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ON A SUBCLASS OF STARLIKE FUNCTIONS

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ABSTRACT. Let $R(\beta)$ denote the class of functions $f(z) = z + a_2z^2 + \cdots$ which are analytic in the unit disc $D = \{z : |z| < 1\}$ and satisfy the condition $\operatorname{Re}(f'(z) + zf''(z)) > \beta$, $\beta < 1$, for $z \in D$. We find β so that $R(\beta)$ is a subclass of S^*, the class consisting of univalent starlike functions in D.

1. Introduction. Let A denote the class of functions f which are analytic in the unit disc $D = \{z : |z| < 1\}$ and normalized so that $f(0) = f'(0) - 1 = 0$. Let S be the subclass of A consisting of univalent functions and let K and S^* denote the usual subclasses of S whose members are convex and starlike, respectively. For $\beta < 1$, let

$$R(\beta) = \{f \in A : \operatorname{Re}(f'(z) + zf''(z)) > \beta,\ z \in D\}.$$

The class $R(0)$ will simply be denoted by R. Chichra [**1**] proved that if $f \in R$, then for $z \in D$, $\operatorname{Re} f'(z) > 0$, and hence $R \subset S$. Singh and Singh [**7**] showed that $f \in R$ would imply $f \in S^*$ and Krzyz [**2**] gave an example to show that R is not a subset of K.

Let

$$\beta_S = \inf\{\beta : R(\beta) \subset S\},$$

and

$$\beta_{S^*} = \inf\{\beta : R(\beta) \subset S^*\}.$$

In a later paper, Singh and Singh [**8**] showed that $\beta_{S^*} \le -1/4$. More recently, the estimate on β_{S^*} was further improved by Nunokawa and Thomas [**3**]. They proved that $R(\beta_0) \subset S^*$ if β_0 satisfies the equation

$$3\beta + (1-\beta)(2 - \log(4/e))\log(4/e) = 0,$$

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and thus $\beta_{S^*} \le \beta_0$, where $\beta_0 \cong -0.262$.

In this paper, we further improve an upper bound for β_{S^*}.

2. Preliminaries. We shall need the following lemmas.

Lemma 1 [**3**]. *Let p be analytic in D with $p(0) = 1$. Suppose that $\alpha > 0$, $\beta < 1$ and that for $z \in D$, $\mathrm{Re}\,(p(z) + \alpha z p'(z)) > \beta$. Then for $z \in D$,*

$$\mathrm{Re}\, p(z) > 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\alpha n}.$$

Lemma 2 [**5**]. *Let w be meromorphic in D with $w(0) = 0$. If, for a certain $z_0 \in D$, the inequality $|w(z)| \le |w(z_0)|$ holds for $|z| \le |z_0|$, then $z_0 w'(z_0)/w(z_0) \ge 1$.*

Lemma 3 [**8**]. *If P is analytic in D, $P(0) = 1$ and $\mathrm{Re}\, P(z) > 1/2$ for $z \in D$, then for any function F analytic in D, the function P^*F takes its values in the convex hull of $F(D)$, where P^*F denotes the convolution or Hadamard product of P and F.*

3. Results.

Theorem 1. *Let $f \in R(\beta)$. Then*

(a) $\mathrm{Re}\, f'(z) > 1 + 2(1-\beta)(\log 2 - 1)$, $z \in D$.

(b) $\mathrm{Re}\, f(z)/z > (1-\beta)(\pi^2/6 - 1) + \beta$, $z \in D$.

Both constants above are sharp.

Proof. (a) Let $p = f'$ and $\alpha = 1$. It follows easily from Lemma 1 that

$$\mathrm{Re}\, f'(z) > 1 + 2(1-\beta) \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n} = 1 + 2(1-\beta)(\log 2 - 1), \qquad z \in D.$$

(b) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Since $\mathrm{Re}\,(f'(z) + z f''(z)) > \beta$, $z \in D$, we have

$$\text{(1)} \qquad \operatorname{Re}\left[1+\frac{1}{2(1-\beta)}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] > \frac{1}{2}, \qquad z \in D.$$

Consider the function

$$F(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{z^{n-1}}{n^2}.$$

Clearly F is analytic in D. Now

$$g(z) = -\frac{1}{z}\log(1-z) = \sum_{n=1}^{\infty}\frac{z^{n-1}}{n}$$

is a convex univalent mapping [**4**] of D with positive coefficients about the origin. Therefore g^*g is convex [**6**] and has positive coefficients, and so $\operatorname{Re}(g^*g)(z) \geq (g^*g)(-1)$, that is

$$\operatorname{Re}\sum_{n=1}^{\infty}\frac{z^{n-1}}{n^2} \geq \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Thus it follows that

$$\text{(2)} \qquad \operatorname{Re} F(z) > (1-\beta)\left(\frac{\pi^2}{6}-1\right) + \beta, \qquad z \in D.$$

Writing $f(z)/z$ as

$$\frac{f(z)}{z} = \left[1+\frac{1}{2(1-\beta)}\sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right] * \left[1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{z^{n-1}}{n^2}\right],$$

and making use of (1), (2) and Lemma 3, we conclude that for $z \in D$, $\operatorname{Re}(f(z)/z) > (1-\beta)(\pi^2/6-1)+\beta$.

That the constants in both results above are sharp is demonstrated by the function $f_0 \in R(\beta)$ defined by $zf_0'(z) = -(1-2\beta)z - 2(1-\beta)\log(1-z)$. □

Remarks . 1. From Theorem 1(a), we see that $\beta_S = -(2\log 2 - 1)/2(1-\log 2)$.

2. The estimate in Theorem 1(b) improves the constant obtained by Singh and Singh [**8**].

Theorem 2. *If $f \in R(\beta)$ with $\beta \ge (6-\pi^2)/(24-\pi^2)$, then $f \in S^*$.*

Proof. Define w in D by

$$\frac{zf'}{f} = \frac{1+w(z)}{1-w(z)}.$$

Clearly w is meromorphic in D, $w(0)=0$ and since f is univalent, we have $w(z) \ne 1$. Also,

$$f'(z) + zf''(z) = \frac{f(z)}{z}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2 + \frac{2zw'(z)}{(1-w(z))^2}\right]. \tag{3}$$

We need to show that $|w(z)| < 1$ for $z \in D$. Suppose there exists a $z_0 \in D$ such that for $|z| \le |z_0|$, $|w(z)| \le |w(z_0)| = 1$. Then the Clunie-Jack lemma (Lemma 2) implies that $z_0 w'(z_0) = kw(z_0) = ke^{i\theta}$, where $k \ge 1$ and $0 < \theta < 2\pi$. With $z = z_0$, it follows from (3) that

$$\begin{aligned}\mathrm{Re}\,(f'(z_0) + z_0 f''(z_0)) &= \mathrm{Re}\left[\frac{f(z_0)}{z_0}\left\{\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right\}\right] \\ &\le -\frac{k}{2\sin^2(\theta/2)}\mathrm{Re}\,\frac{f(z_0)}{z_0} \\ &\le -\frac{1}{2}\mathrm{Re}\,\frac{f(z_0)}{z_0}\end{aligned}$$

provided $\mathrm{Re}\,(f(z_0)/z_0) > 0$. In view of Theorem 1(b), if

$$(1-\beta)(\pi^2/6 - 1) + \beta > 0, \tag{4}$$

then

$$\mathrm{Re}\,(f'(z_0) + z_0 f''(z_0)) \le -(1/2)[(1-\beta)(\pi^2/6-1)+\beta].$$

Thus, if β also satisfies the inequality

$$-(1/2)[(1-\beta)(\pi^2/6-1)+\beta)] \leq \beta, \tag{5}$$

then we have a contradiction to $\mathrm{Re}\,(f'(z)+zf''(z)) > \beta$ at $z = z_0$. The smallest β that satisfies (4) and (5) is $\beta_0 = (6-\pi^2)/(24-\pi^2) = -0.2739\cdots$. Thus, if $\beta \geq \beta_0$, we see that $|w(z)| < 1$ and hence $f \in S^*$. □

The result above shows that $\beta_{S^*} \leq (6-\pi^2)/(24-\pi^2)$, which improves the upper bound obtained by Nunokawa and Thomas. Obviously, $\beta_S \leq \beta_{S^*}$. For the wider class

$$\{f \in A : \mathrm{Re}\,[e^{i\alpha}(zf''(z)+f'(z)-\beta)] > 0, \quad z \in D, \quad \alpha = \alpha(f)\},$$

numerical evidence suggests that indeed $\beta_{S^*} = \beta_{S^*} = (1-2\log 2)/2(1-\log 2)$. However, a proof of this conjecture is missing and is unlikely to be obtained by employing our techniques above.

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