambda : [0,1] \to [0,2] \quad \text{and} \quad w : [0,1] \to [0, \pi/8]$$

be smooth maps with

$$\begin{array}{lll} \lambda([0,1/2]) = 1/2, & \lambda(3/4) = 3/4, & \lambda(1) = 2 \\ w([0,1/4]) = 0, & w(3/4) = \pi/8, & w(1) = 0. \end{array}$$

Also, assume that λ is increasing on $[1/2, 1]$ and that w has exactly one maximum occurring at $3/4$.

For $t \in \mathbf{R}$ let $M(t)$ denote the 2×2 matrix

$$\begin{pmatrix} \cos(w(t)) & \sin(w(t)) \\ -\sin(w(t)) & \cos(w(t)) \end{pmatrix}.$$

Define $\tau : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\tau(\nu) = \begin{cases} \Theta \cdot f \cdot \Theta^{-1}(\nu) & \text{if } |\nu| \geq 1 \\ \lambda(|\nu|) \cdot M(|\nu|) \cdot \nu & \text{if } |\nu| < 1 \end{cases}$$

and set $g = \Theta^{-1} \tau \Theta$.

Since g and f agree off of N, g is isotopic to f. By construction, τ only has one fixed point on $|\nu| \leq 1$ and so $\mathrm{Fix}\,(g) = \mathrm{Fix}\,(f)$. Finally, since Dg_x is conjugate to $D\tau_{\mathbf{O}}$ its eigenvalues have modulus less than one. Then $g \equiv h$ is the desired map. Note that the orientation preserving hypothesis in the two-dimensional case is only used to obtain the o.p. condition given in (c2) of the definition of isotopy-standard.

For the general case, $\dim(M) > 2$, let's first suppose that $\dim(E^u_f) \geq 2$. Apply Lemma 3.4 to get $\hat{f}$ isotopic to f and S, a two-dimensional invariant subspace of $E^u_{\hat{f}}$. Write $TM_x = S \oplus S^{\perp}$. By use of the exponential map, we can find disks N, U and $(n-2)$-balls P and V such that $N \subset U \subset \exp_x(S)$ and $P \subset V \subset \exp_x(S^{\perp})$. Moreover, we arrange that $\hat{f}(N \times P) \subset U \times V$. Use the previous construction (when $\dim M = 2$) to obtain $g_1 : N \to U$ replacing $\hat{f}|N$. Then $g = g_1 \times (\hat{f}|P)$ is isotopic to f with $\mathrm{Fix}\,(g) = \mathrm{Fix}\,(f)$, and $D(g|P)_x = D(f|P)_x$. Also, $\dim(E^u_g) = \dim(E^u_f) - 2$. Repeat, if necessary, this construction until we obtain $h : M \to M$ isotopic to f with $\dim(E^u_h) \leq 1$.

If $\dim(E^u_h) = 0$, apply Lemma 3.4 to get two distinct one-dimensional invariant subspaces of E^s_h on which Dh_x is orientation preserving (by abuse of notation we still call the map h). Let ξ denote the direct sum of these subspaces and using the exponential map, $\exp_x : TM_x \to M$, set $D = \exp_x(\{e \in \xi \mid ||e|| \leq 2\})$ and $B = \exp_x(\{e \in TM_x \mid ||e|| \leq 1)$. Then B is an isotopy-standard ball with $\phi \circ h(B) = Q_0$.

If $\dim(E^u_h) = 1$, we may assume that Dh_x is orientation preserving on this subspace. Otherwise, by an isotopy of h which does not change the fixed point set, we could arrange that Dh_x is contracting on this subspace and then appeal to the previous case. So, by Lemma 3.4, there is a one-dimensional invariant subspace, L, of E^s_h on which Dh_x is orientation preserving. Let $\xi = E^u_h \oplus L$ and, as in the preceding case, construct D and B. Here $\phi \circ h(B) = Q_2$. □

Lemma 3.4. *Let x be a hyperbolic fixed point of $f : M \to M$, and suppose that $\dim(E^t_f) \geq k$ where k is a positive integer and t denotes either s or u. Then f is isotopic to g with support on a regular neighborhood of x such that $\mathrm{Fix}\,(g) = \mathrm{Fix}\,(f)$, and E^t_g contains $k-1$ distinct one-dimensional invariant subspaces such that Dg_x is orientation preserving on each.*

Proof. Suppose that $\nu \in E_f^t$ is a nonzero vector such that the subspace W spanned by $\{\nu, Df_x(\nu)\}$ has dimension 2. Use the exponential map, $\exp_x : TM_x \to M$, to identify TM_x with a neighborhood of x. Let (N, N_0) be a regular neighborhood pair of x contained in $(\exp_x(TM_x), \exp_x(W))$ with $\mathrm{Fix}\,(f) \cap N = \{x\}$. Consider the smooth disk N_0 as the unit disk in $\mathbf{R}^2$, x as the origin. By a suitable rotation of N_0 which is tapered to the identity on ∂N_0, there is a diffeomorphism $\psi : N_0 \to N_0$, isotopic to the identity such that $\psi|\partial N_0 =$ identity, $||\psi(p)|| = ||p||$ for all p and $D\psi_x(Df_x(\nu)) = ||Df_x(\nu)|| \cdot \nu$. Extend ψ to all of N by using the identity on the factor $\exp_x(W^\perp)$. Let $g = \psi \circ f$. Since $Dg_x = D\psi_x \cdot Df_x$, it follows that the subspace spanned by ν is invariant under Dg_x and that Dg_x is orientation preserving on this subspace.

If no such vector ν exists, then one automatically obtains a one-dimensional subspace of E_f^t on which Df_x is invariant and orientation preserving except in the case where Df_x acts in an orientation reversing manner on the span of ν for every $\nu \in E_f^t$. In this case let W denote the span of any two linearly independent vectors in E_f^t and apply the argument of the preceding paragraph to obtain the desired subspace. Now, in each case, as $W \subset E_f^t$, $\mathrm{Fix}\,(g) \cap N = \{x\}$. This completes the proof in the case $k = 2$.

Proceeding inductively, suppose that E_g^t contains p distinct one-dimensional D_f-invariant subspaces (with D_f orientation preserving on each) where $0 < p < k - 1$. Let V denote the span of the union of these subspaces. Then the orthogonal complement $V^\perp$ to V in E_f^t has dimension at least 2 and so the argument given above can be applied to $V^\perp$ to produce a new map g and a one-dimensional invariant subspace of $V^\perp$. As the construction leaves V unchanged, we now have $p + 1$ invariant subspaces on which D_g is orientation preserving. The result now follows. □

4. Proof of Theorem 2.2. In order to prove our main result we first give the following proposition which is essentially the statement of Theorem 2.2 in the two-dimensional case.

Proposition 4.1. *Let* $N = \{x \in \mathbf{R}^2 \mid |x| \leq 1\}$ *and* $S = \partial N$. *Let* C *be a simple closed curve in* $\mathbf{R}^2$*, transverse to* S*, bounding a disk*

M such that $N \cap M$ is a disk. If $h : S \to C$ is a fixed point free homeomorphism with index $(h, S) = 0$*, then h extends to a fixed point free homeomorphism $h : N \to M$.*

Remarks. (1) index $(h, S) \equiv$ index $(\bar{h}, N)$ where $\bar{h}$ is any extension of h to N. (2) If h is orientation preserving, then N is an isotopy-standard ball for h in $\mathbf{R}^2$.

Proof of Proposition 4.1. The proof proceeds as follows. First, two constructions are given which allow us to reduce $h : S \to C$ to a standard form, independent of index (h, S). Then it is shown that a fixed point free extension exists when index $(h, S) = 0$. Fix an orientation on $\mathbf{R}^2$, and let each of N, M, S, and C have the induced orientation. The proof will depend on whether h is orientation preserving or reversing.

First, we can assume that $S \cap C \neq \varnothing$. For, if not, then either $N \subset M$ or $M \subset N$, in which case index $(h, S) \neq 0$, or $N \cap M = \varnothing$ and the fixed point free extension is trivial. Let Z denote the finite set $S \cap C$ and, without loss of generality, we assume that $h(Z) \cap Z = \varnothing$. Let $[S]$ denote the set of components of $S \backslash Z$ and $[C]$ the components of $C \backslash Z$. Since $N \cap M$ is a disk, there is a natural pairing between $[S]$ and $[C]$ as follows: for $\alpha \in [S]$, let $\bar{\alpha} \in [C]$ be such that the closure of $\alpha \cup \bar{\alpha}$ is a simple closed curve and the disk $D(\alpha)$ bounded by this curve is not contained in $N \cap M$. The latter condition is only needed when Z consists of two points. Note that if $\partial D(\alpha)$ is oriented so that its orientation agrees with the orientation induced by S, then it has the opposite orientation as that induced by C.

For the first construction we assume that for some $\alpha \in [S]$, $h(\alpha) \cap \bar{\alpha} = \varnothing$. In the case $\alpha \cap M = \varnothing$, choose a regular neighborhood P of $D(\alpha)$ in N which satisfies: (1) $(P \cap C) = \mathrm{cl}\,(\bar{\alpha})$, (2) $(P \backslash D(\alpha)) \subset (N \cap M)$ and (3) $(P \cap S)$ is an arc whose image under h is disjoint from P. The existence of such a neighborhood follows because C meets S transversally and $h(\alpha)$ is disjoint from $\bar{\alpha}$. Note that P is a disk which meets $C \cap S$ in exactly two points. Now, let Q be a disk in $M \backslash P$ chosen so that $Q \cap C = h(P \cap S)$ and $\mathrm{cl}\,(\partial Q \backslash C)$ meets S transversally in the same number of points as does $Q \cap C$.

Extend h by mapping P to Q and observe that with $N' = \mathrm{cl}\,(N \backslash P)$, $M' = \mathrm{cl}\,(M \backslash Q)$, $S' = \partial N'$, and $C' = \partial M'$ then $N' \cap M'$ is a disk and

$[S']$ has two fewer elements than $[S]$. In the case $\alpha \subset M$, apply the same construction to $h^{-1} : C \to S$ and the arc $\bar{\alpha}$. Here, $\bar{\alpha} \cap N = \varnothing$ and $h^{-1}(\bar{\alpha}) \cap \alpha = \varnothing$.

For the second construction we assume that, for some $\alpha \in [S]$, $h(\alpha) \cap \bar{\alpha} \neq \varnothing$ and $\bar{\alpha} \backslash h(\alpha)$ has exactly one component. Let y denote the endpoint of α which is contained in $h(\alpha)$, and x the other endpoint. As $M \cap N$ is a disk, if h is orientation preserving it must be that $h(x)$ lies on $\bar{\alpha}$ while $h(y)$ is on $C \backslash \mathrm{cl}\,(\bar{\alpha})$. The opposite happens when h is orientation reversing. Choose a point $\tilde{x}$ on $(S \backslash \alpha)$ near x and let P_0 denote the arc from $\tilde{x}$ to y which contains α. Let P be a disk embedded in N so that $P \cap S = P_0$. Also, with P identified with $P_0 \times I$ we assume that $P \cap C = \{x, y\} \times I$. Let $Q_0 = h(P_0)$ and choose Q in M (identified with $Q_0 \times I$) so that either

(i) if $\alpha \cap M = \varnothing$, then $P \cap Q$ is the arc $\{y\} \times I \subset P$. In this case let $\hat{y} = \{y\} \times \{1\} \subset Q$.

(ii) if $\alpha \subset M$, then there is a point y' on α and an arc $\beta \subset \alpha$, joining y' to y, so that $P \cap Q = \beta \times I \subset P$. In this case, let $\hat{y} = \{y'\} \times \{1\} \subset P$.

In either case $\hat{y}$ is contained in $Q_0 \times \{1\}$. Let $\hat{x} = \{x\} \times \{1\}$, and let δ be an arc in $Q_0 \times \{1\}$ which has $h(y) \times \{1\}$ as an endpoint and which does not contain $\hat{y}$.

Now, define $h : P \to Q$ extending $h : P_0 \to Q_0$ by sending

(i) $\tilde{x} \times I$ to $(h(\tilde{x}) \times I) \cup \mathrm{cl}\,((Q_0 \times \{1\}) \backslash \delta)$

(ii) $y \times I$ to $h(y) \times I$

(iii) $P_0 \times \{1\}$ to δ

(iv) $\mathrm{int}\, P$ to $\mathrm{int}\, Q$.

Note that if P and Q are chosen in small neighborhoods of P_0 and Q_0, respectively, then $P \cap Q$ is in a neighborhood of y and $h(P \cap Q)$ is in a neighborhood of $h(y)$. Thus, no fixed points occur. Setting $N' = \mathrm{cl}\,(N \backslash P)$, $M' = \mathrm{cl}\,(M \backslash Q)$, $S' = \partial N'$ and $C' = \partial M'$, then

$$S' \cap C' = ((S \cap C) \backslash \{x, y\}) \cup \{\hat{x}, \hat{y}\}.$$

Finally, with $\alpha' \in [S']$ being the arc joining $\hat{x}$ to $\hat{y}$, then $h(\alpha') \cap \bar{\alpha}' = \varnothing$ when h is orientation preserving, and $h(\alpha') \subset \overline{\alpha'}$ when h is orientation reversing.

By a repeated application of the above two constructions, we reduce to the case where $h : S \to C$ satisfies $h(\alpha) \subset \bar{\alpha}$ or $\bar{\alpha} \subset h(\alpha)$ for each $\alpha \in [S]$. Observe that if α and β are adjacent and $h(\alpha) \subset \bar{\alpha}$, then $\bar{\beta} \subset h(\beta)$. To finish the proof, we consider each of the two cases.

If h is orientation preserving, set $\lambda = 1$ if $h(\alpha) \subset \bar{\alpha}$ occurs when $\alpha \subset M$; otherwise set $\lambda = 0$. Let l be the number of components of $M \backslash N$. By a direct index calculation (see [**5**, p. 13] it can be shown that, under the hypothesis above,

$$\text{index}\,(h, S) = \begin{cases} 1 - l & \text{if } \lambda = 1 \\ 1 + l & \text{if } \lambda = 0. \end{cases}$$

Thus, $\text{index}\,(h, S) = 0$ only when $l = 1$ and $\lambda = 1$. In other words, the closures of $M \backslash N$ and $N \backslash M$ are disks and, with $\alpha = S \cap M$, $h(\alpha) \subset \bar{\alpha} = C \backslash N$. One easily obtains a fixed point free extension to N by choosing an arc $\omega \subset M \backslash N$ with $\omega \cap C = h(\alpha \cap C)$ and mapping $C \cap N$ homeomorphically onto ω.

Now suppose that h is orientation reversing. First, observe that after applying the two constructions, Z consists of at most two points. For, if not, there certainly exists $\alpha \in [S]$ such that $h(\alpha) \subset \bar{\alpha}$. But, in this case, for each of the components $\beta \in [S]$ adjacent to α, the orientation reversing hypothesis ensures that $h(\beta) \cap \bar{\beta} = \varnothing$. Thus, the first construction applies.

Suppose $Z \neq \varnothing$, and let $\alpha \in [S]$ be the unique component such that $\alpha \subset M$. There are two possibilities to consider: either $h(\alpha) \subset \bar{\alpha}$ or vice versa. In either case $\text{index}\,(h, S) = 0$ and a fixed point free extension can be obtained as follows. In the former, the extension mimics that in the orientation preserving case above, while the latter is slightly different. Here choose an arc ω in M which is near $h(\alpha)$ and has the same endpoints as $h(\alpha)$. Map $N \cap M$ into the disk bounded by $\omega \cup h(\alpha)$ and send $N \backslash M$ into the complementary disk in M. The orientation reversing hypothesis guarantees (for a good choice of ω) that the extension is fixed point free. □

Proof of Theorem 2.2. Let D be the disk associated with B. Let $D_0 = D \cap B$. Also let $\phi : (B \cup h(B)) \to (V \cup Q_k)$ be as in the definition of isotopy-standard. First, observe that by Lemma 3.2 and Proposition 4.1, $h|D_0$ is homotopic $\text{rel}\,\partial D_0$ to a homeomorphism $h' : D_0 \to h(D_0)$

such that $\text{Fix}\,(h') = \varnothing$. As homotopy implies isotopy in dimension two, we have an isotopy (after conjugating with ϕ)

$$H : V_1 \times I \to V_1 \times I$$

such that $H_0 = \text{identity}$, $H_1 = \phi \circ (h|D_0)^{-1} \circ h' \circ \phi^{-1}$, and $H|(\partial V_1 \times 1) = \text{identity}$. Let $H_t = H|(V_1 \times \{t\}) : V_1 \to V_1$.

Given any $x \in V$, it can be expressed uniquely in the form $x = (z, t)$ where $z \in V_1$ and $t \in J^{n-2}$. If $t \neq \mathbf{O}$, let s_x denote the segment in J^{n-2} from $\mathbf{O}$ to t and l_x the maximal segment from $\mathbf{O}$ to ∂J^{n-2} containing s_x. Define

$$\tau(x) = \begin{cases} 1 - \dfrac{\text{length }(s_x)}{\text{length }(l_x)} & \text{when } t \neq \mathbf{O} \\ 1 & \text{when } t = \mathbf{O}. \end{cases}$$

Let $\pi_t : V_1 \times \{t\} \to V_1 \equiv V_1 \times \{0\}$ denote the projection map.

Define a map $\mu : V \to Q_k$ by

$$\mu(x) = \phi \circ h \circ \phi^{-1} \circ \pi_t^{-1} \circ H_{\tau(x)} \circ \pi_t(z).$$

This map is clearly one-to-one and onto. Also, if $x \in \partial V$, then $\tau(x) = 0$ and $\mu(x) = \phi \circ h \circ \phi^{-1}(x)$. So we may define a homeomorphism $\tilde{h} : M \to M$ by

$$\tilde{h}(p) = \begin{cases} h(p) & \text{if } p \notin B \\ \phi^{-1} \circ \mu \circ \phi(p) & \text{if } p \in B. \end{cases}$$

In the above, if $\tau(x)$ is replaced by $\rho \cdot \tau(x)$ where $0 \leq \rho \leq 1$, then we get a natural isotopy between h and $\tilde{h}$. Furthermore, we have

(1) For $p \in D_0$ we have $\tau(\phi(p)) = 1$. Thus, since $\pi_0 = \text{identity}$,

$$\mu = \phi h \phi^{-1} \phi (h|D_0)^{-1} h' \phi^{-1} = \phi h \phi^{-1}$$

and so $\tilde{h} = h'$.

(2) Since $\mu(V_1 \times \{t\}) = \phi h \phi^{-1}(V_1 \times \{t\})$ it follows that

$$\tilde{h}(\phi^{-1}(V_1 \times \{t\}) = \phi^{-1} \circ \mu(V_1 \times \{t\}) = h(\phi^{-1}(V_1 \times \{t\})).$$

(3) By condition (c3), if $t \neq 0$, then $\phi h \phi^{-1}(V_1 \times \{t\}) \cap (V_1 \times \{t\}) = \varnothing$. Thus, $h(\phi^{-1}(V_1 \times \{t\})) \cap \phi^{-1}(V_1 \times \{t\}) = \varnothing$.

Putting (1), (2) and (3) together, we see that $\text{Fix}(\tilde{h}) \cap B = \varnothing$ which establishes Theorem 2.2. $\square$

As a final note to this paper, we give a few comments. First, we note that the proof of Proposition 4.1 can be adapted to give an alternate proof of Proposition 3.1. Namely, if (in 4.1) the index is nonzero, then there is an extension of h to the disk N which has exactly one fixed point. Use this extension in the proof of 2.2 to obtain a homeomorphism with exactly one fixed point in the isotopy-standard ball.

The remaining comments are concerned with the use of the smooth category in obtaining the various results. For instance, if the smooth hypothesis in the definition of isotopy-standard is replaced by piecewise-linear (PL), then the proofs of 3.1, 4.1, and 2.2 can each be modified so as to obtain the same conclusion. The corresponding PL tools needed can all be found in [**11**]. On the other hand, the existence result (3.3) uses transversality and so the smooth hypothesis is needed. Also, in the proof of 2.2, if the input data (M and h) is smooth (or PL) then we can arrange for the output homeomorphism $\tilde{h}$ to be smooth (respectively, PL) as well. In 3.1, a coning construction is used to obtain a homeomorphism with one fixed point and, as a result, it is not clear whether or not this homeomorphism can be taken to be smooth.

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