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SOLVING $(I - S)g = f$ WHEN S IS A GENERALIZED SHIFT OPERATOR

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ABSTRACT. Solutions to the equation $(I-S)g = f$ include Weierstrass functions and fractal interpolation functions of Barnsley. Closure of the range of $I - S$ in C and L^r is characterized when $||S|| = 1$ and solutions g are represented as weak Abel-like limits.

1. Introduction. Solutions to the equation

$$(I - S)g = f \tag{1.1}$$

are studied, where S is a generalized shift operator defined in Section 2. The closures of the ranges of the operators $I - S$ in the spaces C and L^p depend upon parameters in S. They are characterized simply, and it is shown that solutions g can be obtained as Abel limits.

In the case of the ordinary shift operator $S = \Sigma$ defined by $\Sigma f(t) = f(2t)$ Fortet [**3**] stated that if f is a Lip (α), $\alpha > 1/2$, periodic function with period 1 and with $\int_0^1 f(t)\, dt = 0$, then the equation (1.1) has a solution g in L^2 if and only if

$$\frac{1}{n}\int_0^1 \left|\sum_{i=0}^n f(2^i t)\right|^2 dt \to 0$$

as $n \to \infty$. Kac [**5**] proved the theorem and Cieselski [**2**] proved it for all $\alpha > 0$. Rochberg [**6**] studied a more general equation in the context of shift operators on a Hilbert space and showed that Kac's result is an immediate consequence of his results.

When $||S|| < 1$ there is for each right hand side f of (1.1) a unique solution given by the Neumann series

$$g = \sum_{j\geq 0} S^j f. \tag{1.2}$$

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When $||S|| = 1$ solutions may not exist for certain right hand sides since $I - S$ need not be invertible and, when they exist, are not generally obtainable as in (1.2) since the series may no longer converge. It is the latter case, $||S|| = 1$, investigated in this note. The FKC theorem addresses the problem when f is Hölder continuous, solutions are sought in L^2, and $S = \Sigma$. We prove that the norm closure of the set of functions f for which a solution g exists is the set of f's for which $\int_0^1 f(x)\,dx = 0$. Although it is not generally true that the range of the operator $I - \Sigma$ is closed, this entails (see Lemma 3.4) the existence of many more solutions to the equation than those covered by the FKC theorem. It is shown in the general case that when $||S|| = 1$, $0 < a_n < 1$, and f is in the range of $I - S$ the functions

$$g_n = \sum_{j \geq 0} a_n^j S^j f \tag{1.3}$$

are approximate solutions in the sense that $||f - (I - S)g_n|| \to 0$ and the g_n converge weakly to a solution of (1.1) as $a_n \uparrow 1$.

The fractal interpolation functions studied by Barnsley [**1**] can be expressed as solutions of an equation (1.1) on the space $C[0, 1]$ for $||S|| < 1$. Here we extend the notion to what one could call L^r fractal interpolation functions and extend the notion in both L^r and C to the case of $||S|| = 1$.

The functional equation

$$\frac{1}{p} \sum_{i=0}^{p-1} f\left(\frac{x+i}{p}\right) = \lambda f(x) \tag{1.4}$$

has been studied by Artin (see [**4**]) for $\lambda = p^{-1}$ and more generally by Hata [**4**] for $\lambda \neq 1$. A generalization of this equation (see (3.2) and Lemma 3.3 below) arises in consideration of the adjoint of the generalized shift operator S when $\lambda = 1$.

2. The generalized shift operator. Let F be a collection of real valued functions on $(-\infty, \infty)$, periodic with period 1, and having restrictions to $[0, 1]$ which are elements of a normed linear space B of functions. The norm of $f \in F$ is the B-norm of its restriction to $[0, 1]$.

Let p be a positive integer and $d_0, \ldots, d_{p-1}$ be real numbers. Define the generalized shift operator S on F by its restriction to $[0,1]$,

$$Sf(t) = D(t)f(pt),$$

where

$$D(t) = \sum_{i=0}^{p-1} d_i I_i(t) \tag{2.1}$$

and $I_i(t)$ is the indicator of the interval $[i/p, (i+1)/p)$. The usual shift operator Σ is obtained by taking $p = 2$ and $d_0 = d_1 = 1$. It is assumed that S maps F into itself and that S has norm no more than 1.

If B is the space $L^r[0,1]$, $1 \le r < \infty$, then straightforward computation shows

$$||Sf||_r^r = \left(\frac{1}{p}\sum_{i=0}^{p-1} |d_i|^r\right)||f||_r^r.$$

If B is $C[0,1]$, then one must generally have $f(0) = f(1) = 0$ to make S map F into F. Denoting by C the subspace of $C[0,1]$ with $f(0) = f(1) = 0$, one has in this case

$$||S|| = \max_{0 \le i \le p-1} |d_i|.$$

For B the space of normalized functions of bounded variation NBV[0,1]

$$||S|| = \left(\sum_{i=0}^{p-1} |d_i|\right).$$

For example, in the case of $S = a\Sigma$ applying this to the function $f(t) = \sin(2\pi t)$ yields the Weierstrass function $g(t) = \sum_{j\ge 0} a^j \sin(2^{j+1}\pi t)$ and it is seen that if $|a| \in [0,1)$ the series representing g converges uniformly to g, a continuous function, and if $|a| < 1/2$, g is of bounded variation. If $|a| \in [1/2, 1)$, then the series does not converge in NBV. It is well known that g is not of bounded variation on any subinterval in this case, but this does not follow from these arguments.

The usual definitions and notations regarding a normed linear space X and its dual X^* are observed below including weak and weak* convergence. For a subset A of X the set $A^\perp$ is the collection of elements x^* in X^* for which $\langle x^*, x\rangle$ vanishes on A while if A is a subset of X^*, $A^\perp$ is the set of $x \in X$ for which $\langle x^*, x\rangle$ vanishes on A. The range of a linear mapping T is $R(T)$, its null space is $N(T)$, and $\overline{A}$ denotes the norm closure of subset A. Introduce

$$\overline{d} = \frac{1}{p}\sum_{i=0}^{p-1} d_i. \tag{2.2}$$

Lemma 2.1. *If $f \in R(I-S)$, $||S|| = 1$, $|a_n| < 1$, and g_n is given in* (1.3), *then whenever $a_n \uparrow 1$,*

$$||f - (I-S)g_n|| \to 0.$$

Proof. Let $f = (I-S)g$ and $a_n \uparrow 1$. Then

$$||k_n - f|| \le |a_n - 1|\,||S||\,||g||,$$

where $k_n = (I - a_n S)g$, and observing that

$$g = \sum_{j\ge 0} a_n^j S^j k_n$$

yields

$$\begin{aligned} ||f - (I-S)g_n|| &= ||(I-S)(g-g_n)|| \\ &= ||k_n - f + (a_n - 1)S\sum_{j\ge 0} a_n^j S^j (k_n - f)|| \\ &\le ||f - k_n|| + ||a_n - 1||\sum_{j\ge 0} |a_n|^j ||f - k_n|| \\ &\le 2||f - k_n||. \qquad \square \end{aligned}$$

3. The range of $I - S$ in L^r. When $||S||_r < 1$, $R(I-S) = L^r$ and $N(I-S) = \{0\}$, $1 \le r \le \infty$. It is assumed throughout this section that $1 = ||S||$, or equivalently when $1 \le r < \infty$, that

$$1 = \left[\frac{1}{p}\sum_{i=0}^{p-1} |d|_i^r\right]^{1/r}. \tag{3.1}$$

Here $\langle g, f\rangle = \int_0^1 f(t)g(t)\,dt$ for $f \in L^r$, $g \in L^s$ and $1/r + 1/s = 1$.

Theorem 3.1. *If $1 < r < \infty$, (3.1) holds and all $d_i = 1$, then*

$$\overline{R}(I - S) = \{f \in L^r : \langle 1, f\rangle = 0\},$$

$$N(I - S) = \{c1 : -\infty < c < +\infty\},$$

and if $f = (I - S)g$ and $a_n \uparrow 1$ then g_n converges weakly to $g - \langle 1, g\rangle$. If $1 < r < \infty$, (3.1) holds and not all d_i are 1, then

$$\overline{R}(I - S) = L^r,$$
$$N(I - S) = \{0\},$$

and if $f = (I - S)g$ the functions g_n of (1.3) converge weakly to g as $a_n \uparrow 1$.

The proof is accomplished by invoking Lemmas 3.2, 3.5 and 3.7 below.

Lemma 3.2. *With S given by (2.1) as a mapping on $L^r[0, 1]$, $1 < r < \infty$, and under the condition (3.1)*

$$\overline{R}(I - S) = \{f \in L^r : \langle 1, f\rangle = 0\}$$

if all d_i are 1; otherwise, $\overline{R}(I - S) = L^r$.

Proof. Let

$$Af(t) = S^* f(t) = \frac{1}{p}\sum_{i=0}^{p-1} d_i f\left(\frac{t + i}{p}\right) \tag{3.2}$$

and introduce the step-function approximations h_n to $h \in L^s$ defined by

$$h_n(s) = \sum_{i=0}^{p^n-1} \xi(n, i) I_{n,i}(s), \tag{3.3}$$

where $I_{n,i}(s)$ is the indicator function which is one on $(i/p^n, (i+1)/p^n]$ and zero elsewhere and

$$\xi(n,i) = p^n \int h(s) I_{n,i}(s)\, ds.$$

Under Lebesgue measure the h_ns form a martingale, and the martingale convergence theorem shows that h_n converges to h a.e. and in L^s.

Under (3.1), Hölder's inequality shows that (see (2.2)) $|\overline{d}| \leq 1$ with equality if and only if all d's are 1 (or all are -1). There are three possibilities: (a) $d_i \equiv 1$, (b) $d_i \equiv -1$, or (c) $|\overline{d}| < 1$.

(a) Noting that for $m \geq n$,

$$A^m h_n = \langle h, 1 \rangle 1$$

and using $||A|| = 1$ it follows that $A^m h$ converges a.e. and in L^s to the constant $\langle h, 1\rangle 1$. The null space of the operator T^*, where $T = I - S$, is the set of functions h such that $Ah = h$ and since $N(T^*)^{\perp} = \overline{R}(T)$, $f \in \overline{R}(T)$ if and only if $\langle f, h\rangle = 0$ for all constant h. Therefore $f \in \overline{R}(T)$ if and only if $\int_0^1 f(x)\, dx = 0$.

(b) The proof is similar to that in (a) except one examines the two subsequences $m = 2j$ and $m = 2j+1$ to conclude that if h is in L^s then $A^{2j}h$ converges to $\langle h, 1\rangle 1$ while $A^{2j+1}h$ converges to $-\langle h, 1\rangle 1$; so if h is in the null space of $I - A$, $A^m h \to 0$ and $\overline{R}(T) = L^r$.

(c) The proof is similar to (b)'s. In this case observe that if m is a sufficiently large integer then $A^m\ h_n$ is a constant, say c_m. Then $A^{m+1}h_n = \overline{d}c_m$. Since $||A|| = 1$ and $|\overline{d}| < 1$, one has $A^m h$ converging to 0 and $\overline{R}(T) = L^r$. □

Equation (3.2) shows that Hata's equation (1.4) with $\lambda = 1$ is a special case of the equation $(I - S^*)f = 0$ and we have the following.

Lemma 3.3. *If $||S^*|| = 1$ and $1 < s < \infty$, then the set of solutions f in L^s to $(I - S^*)f = 0$ consists precisely of all constants if all d_i are 1 and otherwise the unique solution is $f = 0$.*

That $R(I - S)$ is not generally closed can be seen by applying the FKC theorem to the function $f(t) = \cos(2\pi t)$. For this function,

$\int_0^1 f(t)\,dt = 0$ but there is no solution in L^2 to $Tg = f$. Even so, there are many solutions to the equation (1.1) when $||S|| = 1$ which are not covered by the FKC theorem as can be seen by the fact that the f's of the FKC theorem are a meager set in the closure of the range of T. We state this as follows.

Lemma 3.4. *The set of solutions E covered by the* FKC *theorem is a set of first category in $\overline{R}(I - S)$.*

Lemma 3.5. *If $1 \le r < \infty$, then the solutions to* (1.1) *are unique except in the case all d_i are one and then the solutions are unique up to an additive constant.*

Proof. There are the three cases (a) $d_i \equiv 1$, (b) $d_i \equiv -1$ or (c) $|\overline{d}| < 1$. Let h be in $N(T) \subset L^r$. Then for all $g \in L^s$ one has $\langle (I - S)h, g\rangle = 0$. Therefore, for all g,

$$\langle h, g\rangle = \langle h, S^* g\rangle = \langle h, Ag\rangle = \langle h, A^2 g\rangle = \cdots .$$

In case (a), by continuity of the inner product and what has been shown above it follows that if h is in the null space of T, then

$$\langle h, A^n g\rangle \to \langle h, 1\rangle\langle g, 1\rangle$$

so that for all g,

$$\langle h, g\rangle = \langle h, 1\rangle\langle g, 1\rangle.$$

Therefore, if $g \in V = \{f \in L^s : \langle f, 1\rangle = 0\}$ one has $\langle g, 1\rangle = 0$ and hence $\langle h, g\rangle = 0$. This shows that $N(T) \subset V^{\perp}$. Since $V^{\perp}$ consists of the constants and these are clearly in $N(T)$ it follows that the null space consists of the constants.

In case (b), taking limits twice, once for $n = 2j$ and once for $n = 2j + 1$, it follows that if $h \in N(T)$, then for all $g \in L^r$, $\langle h, g\rangle = 0$. So $N(T) = \{0\}$.

The argument in case (c) also shows that if $h \in N(T)$ then $\langle h, g\rangle = 0$ for all g since it has previously been shown that $||A^m h|| \to 0$. So $N(T) = \{0\}$. □

Lemma 3.6. *Assume $f = (I - S)g$ and $1 < r < \infty$. If not all d_i are 1, then* (i) *and* (iii) *are equivalent. If all d_i are 1, then* (ii) *and* (iii) *are equivalent.*

(i) *The approximate solutions g_n of* (1.3) *converge weakly to the unique solution g of equation* (1.1).

(ii) *Every subsequence $g_{n'}$ has a further subsequence $g_{n''}$ and there is a constant c such that $g_{n''}$ converges weakly to $g + c$.*

(iii) $||g_n||$ *is a bounded sequence.*

Proof. Assume that not all d_is are 1. It is proven that if (iii) holds, then for every $q \in L^s$, $\langle q, g - g_n\rangle \to 0$. Let $\varepsilon > 0$ be given. Since in the present case under consideration $\overline{R}(T^*) = N(T)^\perp = L^s$, let $v \in R(T^*)$ be such that $||v - q|| < \varepsilon/M$, where

$$||g|| + \sup ||g_n|| \le M \tag{3.4}$$

and

$$|\langle q, g - g_n\rangle - \langle v, g - g_n\rangle| \le \varepsilon \tag{3.5}$$

for all n. It suffices to prove that $\langle v, g - g_n\rangle \to 0$; but this follows immediately from Lemma 2.1 and

$$\begin{aligned} |\langle v, g - g_n\rangle| &= |\langle T^* w, g - g_n\rangle| = |\langle w, T(g - g_n)\rangle| \\ &\le ||w||\, ||f - (I - S)g_n||. \end{aligned}$$

Still assuming that not all d_i are 1, suppose that (i) holds. Then $||g - g_n||$ is a bounded sequence and $||g_n|| \le ||g - g_n|| + ||g||$.

Now suppose that all d_i are 1. First, assume that (iii) holds. We require the following fact when $||g_n||$ is a bounded sequence. A point x in a normed linear space X is in the closed subspace F if and only if $\langle x^*, x\rangle = 0$ for all points $x^* \in F^\perp$. Let $q \in L^s$ be such that $\langle q, 1\rangle = 0$. Then q is in the closure of the range of T^*. Letting $\varepsilon > 0$ be arbitrary and $v = T^* w$ be such that $||q - v|| < \varepsilon/M$, where M is chosen as in (3.4), observe that (3.5) holds and that to prove $\langle q, g - g_n\rangle \to 0$ it suffices to prove that $\langle v, g - g_n\rangle \to 0$. This is true by the same argument as above. Now since $||g - g_n||$ is a bounded sequence, for every sequence n' there is a weakly convergent subsequence $g - g_{n''}$ and an element k

to which it weakly converges. We conclude that $\langle v, k\rangle = 0$ for all v in $\{q \in L^s : \langle q, 1\rangle = 0\}$. Therefore, k is a constant. Still assuming that all d_i are 1, suppose now that (ii) holds. Then, for every subsequence n', there is a subsequence $||g - g_{n''}||$ which is bounded so (iii) holds. □

Lemma 3.7. *Assume* $f = (I - S)g$ *and* $1 < r < \infty$. *If not all* d_i *are* 1, *then the* g_n *in* (1.3) *converge weakly to the unique solution* g *of* (1.1). *If all* d_i *are* 1, *then* g_n *converges weakly to* $g - \langle 1, g\rangle$.

Proof. Observing that

$$||g_n|| \le ||g_n - g|| + ||g||$$

where

$$\begin{aligned} ||g_n - g|| &= \left\| \sum_{j\ge 0} a_n^j S^j (k_n - f)\right\| \le \frac{1}{1-a_n} ||f - k_n|| \\ &\le \frac{1}{1-a_n} |a_n - 1|\, ||S||\, ||g|| \end{aligned} \tag{3.6}$$

shows the norms of the g_n remain bounded. If not all d_i are 1 the claim is immediate from Lemma 3.6.

If all d_i are 1 we argue as follows. If g_n does not converge weakly to $g - \langle 1, g\rangle 1$, then there is a $v \in L^s$ and a subsequence n' such that

$$\langle v, g_{n'}\rangle \to \lambda > \langle v, g - \langle 1, g\rangle 1\rangle = \langle v, g\rangle - \langle 1, g\rangle\langle 1, v\rangle.$$

Noting that $\langle 1, g_n\rangle = 1/(1 - \overline{d}a_n)\langle 1, f\rangle = 0$, one also has $\langle v - \langle 1, v\rangle 1, g_{n'}\rangle \to \lambda$. By Lemma 3.6 there is a subsequence $g_{n''}$ and a constant c for which

$$\langle v - \langle 1, v\rangle 1, g_{n''}\rangle \to \langle v - \langle 1, v\rangle 1, g + c\rangle = \langle v, g\rangle - \langle 1, v\rangle\langle 1, g\rangle.$$

This contradiction establishes the result. □

4. The range of $I - S$ in C. Assume that the mapping S is defined on periodic functions f of period 1 whose restriction to $[0, 1]$ is in $C[0, 1]$ and satisfies $f(0) = f(1) = 0$. Our interest centers on the case $||S|| = 1$.

Barnsley [**1**] defines a fractal interpolation function to the initial data $(i/p, y_i)$, $i = 0, \dots, p$, as follows. Define the p mappings $w_i : [0,1] \times R^1 \to [i/p, (i+1)/p] \times R^1$ for $i = 0, \dots, p-1$ by

$$w_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix},$$

where $e_i = i/p$ and $a_i = 1/p$ and the remaining constants, except the d_i which are free parameters, are chosen to satisfy

$$w_i \begin{pmatrix} 0 \\ y_0 \end{pmatrix} = \begin{pmatrix} i/p \\ y_i \end{pmatrix}, \qquad w_i \begin{pmatrix} 1 \\ y_p \end{pmatrix} = \begin{pmatrix} (i+1)/p \\ y_{i+1} \end{pmatrix}$$

for $i = 0, \dots, p-1$. If $W : [0,1] \times R^1 \to [0,1] \times R^1$ is defined by

$$W(A) = \bigcup_{i=0}^{p-1} w_i(A)$$

and all $|d_i| < 1$, then there is a fixed set G of the transformation W which is shown to be the graph of a continuous fucntion g on $[0,1]$. The function g is called the fractal interpolation function to the initial data and it can be shown that for all bounded A, $W^n(A) \to G$ as $n \to \infty$. Consider $W([0,1] \times \{0\})$. This is the graph of a piecewise linear function f which satisfies $f(i/p) = y_i$ for $i = 0, \dots, p$. One can check that if $y_0 = 0 = y_p$ and the same d_i's are used in the operator S, then for all $n \geq 1$,

$$\sum_{i=0}^{n-1} S^i f = W^n([0,1] \times \{0\}).$$

In Barnsley's treatment of fractal interpolation functions the condition $|d_i| < 1$ is made and for this case, of course, it is immediate that the left hand side converges to a continuous function g and g satisfies $(I - S)g = f$. We seek solutions g in C to the equation (1.1) where $f \in C$ need not be piecewise linear and g need not be of the form $g = \sum_{i \geq 0} S^i f$.

Lemma 4.1. *The adjoint S^* of the mapping $S : C \to C$ is defined on the space BV of signed measures ν of bounded variation by*

$$S^* \nu([0,w]) = \sum_{i=0}^{p-1} d_i \nu\left(\left(\frac{i}{p}, \frac{w+i}{p}\right]\right).$$

Proof. Consider the action of the linear functional corresponding to the signed measure ν on the function Sf, where $f \in C$,

$$\begin{aligned}\langle \nu, Sf\rangle &= \int I_{[0,1]}(t) Sf(t)\, d\nu(t)\\ &= \int \Big(\sum_{i=0}^{p-1} d_i I_{I_i}(t)\Big) f(pt)\, d\nu(t)\\ &= d_0 \int f(w) I_{[0,1]}(w)\, d\nu\Big(\frac{w}{p}\Big)\\ &\quad + \sum_{i=1}^{p-1} d_i \int f(w) I_{(i,i+1]}(w)\, d\nu\Big(\frac{w}{p}\Big).\end{aligned}$$

Therefore, by the periodicity of f and $f(0) = 0$,

$$\langle \nu, Sf\rangle = \int f(w) I_{(0,1]}(w) d\Big(\sum_{i=0}^{p-1} d_i \nu\Big(\frac{w+i}{p}\Big)\Big) = \langle S^*\nu, f\rangle. \qquad \square$$

We need to characterize the collection of signed measures ν in BV which solve the equation $S^*\nu = \nu$. Toward that end we introduce the notion of a real valued stationary stochastic process on $\mathbf{Z} = \{1, 2, \dots\}$. The stochastic process $\{X(t) : t \in \mathbf{Z}\}$ is said to be stationary if for every $k < \infty$, $(t_1, t_2, \dots, t_k) \in \mathbf{Z}^k$, and Borel measurable subset A in R^k one has for all positive integers v

$$P[(X(t_1), \dots, X(t_k)) \in A] = P[(X(t_1 + v), \dots, X(t_k + v)) \in A].$$

A stochastic process $\{X(t) : t \in \mathbf{Z}\}$ taking values in $\{0, 1, \dots, p-1\}$ defines a random variable X taking values in $[0,1]$ by $X(\omega) = \sum_{i\geq 1} X(\omega, i) p^{-i}$ and conversely, once it is decided which representation is to be used, terminating or nonterminating, any Borel random variable X defines a $\{0, 1, \dots, p-1\}$-valued process $\{X(t) : t \in \mathbf{Z}\}$.

Lemma 4.2. *If $d_i \geq 0$ for all i, then a finite nonnull Borel measure μ on $[0,1]$ solves $S^*\mu = \mu$ if and only if*

(i) $\mu(\{0\}) = 0$ *and when*

$$X(\omega) = \sum_{i\geq 1} X(\omega, i)p^{-i}$$

has the probability distribution $\mu(A)/\mu([0,1]) = P[X \in A]$ *then for all indices* i *such that* $d_i < 1$ *one has for all* j

(ii) $P[X(j) = i] = 0$ *and*

(iii) $\{X(k) : k \in \mathbf{Z}\}$ *is a stationary stochastic process.*

Proof. First assume that μ is a nonnull solution. If a finite nonnull Borel measure μ on $[0,1]$ solves $S^*\mu = \mu$, then the probability measure $\eta(A) = \mu(A)/\mu([0,1])$ also solves the equation. Therefore it can be assumed without loss of generality that the solution μ is a probability measure P. That $\mu(\{0\}) = 0$ follows from the representation formula for S^*. One has

$$S^*\mu([0,1]) = S^*\mu((0,1]) = \sum_{i\in J}\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg) + \sum_{i\notin J} d_i\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg),$$

where $J = \{k : k \in \{0,\dots,p-1\}$ and $d_k = 1\}$. By additivity of μ it follows that if J^c is not empty then

$$S^*\mu((0,1]) < \sum_{i\in J}\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg) + \sum_{i\notin J}\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg) = \mu((0,1])$$

so that $S^*\mu((0,1]) = \gamma\mu((0,1])$, where $|\gamma| < 1$. Iterating S^*, this implies $\mu((0,1]) = 0$ which is impossible under our assumptions. Therefore, if $i \notin J$, then $\mu((i/p,(i+1)/p]) = 0$. For $i \notin J$, consider

$$\begin{aligned}\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg) &= S^*\mu\bigg(\bigg(\frac{i}{p}, \frac{i+1}{p}\bigg]\bigg) = \sum_{j\in J}\mu\bigg(\bigg(\frac{j}{p}+\frac{i}{p^2}, \frac{j}{p}+\frac{i+1}{p^2}\bigg]\bigg)\\ &\quad + \sum_{j\notin J} d_j\mu\bigg(\bigg(\frac{j}{p}+\frac{i}{p^2}, \frac{j}{p}+\frac{i+1}{p^2}\bigg]\bigg)\\ &= \sum_{j\in J}\mu\bigg(\bigg(\frac{j}{p}+\frac{i}{p^2}, \frac{j}{p}+\frac{i+1}{p^2}\bigg]\bigg).\end{aligned}$$

Since $0 = \mu((i/p, (i+1)/p]) = \sum_{j \in J} \mu((j/p + i/p^2, j/p + (i+1)/p^2])$ it follows that for all $j \in \{0, \ldots, p-1\}$ $\mu((j/p+i/p^2, j/p+(i+1)/p^2]) = 0$. The argument can be repeated showing that if $i \notin J$, then for all $j \geq 1$ and all $x = \sum_{i=1}^{j-1} x_i p^{-i}$, $\mu((x + ip^{-j}, x + (i+1)p^{-j}]) = 0$. In terms of the probability measure this means that for all indices i such that $d_i < 1$ one has for all j, $P[X(j) = i] = 0$.

To see that the measure μ must correspond to a stationary process, consider

$$\begin{aligned} &P(\{X : X(i_1) = k_1, \ldots, X(i_m) = k_m\}) \\ &\quad = \sum_{i \in J} P(\{X : X(1) = i, X(i_1 + 1) = k_1, \ldots, X(i_m + 1) = k_m\}) \\ &\quad = P(\{X : X(i_1 + 1) = k_1, \ldots, X(i_m + 1) = k_m\}). \end{aligned}$$

The converse follows by the same arguments. □

For any S, denote the collection of probability measures which solve $S^*\mu = \mu$ by M_S.

Lemma 4.3. *Under the conditions of Lemma* 4.2 *the solutions* $\nu \in BV(0,1]$ *to* $S^*\nu = \nu$ *consist of all signed measures of the form*

$$\nu = c_1\mu_1 - c_2\mu_2, \tag{4.1}$$

where the μ*'s are in* M_S *and the* c*'s are nonnegative real numbers.*

Proof. Clearly all signed measures in (4.1) solve $S^*\nu = \nu$ since S^* is linear. Now let ν be a solution. By the Hahn decomposition there is a Borel set D such that $\nu^+(A) = \nu(A \cap D)$ and $\nu^-(A) = -\nu(A \cap D^c)$ for all measurable sets A. It follows that on D, ν is a measure satisfying $S^*\nu = \nu$ and on D^c the measure $-\nu$ solves $S^*(-\nu) = -\nu$. With the obvious adjustments in case $\nu(D)$ or $\nu(D^c)$ zero, let $\mu_1(A) = \nu(A \cap D)/\nu(D)$, $\mu_2(A) = \nu(A \cap D^c)/\nu(D^c)$, $c_1 = \nu(D)$, and $c_2 = -\nu(D^c)$. □

Theorem 4.4. *Let* $||S|| = 1$ *and* $d_i \geq 0$ *for all* i. *Then* $\overline{R}(I - S) = \{f \in C : E[f(X)] = 0$ *for all* $X \in \mathbf{X}$, *where*

$$X(\omega) = \sum_{i \geq 1} X(\omega, i) p^{-i} \in \mathbf{X}$$

if for all indices i such that $d_i < 1$ and for all j

(i) *$P[X(j) = i] = 0$ and*

(ii) *$\{X(k) : k \in \mathbf{Z}\}$ is a stationary stochastic process.*

The null space is $N(T) = \{0\}$.

Proof. The closure of the range of T is the orthogonal complement of the null space of $I - S^*$ so $f \in \overline{R}(T)$ if and only if $\int f\, d\nu = 0$ for all ν in $BV[0,1]$ solving $S^*\nu = \nu$. Therefore, $f \in \overline{R}(T)$ if and only if $\int f\, d\mu = 0$ for any measure $\mu([0,t]) = P[X \le t]$, where X satisfies (i) and (ii).

To prove the claim regarding $N(T)$, let $v, x \in (0,1)$ and $f \in N(T)$ be arbitrary. Then upon iterating the relation

$$f\left(\frac{x+i}{p}\right) = d_i f(x+i) = d_i f(x)$$

for points $v_k = \sum_{i=1}^k v(i)p^{-i}$ it follows that

$$f(v_k + xp^{-k}) = \prod_{i=1}^{k} d_{v(i)} f(x).$$

Using $v = \sum_{i\ge 1} v(i)p^{-i}$, $xp^{-k} \to 0$, $v_k \to v$, and the continuity of f shows

$$f(v) = f(x) \limsup_{k\to\infty} \prod_{i=1}^{k} d_{v(i)} \le f(x) \liminf_{k\to\infty} \prod_{i=1}^{k} d_{v(i)} = f(v).$$

Taking $x \to 0$ shows $f(v) = 0$. □

Theorem 4.5. *If $||S|| = 1$ and $f \in R(T)$, then the approximate solutions of* (1.3) *converge in the weak* topology of L^∞ to the unique solution g of* (1.1) *unless all d_i's are one, in which case the g_n converge to $g - \langle 1, g\rangle$.*

Proof. Let $b \in L^1$ be such that $\langle b, 1\rangle = 0$. Then b must be in $\overline{R}(T^*)$. This can be seen as follows. If b is not in $\overline{R}(T^*)$, then there is a function

$u \in L^\infty$ such that $\langle u, b\rangle > 0$ and for all $v \in L^1$, $\langle u, T^* v\rangle = 0$. Hence, for all $v \in L^1$, $\langle Tu, v\rangle = 0$ and this entails $Tu = 0$. Now this must mean that u is constant a.e. as is seen by using Lusin's theorem and a simple adaptation of the proof in Theorem 4.4 that $N(T) = \{0\}$. But then $0 < \langle u, b\rangle = c\langle 1, b\rangle = 0$, which is a contradiction; so $b \in \overline{R}(T^*)$.

By the boundedness of the sequence $||g_n - g||$ (see (3.6)) and the fact that for $b = T^* w$, $w \in L^1$, one has

$$|\langle b, g_n - g\rangle| = |\langle T^* w, g_n - g\rangle| = |\langle w, Tg_n - f\rangle| \leq ||w||\,||f - Tg_n|| \to 0,$$

it must be that for all $b \in L^1$ such that $\langle b, 1\rangle = 0$ the sequence $\langle b, g_n - g\rangle$ converges to zero.

Now suppose $b \in L^1$ is arbitrary. We show that $\langle b, g_n - g\rangle \to 0$ when not all d_i are 1. If not all d_i are 1, then for n sufficiently large

$$\langle 1, g_n\rangle = \sum_{j\geq 0} (\overline{d} a_n)^j \int f = \frac{1}{1 - \overline{d} a_n} \int f,$$

while $\int f = \int (I - S)g = \int g - \overline{d} \int g$ shows that $\langle 1, g_n\rangle \to \langle 1, g\rangle$. Consequently, if $b \in L^1$ is arbitrary and $b' = b - \langle 1, b\rangle$, then $0 = \lim\langle b', g_n - g\rangle = \lim\langle b, g_n - g\rangle - 0$.

Suppose $b \in L^1$ is arbitrary and all $d_i = 1$. If all $d_i = 1$, then $\int f = \int (I - S)g = \int g - \int g = 0$. Therefore, $\langle g_{n'} 1\rangle \equiv 0$ and for all $b \in L^1$,

$$\lim\langle b, g_n - (g - \langle 1, g\rangle 1)\rangle = 0. \qquad \square$$

When all d_i's are 1 the assertion of Theorem 4.5 leaves open the question of whether the approximate solutions g_n may converge to a function which is not in C. A simple example shows that they can; let $f(t) = g(t) - g(2t)$ with $g(t) = 2tI_{[0,.5]}(t) + (2 - 2t)I_{(.5,1]}(t)$ so that g_n converges to $g - 1/2$.

REFERENCES

1. M.F. Barnsley, *Fractal functions and interpolation*, Constr. Approx. **2** (1986), 303–329.

2. Z. Cieselski, *On the functional equation* $f(t) = g(t) - g(2t)$, Proc. Amer. Math. Soc. **13** (1962), 388–392.

3. R. Fortet, *Sur une suite egalment repartie*, Studia Math. **9** (1940), 54–69.

4. M. Hata, *On the functional equation* $(1/p)\{f(x/p) + \cdots + f((x+p-1)/p))\} = \lambda f(x)$, J. Math. Kyoto Univ. **25** (1985), 357–364.

5. M. Kac, *On the distribution of sums of the type* $\sum f(2^k t)$, Ann. Math. **47** (1946), 33–49.

6. R. Rochberg, *The equation* $(I - S)g = f$ *for shift operators in Hilbert space*, Proc. Amer. Math. Soc. **19** (1968), 123–129.

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