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RIGHT UNIQUE FACTORIZATION DOMAINS

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ABSTRACT. A right unique factorization domain is an atomic integral domain in which atomic factorization is unique up to order of factors and right unit factors. If the ring is also strongly right Ore (for each pair of nonzero elements a and b there exist nonzero a_1 and b_1 such that $ab_1 = ba_1$; if a and b are atoms there exist such a_1 and b_1 that are atoms), then it is right invariant. For atomic right invariant rings it is shown that uniqueness is equivalent to the right LCM requirement. A right UFD need not be a left UFD unless the ring satisfies Ore conditions.

The notion of unique factorization domain (UFD for short) is a familiar topic in a first abstract algebra course (see [**3**, Chapter 7], for example). To determine if an integral domain is a UFD there are two conditions to verify. One condition requires that every nonzero nonunit be the product of irreducibles (which are being referred to more often today as atoms); when this is the case, R is said to be atomic. The condition of atomicity is usually observed by inspection or verified through an ascending chain condition. The uniqueness condition, on the other hand, has several different equivalent forms. Depending upon which one of these we focus on, various kinds of generalizations to the noncommutative case arise. As expected, the definitions and corresponding theory vary in complexity. Most were described in the survey [**2**], while others have been proposed since then. One of the most basic notions of noncommutative UFD is that of a similarity-UFD: an atomic integral domain R in which whenever

$$a_1 a_2 \dots a_n = b_1 b_2 \dots b_m$$

for atoms a_i and b_j, then $n = m$ and there is a permutation π of the subscripts such that a_i and $b_{\pi(i)}$ are similar (that is, $R/a_iR \simeq R/b_{\pi(i)}R$ as R-modules). As indicated in [**2**] this concept is left-right symmetric. For commutative rings similarity reduces to the usual association. It is

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appropriate to relate any new definition of UFD to that of a similarity-UFD. We do this below. We consider a definition of UFD which follows the commutative case quite closely and yet which can be very one-sided.

Definition. A right UFD is a not-necessarily commutative integral domain satisfying the following conditions:

(i) R is atomic,

(ii) atomic factorization is unique up to order of factors and right unit factors, that is, if

$$a_1 a_2 \dots a_n = b_1 b_2 \dots b_m \tag{1}$$

where the a_i and b_j are atoms, then $n = m$ and $a_i R = b_{\pi(i)} R$ for some permutation π of the subscripts.

It is easily verified that $aR = bR$ if and only if $a = bu$ for some unit u in R, in which case a and b are said to be right associates. Thus, condition (ii) may be phrased as uniqueness up to order of factors and right unit factors (or right association).

Obviously, a right UFD is a similarity-UFD. The free associative algebra $F\langle x, y\rangle$ over a field F is an example of a similarity-UFD [**2**, p. 9] which is not a right-UFD: consider the equation

$$x(yx+1) = (xy+1)x.$$

In this case R is not a right Ore domain; specifically, we have $xR \cap yR = 0$. In order to establish the kind of properties that we want, we shall need the following strong form of the right Ore condition.

Definition. A right Ore domain R is strongly right Ore if for any two atoms a and b in R there exist atoms a_1 and b_1 in R such that $ab_1 = ba_1$.

A right UFD R has the following five properties.

Property 1. *Every atom a in R is prime, that is, if a divides a product, then it must divide one of the factors.*

Proof. Follows immediately from uniqueness. □

Property 2. *If a is an atom and u is a unit in R, then there exists a unit v in R such that $ua = av$.*

Proof. Applying uniqueness to the equation $a^2 = (au^{-1})(ua)$ we obtain $aR = uaR$ as desired. □

Property 3. *Assume that R is strongly right Ore. For each a and b in R, $abR = baR$.*

Proof. We first assume that a and b are atoms. If $aR = bR$, then $b = au$ for some unit u and (by Property 2) $ua = av$ for some unit v; thus, $b^2 = auau = a^2vu$ from which we obtain $abR = a^2R = b^2R = baR$. For the other case we assume $aR \neq bR$. Choose atoms a_1 and b_1 in R such that $ab_1 = ba_1$. Since $aR \neq bR$, we must have $aR = a_1R$ and $bR = b_1R$ by uniqueness. Thus, $abR = ab_1R = ba_1R = baR$. Using Property 2 and induction we see that every order of atomic factors is possible in the expression $a_1a_2 \ldots a_nR$. Thus the result follows. □

Recall that a ring R is right invariant if every principal right ideal is two sided, that is, $Ra \subseteq aR$ for each $a \in R$. It is noteworthy and clear that in a right invariant ring every divisor is a left divisor.

Property 4. *The ring R is strongly right Ore if and only if R is right invariant.*

Proof. If R is strongly right Ore and $a \in R$, then for any $b \in R$, Property 3 shows that there is a unit u in R such that $ba = abu \in aR$ so that $Ra \subseteq aR$. Conversely, if R is right invariant and if a and b are atoms in R, then $ba \in aR$ so that $ba = ab_1$ for some $b_1 \in R$ and b_1 must be an atom by unique factorization. □

Property 5. *Assume that R is strongly right Ore. Then R is a right LCM domain, that is, if a and b are nonzero elements of R, then $aR \cap bR = mR$ for some m which is then the least common right multiple* *of a and b; (the greatest common right factor also exists).*

Proof. This follows just as in the commutative case since Property 3 shows that each nonzero element of R has a factorization of the form

$$a_1^{k_1} \dots a_n^{k_n} u$$

where u is a unit and the a_i are atoms. □

We have established part of the following.

Theorem 1. *Let R be an atomic right invariant ring. The following are equivalent.*

(i) *R is a right* LCM *domain,*

(ii) *R is a right* UFD,

(iii) *R is a similarity*-UFD.

Proof. Properties 4 and 5 show that (ii) implies (i). Assume that R is a right LCM domain. Recall that in a right invariant ring all divisors are left divisors. We claim that if a is an atom in R that does not divide a nonzero element b in R, then $aR \cap bR = baR$. First observe that by right invariance we have $ba \in aR \cap bR$ which is equal to $ba'R$ for some a' and so $a \in a'R$ since a does not divide b we have $a'R \neq R$ (that is, $ba'R = aR \cap bR \neq bR$) and since a is an atom we obtain $aR = a'R$ (which gives $baR = ba'R$), proving the claim. Now we can show that each atom a of R is prime. Suppose a divides bc and a does not divide b. Right invariance shows that $bc \in aR$. The preceding claim then yields $bc \in aR \cap bR = baR$ so that $c \in aR$ as desired. Since atoms are primes, uniqueness in the definition of right UFD follows by induction as in the commutative case. Thus, (i) implies (ii).

To show that (iii) implies (ii), let R be a similarity-UFD and let $R/aR \simeq R/a'R$ as R-modules. It is well known (see [**2**, Theorem 5] and the references given there) that this isomorphism corresponds to the existence of $b \in R$ such that $aR + bR = R$ and $aR \cap bR = ba'R$. Right invariance shows that $ba \in aR \cap bR = ba'R$ so that $aR \subseteq a'R$. Thus, $a = a'r$ for some $r \in R$. Since a and a' are similar, r must be a unit so that $aR = a'R$. Since it is clear that (ii) implies (iii), the proof is complete. □

A right UFD need not be a left UFD even in the right invariant case as the following example shows.

Example. Let $R = F[[x, \sigma]]$, the ring of "twisted" formal power series in x over F, where F is a field and $t \to t^\sigma = \sigma(t)$ is a monomorphism of F into a proper subfield of F. (For example, we could let $F = Q(y_1, y_2, \dots)$ be the ring of rational functions in an infinite number of indeterminates y_i over the field Q of rationals and define σ so that $\sigma(y_i) = y_{i+1}$.) Coefficients of powers of x are written on the right hand side. Addition in R is as usual and multiplication follows the usual rules except that $tx = xt^\sigma$ for each $t \in F$. Thus, each element in R has the form $x^n u$ where u is a unit (that is, a power series with nonzero constant term). From this observation it follows easily that R is a right invariant right UFD with "a unique prime." However, R is not a left UFD, since if $u \in F \backslash F^\sigma$, we have

$$x^2 = (xu)(u^{-1}x)$$

but $Rx \neq Rxu$.

The ring $R = F[[x, \sigma]]$ is not left Ore; we have $Rx \cap Rxu = 0$ for $u \in F \backslash F^\sigma$. It is the Ore condition which provides for the left-right symmetry.

Theorem 2. *Let R be a right-invariant right* UFD. *If R is left Ore then R is a left-invariant left* UFD.

Proof. To show that R is a left UFD it suffices to show that, for atoms a and b in R, if $aR = bR$, then $Ra = Rb$. Choose atoms a and b in R. Using the left Ore condition we may select nonzero r and s in R such that

$$ra = sb. \tag{2}$$

Let $r = p_1 p_2 \dots p_n$ where the p_i are atoms. If some p_i divides s, then it divides an atomic factor of s; these may be brought to the left side in Equation (2) and cancelled (Properties 1 and 3). In this way, we may assume that no p_i divides s and so $p_iR = bR$ for each i. Thus, $rR = b^nR$. Similarly, $sR = a^nR$. Equation (2) takes the form

$$b^n ua = a^n vb \tag{3}$$

for units u and v in R. Assuming that $aR = bR$ we obtain $a^nR = b^nR$ as in the proof of Property 3. Thus, $a^n = b^nt$ for some unit t in R. Substituting this into Equation (3) and cancelling common factors we obtain $Ra = Rb$. Thus, R is a left UFD.

Unique factorization applied to Equation (3) shows that either $aR = bR$ or $n = 1$. In any case it follows that R is strongly left Ore. Applying the left-right analog of Property 4 we conclude that R is left invariant. □

We close by noting that much of the commutative theory does not go through for noncommutative UFDs. For example (see [**2**, p. 13] for other illustrations), the polynomial ring $F[x]$ in a central indeterminate x over a skew field F is not a right UFD since Property 2 fails: for $ua \neq au$ and v in F consider the atom $1 + ax$ in the equation

$$u(1 + ax) = (1 + ax)v.$$

Fortunately, $F[x]$ is a similarity-UFD [**2**]. With two central indeterminates the situation is much worse. It can be shown that $F[x, y]$ is not a unique factorization domain in any reasonable sense when $F = Q(1, i, j, k)$ is the skew field of rational quaternions. In [**1**] it is shown that

$$f = (x^2y^2 - 1) + (x^2 - y^2)i + 2xyj$$

and its conjugate

$$\bar{f} = (x^2y^2 - 1) - (x^2 - y^2)i - 2xyj$$

are atoms whose product factors into four atoms, namely,

$$f\bar{f} = (x^2 + i)(x^2 - i)(y^2 + i)(y^2 - i).$$

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