ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 24, Number 2, Spring 1994

# ON GENERALIZED HOMOGENEITY OF CURVES OF A CONSTANT ORDER

JANUSZ J. CHARATONIK

ABSTRACT. Locally connected plane curves with the property that all their points are of the same Menger-Urysohn order are considered. The problem of their generalized homogeneity is discussed with respect to various classes of mappings.

All spaces considered in this paper are assumed to be metric, and all mappings are continuous. By a *continuum* we mean a compact connected space. A *curve* means a 1-dimensional continuum.

The concept of the *order* of a point $p$ in a space $X$, denoted by $\operatorname{ord}_p X$, is used in the sense of Menger-Urysohn [**22**, p. 48; **11**, Section 51, I, p. 274]. Roughly speaking, for a point $p$ of a space $X$ we write $\operatorname{ord}_p X = \mathfrak{n}$, provided $\mathfrak{n}$ is the minimal cardinal number such that there is a local base at $p$ whose elements have boundaries of cardinality $\mathfrak{n}$. If a point $p$ has a local base with finite boundaries, and if the supremum of the cardinalities of the boundaries of elements of the base tends to infinity if their diameters tend to zero, we write $\operatorname{ord}_p X = \omega$.

A surjective mapping $f : X \to Y$ between spaces $X$ and $Y$ is said to be:

—a *local homeomorphism in the large sense* provided that for every point $x \in X$ there is an open neighborhood $U$ such that the partial mapping $f|U : U \to f(U)$ is a homeomorphism;

—*monotone* provided that for each point $y \in Y$ the set $f^{-1}(y)$ is connected;

—*open* if images of open sets under $f$ are open;

Received by the editors on February 29, 1991, and in revised form on July 25, 1992.
*Key words and phrases.* Confluent, continuum, curve, homogeneous, light, local homeomorphism, locally connected, monotone, open, order, plane, simple closed curve, universal plane curve.
1991 *Mathematical Subject Classification.* 54C10, 54F15, 54F50.

—*confluent* provided that for each subcontinuum $Q$ in $Y$ each component of $f^{-1}(Q)$ is mapped onto $Q$ under $f$.

A topological space $X$ is said to be *homogeneous* provided that for every two points $p, q \in X$, there is a homeomorphism $f$ of $X$ onto itself such that $f(p) = q$. This notion was introduced by W. Sierpiński in 1920 [**20**, p. 16]. In the same journal, viz. in the first volume of Fundamental Mathematicae, B. Knaster and K. Kuratowski asked if the simple closed curve is the only plane homogeneous continuum. A partial answer was given four years later by S. Mazurkiewicz [**13**, p. 137] under an additional assumption of local connectedness of the space.

**Theorem 1** (Mazurkiewicz). *Each nondegenerate locally connected plane homogeneous continuum is a simple closed curve.*

Local connectedness is essential in this result because the pseudo-arc is a plane homogeneous continuum which is not locally connected [**3**, **15**], and also planability is an indispensable condition because the Menger universal curve is a locally connected continuum which is not embeddable in the plane [**1**, **2**].

It is observed below that the two assumptions, i.e., local connectedness and planability of the continuum can be replaced in the result by only one condition, namely saying that the continuum contains a point of order 2, and that homogeneity can then be relaxed to homogeneity with respect to some larger classes of mappings. To formulate this rigorously, we recall some definitions.

A class $\mathcal{M}$ of mappings is said to be *admissible* if it contains all homeomorphisms and if the composition of any two mappings in $\mathcal{M}$ is also in $\mathcal{M}$. A space $X$ is said to be *homogeneous with respect to a class* $\mathcal{M}$ of mappings of $X$ onto itself provided that, for every two points $p, q \in X$, there is a mapping $f \in \mathcal{M}$ such that $f(p) = q$.

We say that a surjective mapping $f : X \to Y$ *does not decrease* (*increase*) *order of points* provided that for each point $x$ of $X$ the inequality $\mathrm{ord}_x X \leq \mathrm{ord}_{f(x)} Y$ (respectively, $\mathrm{ord}_x X \geq \mathrm{ord}_{f(x)} Y$) is satisfied.

**Proposition 2.** *If a space $X$ is homogeneous with respect to a class* *of mappings that does not either decrease or increase order of points, then all points of $X$ are of the same order.*

*Proof.* Indeed, assume $X$ is homogeneous with respect to a class of mappings that does not decrease order of points. If there are two distinct points $x$ and $y$ in $X$ of different orders, say $\operatorname{ord}_x X < \operatorname{ord}_y X$, then taking a mapping $f$ from the considered class for which $f(x) = y$ we get a contradiction. For the other class of mappings we take a mapping $g$ with $g(y) = x$ and argue in a similar way. □

Conversely, the following observation is evident.

*Observation* 3. If all points of a space $X$ are of the same order, then any mapping of $X$ onto itself does not decrease and does not increase order of points.

Let us recall that if all points of a continuum $X$ are of a finite order, and if all are of order at least $n$, where $n$ is a positive integer, then there exists in $X$ a point of order at least $2n - 2$ ([**21**, Chapter VI, Section 2, the Fundamental Theorem, p. 105]; compare [**4**, Theorem 13.19, p. 473]. It implies that if all points of a continuum $X$ have the same finite order $n > 0$, then $n = 2$.

As a consequence we conclude with the following result due to P.S. Urysohn (see [**21**, Chapter VI, Section 2, p. 105]).

**Proposition 4** (Urysohn). *If all points of a continuum are of the same order* $\mathfrak{n}$*, then this order can take only four values, namely,* $\mathfrak{n} \in \{2, \omega, \aleph_0, c\}$.

In the light of Proposition 4 it is natural to ask about existence and topological properties of continua, all points of which are of the same order. It is widely known that if all points of a continuum $X$ are of the same order $\mathfrak{n} = 2$, then $X$ is the simple closed curve (see, e.g., [**11**, Section 51, V, Theorem 9, p. 298]). With regard to $\mathfrak{n} = \omega$ and $\mathfrak{n} = \aleph_0$, Urysohn constructs (see [**21**, Chapter VI, Sections 6-8 and 9-10, pp. 109–115], examples of curves whose points are all of the same order $\omega$ and $\aleph_0$ (for order $\omega$ see also [**14**, Chapter VIII, Section 5, p. 279].

However, his description of the examples is rather complicated and cumbersome. So, we give here another construction of two examples with the same basic attributes that all their points are of the same order $\aleph_0$ or $\omega$ and with some additional properties. But it should be stressed that the idea of this construction is due to P.S. Urysohn.

It will be more convenient to us to start with $\mathfrak{n} = \aleph_0$ and to consider the case of $\mathfrak{n} = \omega$ later.

*Example* 5. There exists a plane locally connected curve $X(\aleph_0)$ such that each point of $X(\aleph_0)$ is of order $\aleph_0$.

*Proof.* Let $\mathbf{R}^2$ denote the Euclidean plane. By a disc we mean a homeomorphic image of the closed unit square. The set of all nonnegative integers is denoted by $\mathbf{N}$. Let a fixed disc $D_0$ in the plane $\mathbf{R}^2$ be given. Assume that for some $k \in \mathbf{N}$ a disc $D_k$ is defined. We perform the following construction. Consider a chain of discs $B_{k,n}$ such that for each $n \in \mathbf{N}$ we have

(6) $B_{k,n} \subset D_k$;

(7) $B_{k,n} \subset \operatorname{int} D_k$ if $n > 0$;

(8) $\operatorname{bd} B_{k,0} \cap \operatorname{bd} D_k = \{p_{k,0}\}$ for some point $p_{k,0}$;

(9) $B_{k,n} \cap B_{k,n+1} = \{p_{k,n+1}\}$ for some point $p_{k,n+1}$;

(10) the topological limit of the sequence of discs $B_{k,n}$ for the fixed $k$ and $n$ tending to infinity is a singleton $\{q_k\} \subset \operatorname{bd} D_k \backslash \{p_{k,0}\}$.

It follows from (9) and (10) that if $k$ is fixed and $n \to \infty$, then

$$\lim \operatorname{diam} B_{k,n} = 0 \quad \text{and} \quad \lim p_{k,n} = q_k \neq p_{k,0}. \tag{11}$$

Then $D_k \backslash \cup \{\operatorname{int} B_{k,n} : n \in \mathbf{N}\}$ is the union of two discs which are labelled as $D_{2k+1}$ and $D_{2k+2}$, and we see that

$$D_{2k+1} \cap D_{2k+2} = \{q_k\} \cup \{p_{k,n} : n \in \mathbf{N}\} \tag{12}$$

is a closed countable set with exactly one limit point, $q_k$, where $q_k \in \operatorname{bd} D_k$. Put

$$A_{2k+1} = \operatorname{bd} D_k \cap \operatorname{bd} D_{2k+1} \quad \text{and} \quad A_{2k+2} = \operatorname{bd} D_k \cap \operatorname{bd} D_{2k+2}.$$

Hence $A_{2k+1}$ and $A_{2k+2}$ are two arcs with end points $p_{k,0}$ and $q_k$ such that

$$A_{2k+1} \cap A_{2k+2} = \{p_{k,0}, q_k\} \quad \text{and} \quad A_{2k+1} \cup A_{2k+2} = \operatorname{bd} D_k.$$

Moreover,

$$(A_{2k+1} \cup A_{2k+2}) \cap (D_{2k+1} \cap D_{2k+2}) = \{p_{k,0}, q_k\}.$$

The above inductive procedure can be done in such a way that:

(13) if we take in $D_k$ a chain of discs $B_{k,n}$ for $n \in \mathbf{N}$ such that $D_k \setminus \cup \{\operatorname{int} B_{k,n} : n \in \mathbf{N}\} = D_{2k+1} \cup D_{2k+2}$ with (12), then $q_{2k+1} \in A_{2k+1}$ and $q_{2k+2} \in A_{2k+2}$;

(14) $\lim \operatorname{diam} D_k = 0$ if $k \to \infty$;

(15) $\cup\{\operatorname{int} B_{k,n} : k, n \in \mathbf{N}\}$ is a dense subset of $D_0$;

(16) for distinct indices $k$ the sets $D_{2k+1} \cap D_{2k+2}$ are pairwise disjoint.

Let $U_m$ be the union of $2^m$ discs $D_k$, where $k$ runs over nonnegative integers from $2^m - 1$ to $2^{m+1} - 2$. Note that

$$(17) \qquad U_m = D_0 \setminus \cup \{\cup\{\operatorname{int} B_{k,n} : n \in \mathbf{N}\} : k \in \{0, \dots, 2^{m+1} - 2\}\};$$

hence it follows that each $U_m$ is connected, thus a continuum. Furthermore, (17) implies that $U_{m+1} \subset U_m$ for each $m \in \mathbf{N}$.

Therefore, putting

$$(18) \qquad X(\aleph_0) = D_0 \setminus \cup \{\operatorname{int} B_{k,n} : k, n \in \mathbf{N}\} = \cap\{U_m : m \in \mathbf{N}\}$$

we see that $X(\aleph_0)$ is a continuum as the intersection of a decreasing sequence of continua (see, e.g., [**11**, Section 47, II, Theorem 5, p. 170]). Condition (15) implies that $X(\aleph_0)$ is a nowhere dense subset of $D_0$, and hence it is 1-dimensional, i.e., $X(\aleph_0)$ is a curve.

It follows from the above construction that

$$(19) \qquad D_k \cap X(\aleph_0) \quad \text{is connected for each } k \in \mathbf{N},$$

and that

(20) the boundary of $D_k \cap X(\aleph_0)$ with respect to $X(\aleph_0)$ consists of countably many points of the form $q_i$ and $p_{j,n}$ for some $i, j, n \in \mathbf{N}$.

Now we conclude from condition (14) that

(21) for each $\varepsilon > 0$ there is an $m \in \mathbf{N}$ such that for every $k \in \{2^m - 1, \dots, 2^{m+1} - 2\}$ we have $\operatorname{diam} D_k < \varepsilon$.

Since

$$X(\aleph_0) = \cup\{D_k \cap X(\aleph_0) : k \in \{2^m - 1, \dots, 2^{m+1} - 2\}\}, \tag{22}$$

we see by (19) that for each $\varepsilon > 0$ the continuum $X(\aleph_0)$ is the union of finitely many subcontinua of diameters less than $\varepsilon$, from which local connectedness of $X(\aleph_0)$ follows by the Sierpiński characterization (see [**11**, Section 50, II, Theorem 2, p. 256]).

Finally, observe that no point of $X(\aleph_0)$ has a local base consisting of sets with finite boundaries, since no finite set separates $X(\aleph_0)$. Further, conditions (19)–(22) imply that interiors of subcontinua of $X(\aleph_0)$ having the form $D_k \cap X(\aleph_0)$ form an open basis with countable boundaries for $X(\aleph_0)$. Hence it follows that $\operatorname{ord}_p X(\aleph_0) = \aleph_0$ for each $p \in X(\aleph_0)$. The proof is complete. □

*Example* 23. There exists a plane locally connected curve $X(\omega)$ such that each point of $X(\omega)$ is of order $\omega$.

*Proof.* The present construction is a modification of the previous one, and thus we keep notation from the proof of Example 5. In each disc $D_k$ instead of an infinite sequence of discs $B_{k,n}$ with $n \in \mathbf{N}$ we take a finite one, with $n \in \{0, \dots, n_k\}$, in such a way that the following conditions are satisfied:

(24) $B_{k,n} \subset D_k$ for each $n \in \{0, \dots, n_k\}$;

(25) $B_{k,n} \subset \operatorname{int} D_k$ if $0 \neq n \neq n_k$;

(26) $\operatorname{bd} B_{k,0} \cap \operatorname{bd} D_k = \{p_{k,0}\}$ for some point $p_{k,0}$;

(27) if $n_k > 0$ and $n \in \{1, \dots, n_k - 1\}$, then $B_{k,n} \cap B_{k,n+1} = \{p_{k,n+1}\}$ for some point $p_{k,n+1}$;

(28) $\operatorname{bd} B_{k,n_k} \cap \operatorname{bd} D_k = \{q_k\}$ for some point $q_k \neq p_{k,0}$;

(29) $\lim n_k = \infty$ if $k \to \infty$.

Then $D_k \setminus \cup \{\operatorname{int} B_{k,n} : n \in \{0, \ldots, n_k\}\}$ is the union of two discs which are labelled as $D_{2k+1}$ and $D_{2k+2}$, and we see that

$$D_{2k+1} \cap D_{2k+2} = \{q_k\} \cup \{p_{k,n} : n \in \{0, \ldots, n_k\}\} \tag{30}$$

is finite. Let $A_{2k+1}$ and $A_{2k+2}$ have the same meaning as before. All this is done in such a way that conditions (13)–(16) hold true with the only change that $n \in \mathbf{N}$ is replaced by $n \in \{0, \ldots, n_k\}$. Defining $U_m$ for each $m \in \mathbf{N}$ as previously, we put

$$\begin{aligned} X(\omega) &= D_0 \setminus \cup \{\cup\{\operatorname{int} B_{k,n} : n \in \{0, \ldots, n_k\}\} : k \in \mathbf{N}\} \\ &= \cap\{U_m : m \in \mathbf{N}\}. \end{aligned} \tag{31}$$

Arguing as in the proof of Example 5, we see that $X(\omega)$ is a 1-dimensional continuum, and we observe that further steps of the proof run almost without change (obviously with $X(\omega)$ in place of $X(\aleph_0)$). The only difference is condition (20), which should be replaced by

(32) the boundary of $D_k \cap X(\omega)$ with respect to $X(\omega)$ consists of finitely many points of the form $q_i$ and $p_{j,n}$ for some $i, j, n \in \mathbf{N}$.

Thereby, using (29), we conclude that $\operatorname{ord}_p X(\omega) = \omega$ for each point $p \in X(\omega)$. Thus the proof is finished. □

*Remark* 33. Note that the continua $X(\aleph_0)$ and $X(\omega)$ just defined depend on how the discs $B_{k,n}$ and the points $q_k$ are situated in $D_0$ with respect to other discs and points $p_{k,n}$ and $q_k$. In the case of $X(\omega)$ we have a definition of not any particular (uniquely determined) example, but of a class of examples, each of which enjoys the properties formulated in Example 23. This can be seen by the following argument. If we assumed that $n_i > n_0$ for $i > 0$, then the set $\{p_{0,0}, p_{0,1}, \ldots, p_{0,n_0}, q_0\}$ is the only subset of $X(\omega)$ of cardinality $2 + n_0$ which separates $X(\omega)$. Therefore, taking different values for $n_0$ we obtain topologically distinct continua $X(\omega)$.

*Question* 34. Recall that by $X(\aleph_0)$ we have denoted any continuum defined by (18), where $B_{k,n}$ and $U_m$ satisfy all the conditions (6)–(17). Are the continua $X(\aleph_0)$ homeomorphic?

Passing to the last of the four values of $\mathfrak{n}$ (see Proposition 4 above), i.e., to $\mathfrak{n} = c$, recall that the Sierpiński universal plane curve is a (locally connected) continuum that consists exclusively of points of order $c$ (see, e.g., [**11**, Section 51, I, Theorem 5, p. 275]). So, taking $X(2)$ as a simple closed curve and $X(c)$ as the Sierpiński universal plane curve, we have

**Corollary 35.** *For each* $\mathfrak{n} \in \{2, \omega, \aleph_0, c\}$ *there exists a locally connected plane curve* $X(\mathfrak{n})$ *such that*

(36) $$\mathrm{ord}_p X(\mathfrak{n}) = \mathfrak{n} \quad \textit{for all points } p \in X(\mathfrak{n}).$$

Condition (36) for $\mathfrak{n} = 2$ characterizes $X(2)$ (i.e., a simple closed curve) in the class of all continua (see, e.g., [**11**, Section 51, V, Theorem 9, p. 298]). The same condition for $\mathfrak{n} = c$ is very far from being a characterization of the Sierpiński universal plane curve $X(c)$. Namely the condition $\mathrm{ord}_p X = c$ for all $p \in X$ holds if $X$ is a disc or the cone over the Cantor set. Also, the one point union of two copies of $X(c)$ has the same property. It is quite obvious that we have a similar situation for $\mathfrak{n} = \omega$ and $\mathfrak{n} = \aleph_0$, i.e., for these $\mathfrak{n}$ condition (36) does not characterize $X(\mathfrak{n})$. Note, however, that the constructions of continua $X(\aleph_0)$ and $X(\omega)$ resemble the well-known construction of the Sierpiński universal plane curve. That curve has been topologically characterized by G.T. Whyburn in [**23**].

*Question* 37. What topological properties characterize the curves $X(\omega)$ and $X(\aleph_0)$?

Given a class $\mathcal{L}$ of topological spaces, we say that a space $U$ is *universal* in $\mathcal{L}$ if $U$ is in $\mathcal{L}$ and each member of $\mathcal{L}$ can be embedded into $U$, i.e., for each $X \in \mathcal{L}$ there is a homeomorphism $h : X \to h(X) \subset U$. The simple closed curve $X(2)$ is universal in the class of all continua $X$ such that $\mathrm{ord}_p X \leq 2$. Namely each such $X$ is either an arc or a simple closed curve [**14**, p. 267]. Further, the Sierpiński universal plane curve $X(c)$ is known to be universal in the class of all plane curves ([**19**, p. 629]; see also [**4**, Theorem 12.11, p. 433], compare [**14**, p. 73]). Since for each continuum $X$ we have

(38) $$\mathrm{ord}_p X \leq c \quad \text{for all points } p \in X,$$

we can say that $X(c)$ is universal in the class of all plane curves with (38). Thus it is quite natural to ask if similar statements hold true for

$\mathfrak{n} = \omega$ and $\mathfrak{n} = \aleph_0$. In both cases the answer is no. To show this, recall two definitions and two old results.

A space $X$ is said to be (a) *regular* (b) *rational* (in the sense of the theory of order) if for each point $p \in X$ we have (a) $\operatorname{ord}_p X \leq \omega$ (b) $\operatorname{ord}_p X \leq \aleph_0$. It is evident that each regular or rational continuum is a curve.

In 1931 G. Nöbeling ([**16**, p. 82]; compare [**14**, Theorem p. 290]) proved that for every regular curve $K$ there exists a (plane) regular curve $L$ such that no subcontinuum of $K$ is homeomorphic to $L$. As a consequence, we conclude that there is no universal element in the class of plane regular curves (compare Footnote 7 in [**16**, p. 82]).

In 1930 H. Reschovsky ([**18**, p. 19]; compare [**11**, Section 51, IV, Theorem 13, p. 290]) proved that for every compact rational space $X$ there exists an ordinal number $\alpha$ less than the first uncountable ordinal $\omega_1$ such that for every point $p \in X$ and every $\varepsilon > 0$ there exists an open set $G \subset X$ which satisfies conditions: $p \in G$, $\operatorname{diam} G < \varepsilon$ and the derived set of $\operatorname{bd} G$ of order $\alpha$ is empty. Since $\alpha$ can be chosen arbitrarily close to $\omega_1$ (this can be seen by taking cones over compact countable subsets $C(\alpha) \subset [0,1]$ whose derived sets of order $\beta < \alpha$ are nonempty while one of order $\alpha$ is empty), it follows that there is no universal element in the class of plane rational curves. Thus neither $X(\omega)$ nor $X(\aleph_0)$ is universal in the considered classes of plane curves.

It is worth noting that all four examples of continua having a constant order of points are locally connected. Since each locally connected continuum is homogeneous with respect to the class of all (continuous) mappings according to Theorem 1 of [**10**, p. 347], each of these four examples has this property. The simple closed curve is homogeneous in the strongest sense, i.e., with respect to homeomorphisms and, by the Mazurkiewicz result (Theorem 1 above), it is the only one of the four having this property. Further, it is known that the Sierpiński universal plane curve is homogeneous with respect to monotone mappings [**6**, Theorem 5, p. 131]. Thus it is natural to ask the following question.

*Question* 39. What are admissible classes $\mathcal{M}$ of mappings having the property that if all points of a locally connected (plane) continuum $X$ are of the same order, then $X$ is homogeneous with respect to $\mathcal{M}$?

The rest of the paper addresses this question. We start with a remark that concerns local connectedness of $X$.

*Remark* 40. Let a continuum of $X$ have all its points of the same order $\mathfrak{n}$. Then it follows from Proposition 4 that $\mathfrak{n} \in \{2, \omega, \aleph_0, c\}$. If $\mathfrak{n} = 2$ or $\mathfrak{n} = \omega$, then $X$ is regular in the sense of the theory of order, and hence it is locally connected [**11**, Section 51, IV, Theorem 1, p. 283]. Therefore $X$ is homogeneous with respect to the class of all (continuous) mappings (see [**10**, Theorem 1, p. 347]). Nothing similar holds if $\mathfrak{n} = \aleph_0$ or $\mathfrak{n} = c$. Local connectedness not only does not follow from the assumption, but even no version of homogeneity can be obtained, even in its weakest form, i.e., with respect to all mappings. Thus, local connectedness is essential in Question 39 only in the case of orders $\aleph_0$ or $c$. This is shown by two examples below.

*Examples* 41. There exist plane continua $X_1$ and $X_2$ such that

$$\operatorname{ord}_p X_1 = \aleph_0 \quad \text{for all points } p \in X_1, \tag{42}$$

and

$$\operatorname{ord}_p X_2 = c \quad \text{for all points } p \in X_2, \tag{43}$$

which are not homogeneous with respect to any class of mappings.

*Proof.* In the Cartesian coordinates in the plane take points $a_0 = (0,0)$, $b_0 = (0,1)$, and for each positive integer $n$ put $a_n = (1/n, 0)$ and $b_n = (1/n, 1)$. Let $a_0b_0$ be the straight line segment with end points $a_0$ and $b_0$. For each $k \in \{1,2,3,\dots\}$ let $T_{2k-1}$ and $T_{2k}$ stand for the triangle with vertices $a_{2k-1}, b_{2k-1}, a_{2k}$ and $a_{2k}, b_{2k}$, $a_{2k+1}$, respectively. Then the union

$$U = a_0b_0 \cup \cup\{T_k : k \in \{1,2,3,\dots\}\}$$

is a 2-dimensional not locally connected continuum having two arc components. Now each triangle $T_k$ is considered as a closed disc in which either a copy $C_{1,k}$ of the curve $X(\aleph_0)$ of Example 5, or a copy $C_{2,k}$ of the Sierpiński universal plane curve $X(c)$ is located. Putting for $j = 1$ and $j = 2$

$$X_j = a_0b_0 \cup \cup\{C_{j,k} : k \in \{1,2,3,\dots\}\}$$

we see that conditions (42) and (43) hold true. Note that both $X_1$ and $X_2$ have two arc components, one of which, $a_0b_0$, is compact. It is known that if a continuum is homogeneous with respect to a class of mappings, is not arcwise connected and has a compact arc component, then it has infinitely many arc components [**10**, Proposition 4, p. 354]. Thus the proof is complete. □

Our next propositions are also related to Question 39.

**Proposition 44.** *If a continuum $X$ contains a point of order 2, then the following conditions are equivalent*:

1) *$X$ is homogeneous*;

2) *$X$ is homogeneous with respect to any class of mappings that do not decrease order of points*;

3) *$X$ is homogeneous with respect to any class of mappings that do not increase order of points*;

4) *$X$ is a simple closed curve.*

*Proof.* The implications 1) $\Rightarrow$ 2) and 1) $\Rightarrow$ 3) are obvious. If either 2) or 3) is assumed, then all points of $X$ are of the same order by Proposition 2, thus of order 2 since $X$ contains a point of order 2 by assumption. Since a continuum $X$ is a simple closed curve if and only if each point of $X$ is of order 2 (see [**11**, Section 51, V, Theorem 6, p. 294]) we get 4), which trivially implies 1). □

It is known that each local homeomorphism in the large sense of a compact space does not decrease order or points [**12**, Section 3, Theorem 2, p. 55] and that each open mapping of a compact space does not increase order of points [**22**, Corollary (7.31), p. 147]. Hence Proposition 44 implies the following corollary.

**Corollary 45.** *If a continuum $X$ contains a point of order 2, then the following conditions are equivalent*:

1) *$X$ is homogeneous*;

2) *$X$ is homogeneous with respect to local homeomorphisms in the*

*large sense*;

3) *X is homogeneous with respect to open mappings*;

4) *X is a simple closed curve.*

*Remark* 46. The assumption that $X$ contains a point of order 2 is essential in Proposition 44 and Corollary 45 because, as was recently shown by J.R. Prajs [**17**, Corollary 5], the disc $B_2 = \{(x,y) \in \mathbf{R}^2 : x^2 + y^2 \le 1\}$ is homogeneous with respect to open mappings. This is a negative solution of Problem 4 of [**6**, p. 132]. Note, however, that this counterexample is 2-dimensional.

As a more specific case of Question 39, one can ask the following.

*Questions* 47. Let $X = X(\mathfrak{n})$ where $\mathfrak{n} \in \{\omega, \aleph_0, c\}$. Is then $X$ homogeneous with respect to the class of (i) open mappings, (ii) monotone mappings (if $\mathfrak{n} \neq c$)?

*Remark* 48. Each open mapping of a compact space is confluent [**5**, VI, p. 214; **22**, (7.5), p. 148], and obviously each monotone mapping of a continuum is confluent. However, neither monotone nor confluent mappings can be joined to ones listed in Corollary 45 because each dendrite contains points of order 2 (even as a dense subset, see [**11**, Section 51, VI, Theorem 8, p. 302]), and each standard universal dendrite $D_n$ of order $n \in \{3, 4, \dots, \omega\}$ is homogeneous with respect to the class of monotone, thus of confluent mappings (for $n = 3$ see [**8**, Example 2.4, p. 59] and [**9**, Proposition 2.4, p. 223]; for an arbitrary integer $n \ge 3$ and $n = \omega$ see [**7**, Theorem 7.1, p. 186]).

For $\mathfrak{n} = \omega$ we have the following analog of Proposition 44 and Corollary 45.

**Proposition 49.** *If a continuum $X$ contains a point of order $\omega$ and is homogeneous with respect to any class of mappings that does not either decrease or increase order of points, then $X$ is locally connected.*

*Proof.* If $X$ satisfies the homogeneity assumption, then all its points are of the same order by Proposition 2, hence all are of order $\omega$. Thus

$X$ is locally connected by Theorem 1 of [**11**, Section 51, IV, p. 283]. □

*Remark* 50. In Proposition 44 and Corollary 45 equivalences of the discussed conditions are shown. Unlike in Proposition 49 only the implication in one direction is true because, e.g., the union of countably many straight line segments $pa_n$ of lengths $1/n$ each, emanating from $p$ and pairwise disjoint out of $p$, is a locally connected continuum containing just one point $p$ of order $\omega$, so it is homogeneous with respect to the class of all mappings [**10**, Theorem 1, p. 347], except those which do not decrease or increase order of points.

**Acknowledgment.** The author thanks Dr. K. Omiljanowski and Dr. J.R. Prajs for a fruitful discussion on the topic of this paper.

## REFERENCES

**1.** R.D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. **67** (1958), 313–324.

**2.** ———, *One-dimensional continuous curves and a homogeneity theorem*, Ann. of Math. **68** (1958), 1–16.

**3.** R.H. Bing, *A homogeneous indecomposable plane continuum*, Duke Math. J. **15** (1948), 729–742.

**4.** L.M. Blumenthal and K. Menger, *Studies in geometry*, W.H. Freeman and Co., NY, 1970.

**5.** J.J. Charatonik, *Confluent mappings and unicoherence of continua*, Fund. Math. **56** (1964), 213–220.

**6.** ———, *Mappings of the Sierpiński curve onto itself*, Proc. Amer. Math. Soc. **92** (1984), 125–132.

**7.** ———, *Monotone mappings of universal dendrites*, Topology Appl. **38** (1991), 163–187.

**8.** H. Kato, *Generalized homogeneity of continua and a question of J.J. Charatonik*, Houston J. Math. **13** (1987), 51–63.

**9.** ———, *On problems of H. Cook*, Topology Appl. **26** (1987), 219–228.

**10.** P. Krupski, *Continua which are homogeneous with respect to continuity*, Houston J. Math. **5** (1979), 345–356.

**11.** K. Kuratowski, *Topology,* Vol. 2, Academic Press and PWN, 1968.

**12.** A. Lelek, *Sur l'unicohérence, les homéomorphies locales et les continus irréductibles*, Fund. Math. **45** (1957), 51–63.

**13.** S. Mazurkiewicz, *Sur les continus homogènes*, Fund. Math. **5** (1924), 137–146.

**14.** K. Menger, *Kurventheorie*, Teubner Verlag 1932 (reprinted by Chelsea Publ. Co., NY, 1967).

**15.** E.E. Moise, *A note on the pseudo-arc*, Trans. Amer. Math. Soc. **64** (1949), 57–58.

**16.** G. Nöbeling, *Über regulär-eindimensionale Räume*, Math. Ann. **104** (1931), 81–91.

**17.** J.R. Prajs, *On open homogeneity of closed balls in the Euclidean spaces* (preprint).

**18.** H. Reschovsky, *Über rationale Kurven*, Fund. Math. **15** (1930), 18–37.

**19.** W. Sierpiński, *Sur une courbe cantorienne qui contient une image biunivoque et continue de toute courbe donnée*, C. R. Acad. Sci. Paris **162** (1916), 629–632.

**20.** ———, *Sur une propriété topologique des ensembles dénombrables denses en soi*, Fund. Math. **1** (1920), 11–16.

**21.** P.S. Urysohn, *Memoire sur les multiplicités Cantoriennes Partie* II: *Les lignes Cantoriennes*, Verh. Konink. Acad. Wetensch. Amsterdam, 1 sectie, **13** (1927), 1–172.

**22.** G.T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ. **28** (1942).

**23.** ———, *Topological characterization of the Sierpiński curve*, Fund. Math. **45** (1958), 320–324.

Institute of Mathematics, University of Wrocław, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland