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CLOSED INCOMPRESSIBLE SURFACES OF ARBITRARILY HIGH GENUS IN THE COMPLEMENTS OF CERTAIN STAR KNOTS

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1. Introduction and preliminaries. One of the several results presented by U. Oertel in [**2**] is the determination of the closed incompressible surfaces in the complements of most star knots. Presented in this paper are the cases for some star knots not included in Oertel's discussion. A brief sketch of the terminology and definitions follows.

A "tangle" is a set of two tangled arcs embedded in a 3-ball B, both of whose endpoints lie in $S^2 = \partial B$ (e.g. Figure 1). The construction of a rational tangle originates from "drawing slope p/q lines on a square pillowcase starting at the four corners" [**2**], $p/q \in Q$ with $(p,q) = 1$ (e.g., Figure 2).

The star knots are formed from geometric sums of rational tangles which follow a prescription $K(p_1/q_1, p_2/q_2, \dots, p_k/q_k)$ so that adjacent endpoints of the tangles match in pairs to form the knot (e.g., Figure 3). Cases where K is a knot provide that $q_i \geq 2$.

Oertel constructs his closed incompressible surfaces in $S^3 - K$ in two steps.

The first is the selection of a finite collection of disjoint 4-punctured spheres each of whose intersection with the plane of projection of K consists of simple closed curves which intersect K transversely at four points on the arcs of K joining tangles.

The second step is to complete the closed surface by a sequence of peripheral tubing operations which connect the punctured spheres.

If a 4-punctured sphere bounds a ball which contains more than one rational tangle, the ball is a "Seifert tangle" denoted $(B; p_1/q_1, \dots, p_k/q_k)$ (Figure 4). The sphere $\partial B - K$ is incompressible in $S^3 - K$ if and only if it bounds a Seifert tangle on each side which is not a rational tangle (cf. [**2**, Corollary 2.14]).

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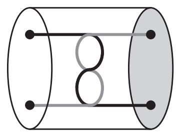

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Figure 1

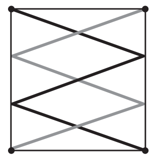

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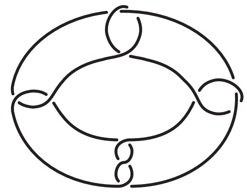
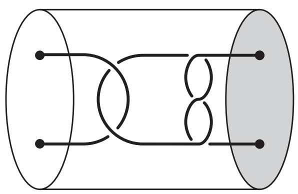
Figure 2



K(-1/2, 1/3, 2/3. 2/3)

Figure 3



(B; -1/2, 1/3)

Figure 4

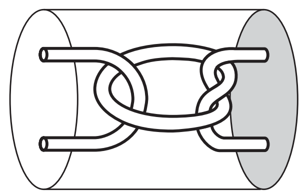


Figure 5(a)

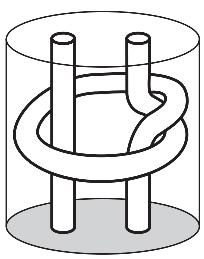


Figure 5(b)

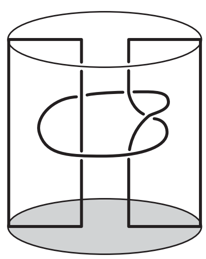


Figure 5(c)

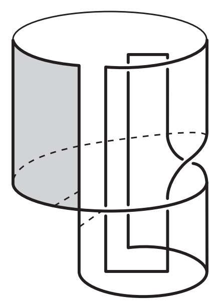


Figure 6

FIGURES 1–6.

Additionally, if $q_i \geq 3$ for each i, then a surface obtained from disjoint incompressible 4-punctured spheres by a sequence of tubing operations is incompressible if and only if each tube passes through at least one rational tangle (cf. [**2**, Theorem 2]).

2. The exceptions. The interest here is one of Oertel's results

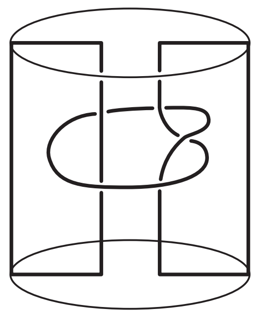


FIGURE 7.

which states that if $q_i \geq 3$ for each i, $k \geq 4$, and K is a star knot, then $S^3 - K$ contains closed incompressible surfaces of every genus ≥ 2 (cf. [**2**, Corollary 3]). Also required is that $\sum_{i=1}^{k} p_i/q_i \neq 0$.

If some $q_i = 2$, then tubing some incompressible spheres can result in a compressible closed surface.

Figure 5 presents such a case. Figure 5(a) shows a tubing scheme for $(B; -1/2, 1/3)$ (Figure 4) and Figure 5(b) presents another view. There are two arcs in $B = (B; -1/2, 1/3)$.

The arc from $p_1/q_1 = -1/2$ can be completed on ∂B to a circle while the other arc completes on ∂B to a trefoil τ. This is illustrated in Figure 5(c). Because the arcs which join points on a sphere are unique up to homotopy preserving the end points, by adding disjoint arcs in the closure of $\partial B \cup \{a\text{-tube}, b\text{-tube}\}$ one obtains a pair of simple closed curves with linking number "one." The tube in Figure 5(a) about the arc $p_1/q_1 = -1/2$ will be called the "a-tube" and the second tube the "b-tube." Figure 6 illustrates compressing disc $D(b)$. In Figure 7 the boundary of $D(b)$ has been broken into eight segments. The four segments with end points labelled 1 & 0, 2 & 3, 1 & 3, 2 & 0 lie in pairs in the discs of ∂B in Figure 5(b). The three segments with end points labelled 1, 2, 3 lie on the a-tube while the last segment with end points labelled 0 lies on the b-tube. Figure 8 illustrates compressing disc $D(a)$. In Figure 8 the boundary of $D(a)$ has been broken into six segments with the same labelling scheme as that for $D(b)$ except that the segment 1 & 3 also has an arc on the annulus of ∂B. There is no arc on the b-tube.

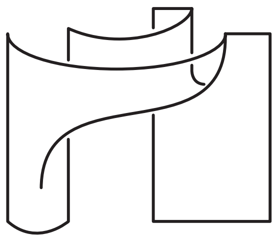


Figure 8

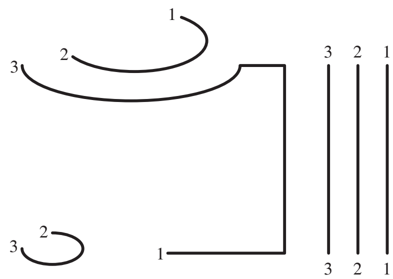


Figure 9

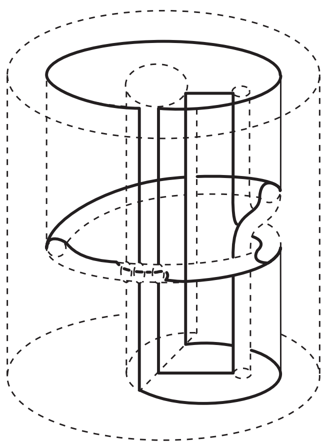


Figure 10(a)

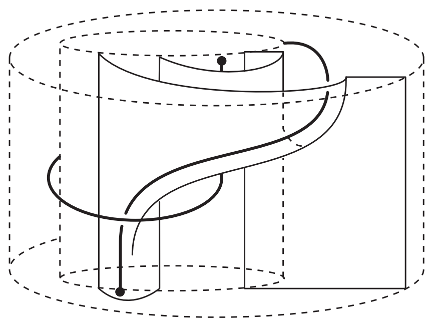


Figure 10(b)

FIGURES 8–10(b).

Note that any arc γ completing the trefoil τ intersects each of $\partial D(a)$ and $\partial D(b)$. Also note that discs $D(a)$ and $D(b)$ can be placed so that they are disjoint. Figures 10(a) and 10(b) show the discs $D(a)$ and $D(b)$ in place in Figure 5(b).

3. In the remaining discussion $K = K(-1/2, 1/3, 2/3, 2/3)$ (cf. Figure 3), $B = B(-1/2, 1/3)$ and $B^* = B(2/3, 2/3) - K$.

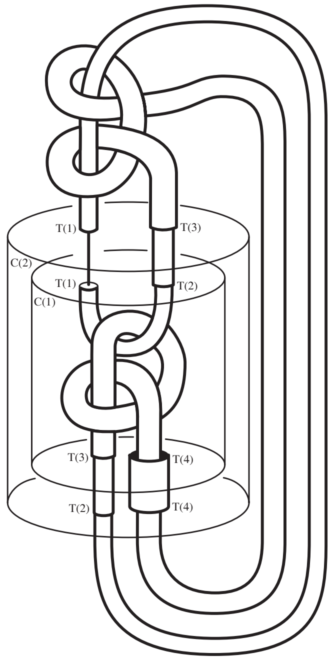


Figure 11

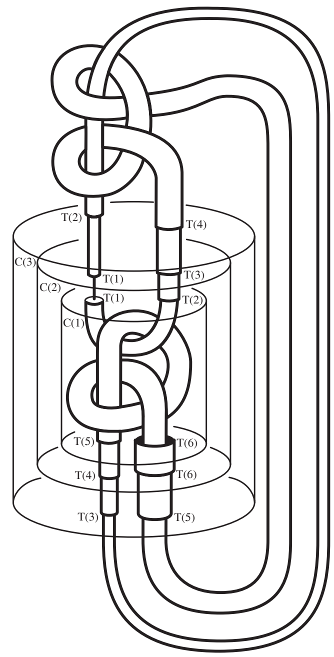


Figure 12

FIGURES 11 and 12.

4. Remark. Proposition 2.13 of [**2**] gives that $\partial B - K$ is incompressible in $B - K$ and also that $\partial B^* - K = \partial(S^3 - B(-1/2, 1/3) - K$ is incompressible in $B^* - K$. Thus, there are no compressing discs in either $B - K$ or $B^* - K$.

5. Some closed surfaces in $S^3 - K$. Figure 11 illustrates surface $S(2)$ of genus $2 + 1 = 3$. $S(2)$ is formed by tubing two copies of $\partial B - K : \partial B(1) - K$, $\partial B(2) - K$ and four tubes $T(1), T(2), T(3), T(4)$.

Figure 12 illustrates $S(3)$ of genus $3 + 1 = 4$, again formed by tubing three copies of $\partial B - K : \partial B(1) - K, \partial B(2) - K, \partial B(3) - K$ and six tubes in this case.

Figure 13 gives the general scheme for continuing constructions in the pattern already established to yield a surface $S(n)$ of genus $n + 1$

formed from n copies of $\partial B - K$ and $2n$ tubes.

6. A view of $s(n)$ and the tubing scheme. In Figure 14 is displayed a regular neighborhood U of K, along with two copies of ∂B and the meridians cut in $H = \partial U$ by these spheres. Figure 15 presents a view of U and the relative positions of the four tubes for $S(2)$. Note that tube $T(4)$ is an annular subset of H while the other three tubes are nested (along with $T(4)$) in U and "flare" (by annuli from meridian discs) to meet $C(1) = \overline{\partial B(1) - U}$ and $C(2) = \overline{\partial B(2) - U}$. The two boundary components of each $T(l)$, $l = 1, 2, 3, 4$ cut H into annular subsets $H(l)$, $l = 1, 2, 3, 4$ which are nested $H(4) \subseteq H(3) \subseteq H(2) \subseteq H(1) \subseteq H$ (Figure 15).

Note that the tubed 2-sphere $\partial(B^* \cap U) \cup (\overline{\partial B^* - H})$ is incompressible by Oertel's Theorem 2.

7. Theorem.

Theorem 7. *For $n = 1, 2, \ldots, S(n)$ is incompressible in $S^3 - K$.*

The structure of the following proof parallels that of Section 6 in [**1**].

Lemma 7.1. *For each $l = 1, 2, \ldots, 2n$, each tube $T(l)$ and each annulus $H(l)$ is incompressible in $S^3 - K$.*

Proof. Each noncontractible simple closed curve γ in $H(l)$ is parallel in $H(l)$ to either component of $\partial H(l)$, hence γ bounds a disc $E(\gamma) \subseteq U$ which intersects K in just one point. If such a simple closed curve γ were to bound a disc E^* in $S^3 - U$, $E^* \cap H(l) = \gamma$, then $E(\gamma) \cup E^*$ is a sphere in S^3 which intersects the simple closed curve K in just one point, which is impossible. The above holds as well for each $T(\gamma)$ for the same reasons. □

Lemma 7.2. *For each $i = 1, 2, 3, \ldots, n$, $C(i)$ is incompressible in $S^3 - K$.*

See Remark 4.

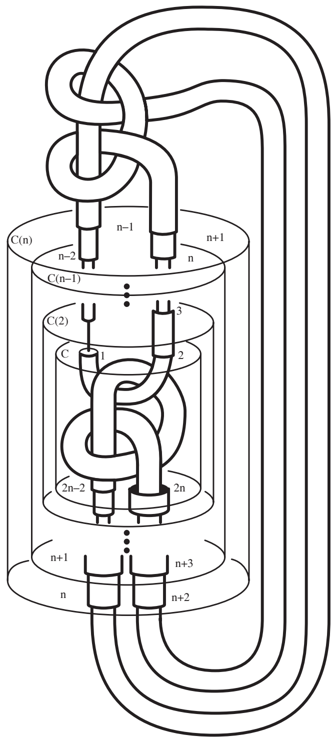


Figure 13

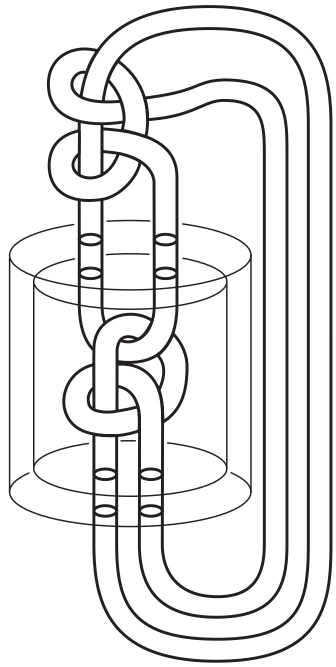


Figure 14

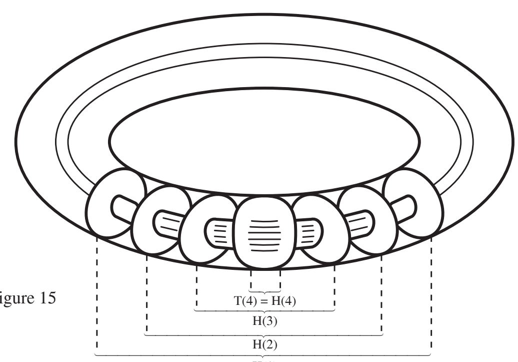


Figure 15

FIGURES 13–15.

Suppose $S(n)$ is compressible, and let E be a compressing disc for $S(n)$. Then $E \cap S(n)$ does not bound a disc on $S(n)$. Assume for each $i = 1, 2, \dots, n$ that $E \cap C(i)$ is minimal and that $E \cap H$ is minimal. If $E \cap H = \varnothing$, then just one of the following can hold:

(a) There is some $l = 1, 2, \dots, 2n$ such that $\partial E \subseteq H(l)$; or

(b) There is some $i = 1, 2, \dots, n$ such that $\partial E \subseteq C(i)$. But (a) and (b) cannot occur because of Lemma 7.1 and Lemma 7.2. Hence, $E \cap H \neq \varnothing$.

From the minimality conditions stated after Remark 7.4 for $E \cap C(i)$ and $E \cap H(l)$ stated above, $H(l) \cap E$ consists of at most disjoint spanning arcs in $H(l)$ and E with boundary in both components of $\partial H(l)$. Call the class of such arcs $\overline{A}$ and suppose that $\overline{A} \neq \varnothing$. Then each arc in $\overline{A}$ separates E into two disjoint subdiscs and among these subdiscs are "outermost discs" which contain no other subdisc. Since $\overline{A} \neq \varnothing$, E must have at least two "outermost discs." If E is an "outermost disc," then $\partial E = \delta \cup \eta$ where δ is a spanning arc in $\overline{A}$ and $\eta \subseteq \partial E$. Neither $D(a)$ nor $D(b)$ can be an "end disc" because each of $\partial D(a)$ and $\partial D(b)$ contains at least three spanning arcs of $\overline{A}$. So there is just one possibility for δ: $\delta \subseteq H(l)$, and η is a subset of one of $C(1), C(2), \dots, C(n)$. But this contradicts: Theorem 2 of [**2**] in the case that $H(l) \subseteq B(2/3, 2/3)$ and, in the case that $H(l) \subseteq B(-1/2, 1/3)$, it contradicts that $\delta \cup \eta$ either forms a trefoil knot or $\delta \cup \eta$ bounds a disc punctured by K. See Figure 5(c). So $\overline{A} = \varnothing$, which contradicts $E \cap H \neq \varnothing$. Thus, $S(n)$ is incompressible in $S^3 - K$.

8. More knots. It should be clear that there are knots other than $K(-1/2, 1/3, 2/3, 2/3)$ to which the previous discussion applies. According to 2.1 all that is really required is that $K = K(-1/2, 1/3, p_3/q_3, \dots p_k/q_k)$, $k \geq 4$, be a knot, that spheres originate from $\partial B(-1/2, 1/3)$,

$$-1/2 + 1/3 = \sum_{k=1}^{3} p_k/q_k \neq 0, \qquad (p_k, q_k) = 1, \quad q_i \geq 3, i \geq 3.$$

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