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CARLESON'S INEQUALITY AND QUASICONFORMAL MAPPINGS

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1.1. Introduction. In his work on the interpolation of analytic functions Carleson characterized certain measures on the unit disc by means of L^p-integral inequalities for functions in H^p. Duren extended Carleson's theorem to exponents $0 < p \leq q < \infty$. We prove here analogues of these results for quasiconformal mappings in $\mathbf{R}^n$.

We denote the unit ball in n-dimensional Euclidean space, R^n, by B^n, and S^{n-1} denotes its boundary. The open ball centered at $x \in R^n$ of radius r is denoted $B(x,r)$. We assume throughout that μ is a positive measure on B^n. We call μ a t-Carleson measure, $0 < t < \infty$, if there exists a constant $N(\mu)$ such that

$$\mu(B(s,r) \cap B^n) \leq N(\mu) r^{t(n-1)} \tag{1.2}$$

for all $s \in S^{n-1}$ and all $0 < r < \infty$. When $n = 2$ and $t = 1$, this is Carleson's original definition [**3**].

The main result of this paper, Theorem 1.3, is a quasiconformal analogue of results of Carleson [**3**] and Duren [**4**] concerning analytic functions. To obtain this result, we use certain integral inequalities for the nontangential maximal function given in [**1**] and [**8**].

When $f : B^n \to R^m$ is measurable and $0 < p < \infty$, we write

$$||f||_{H^p} = \limsup_{r \to 1} \left(\int_{S^{n-1}} |f(rs)|^p \, d\sigma(s) \right)^{1/p}$$

where $d\sigma$ is the surface area measure on S^{n-1}.

We use here the usual definition of a K-quasiconformal mapping as defined in [**7**].

Theorem 1.3. *Suppose that* $0 < p \leq q < \infty$. *If* $t = q/p$ *and if*

$$\mu \textit{ is a } t\textit{-Carleson measure,} \tag{1.4}$$

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then there exists a constant C, depending only on n, p, q, K and $N(\mu)$, such that

$$\left(\int_{B^n} |f|^q \, d\mu\right)^{1/q} \le C||f||_{H^p} \tag{1.5}$$

for all K-quasiconformal $f : B^n \to R^n$.

Conversely, if (1.5) holds for all K-quasiconformal $f : B^n \to R^n$, where K depends only on $p, q,$ and n, then (1.4) holds with $N(\mu)$ depending only on n, p, q and C.

When $n = 2$ and f is analytic, this is Carleson's result [**3**] for $p = q$.

Corollary 1.6. *Theorem 1.3 remains valid if, in (1.5), $||f||_{H^p}$ is replaced by*

$$\min_{1\le i\le n} ||f_i||_{H^p}$$

and we assume that $f(0) = 0$.

This follows from the use of Theorem 2.3. We also have the following corollaries.

Corollary 1.7. *There is a constant C, depending only on p, q, n and K, such that*

$$\left(\int_{B_k} |f(x)|^q (1 - |x|)^{(n-1)q/p-k} \, dm_k\right)^{1/q} \le C||f||_{H^p}$$

for all K-quasiconformal $f : B^n \to R^n$. Here dm_k is k-dimensional Lebesgue measure, $1 \le k \le n$ and $0 < p \le q < \infty$, $B_k = \{(x_1, x_2, \dots, x_n) \mid x_j = 0, k+1 \le j \le n\}$ for $1 \le k \le n-1$ and $B_n = B^n$.

Corollary 1.8. *there is a constant C, depending only on p, q, n and K, such that*

$$\left(\sum_{k=0}^{\infty} |f((1 - 2^{-k})s_k)|^q 2^{-(n-1)qk/p}\right)^{1/q} \le C||f||_{H^p}$$

for all K-quasiconformal $f : B^n \to R^n$. Here $0 < p \leq q < \infty$ and $\{s_k\}_{k=0}^{\infty}$ is any sequence in S^{n-1}.

Define $\mu = \sum 2^{-\gamma k}\delta_{x_k}$ where $\gamma = (n-1)q/p$, $x_k = (1-2^{-k})s_k$ and δ_{x_k} is the unit mass distribution at x_k. Then μ is t-Carleson with $t = q/p$ and Corollary 1.8 follows from Theorem 1.3 using this μ.

Corollary 1.9. *For each p, $0 < p < \infty$, there is a constant C, depending only on p, n, and K such that*

$$|f(x)| \leq C(1-|x|)^{-(n-1)/p}||f||_{H^p}$$

for all K-quasiconformal $f : B^n \to R^n$ and all $x \in B^n$.

Corollary 1.9 follows from Theorem 1.3 with $\mu = (1-|x|)^{q(n-1)/p}\delta_x$.

2.1. Quasiconformal mappings and the nontangential maximal function. We define a Stolz region $\Gamma(s)$ at $s \in S^{n-1}$ by

$$\Gamma(s) = \{x \in B^n \mid |x-s| \leq 3(1-|x|)\}.$$

We denote the nontangential maximal function of $f : B^n \to R^m$ by

$$f^*(s) = \sup_{x \in \Gamma(s)} |f(x)|.$$

Although other expansion factors lead to similar results, we use the factor 3 in the definition of Γ in agreement with [**8**] and [**1**]. In [**8**] and [**1**] the factor 3 simplifies estimates required in the proofs of the theorems. Since the inequalities involving the nontangential maximal function, f^*, are used here only as intermediate results, the actual value of this factor is immaterial.

The following result appears in [**8**] and [**1**].

Theorem 2.2. *Let $0 < p < \infty$. There exists a constant C, depending only on n, p and K, such that*

$$||f||_{H^p}/C \leq \left(\int_{S^{n-1}} f^*(s)^p \, d\sigma(s)\right)^{1/p} \leq C||f||_{H^p}$$

for all K-quasiconformal $f : B^n \to R^n$.

Also, the following analogue of the result in [**2**] is in [**1**].

Theorem 2.3. *Let $0 < p < \infty$. There exists a constant C, depending only on n, p and K, such that*

$$\begin{aligned} ||f||_{H^p}/C &\le \min_{1\le i\le n} \left(\int_{S^{n-1}} f_i^*(s)^p \, d\sigma(s) \right)^{1/p} \\ &\le C||f||_{H^p} \end{aligned}$$

for all K-quasiconformal $f = (f_1, f_2, \ldots, f_n)$: $B^n \to R^n$ with $f(0) = 0$.

3.1. Proof that (1.5) implies (1.4). To verify (1.2) we may assume that $0 < r < 1/2$ and that $s = (0, 0, \ldots, 0, 1)$. Set $a = (0, 0, \ldots, 0, 1 - r)$ and define a quasiconformal mapping as follows:

$$f(x) = (x - a^*)|x - a^*|^{-(\beta/p+1)}$$

where $a^* = a/|a|^2$ and $0 < \beta < \infty$.

Notice that if $x \in B(s, r) \cap B^n$, then

$$\begin{aligned} 1 &= |x - a^*|^{\beta/p}|f(x)| \\ &\le (3r)^{\beta/p}|f(x)|. \end{aligned}$$

Using (1.5) we obtain

$$\begin{aligned} \mu(B(s, r) \cap B^n) &\le \int_{B^n} (3r)^{\beta q/p}|f|^q \, d\mu \\ &\le C_0 r^{\beta q/p}||f||^q_{H^p}. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned} ||f||_{H^p} &= \left(\int_{S^{n-1}} |s - a^*|^{-\beta} \, d\sigma(s) \right)^{1/p} \\ &= \left(c(n) \int_0^{\pi} \frac{\sin^{n-2}\varphi}{(1 + R^2 - 2R\cos\varphi)^{\beta/2}} \, d\varphi \right)^{1/p} \end{aligned}$$

where $R = |a^*|$.

We use the inequality

$$\sin^2 \varphi \leq 1 + R^2 - 2R\cos\varphi$$

for all $0 \leq \varphi \leq 2\pi$.

If the dimension n is even, then set $\beta = n$. We get

$$\begin{aligned}||f||_{H^p} &\leq C_1\left(\int_0^\pi \frac{d\varphi}{1+R^2-2R\cos\varphi}\right)^{1/p} \\ &\leq C_2(R-1)^{-1/p} \leq C_2 r^{-1/p}.\end{aligned}$$

Then (1.2) follows with (3.2).

If n is odd, then set $\beta = n+1$ to obtain

$$\begin{aligned}||f||_{H^p} &\leq C_3\left(\int_0^\pi \frac{\sin\varphi\, d\varphi}{(1+R^2-2R\cos\varphi)^2}\right)^{1/p} \\ &\leq C_3(R-1)^{-2/p} \leq C_3 r^{-2/p}.\end{aligned}$$

Again (1.2) follows with (3.2).

4.1. Proof that (1.4) implies (1.5). We use the notation in [**1**]. We write

$$S(x) = S^{n-1} \cap B(x, 3(1-|x|))$$

for the cap associated with $x \in B^n$. We also define a tent over a set $U \subset S^{n-1}$ as

$$T(U) = \{x \in B^n \mid S(x) \subset U\}.$$

Lemma 4.2. *Let $E \subset B^n$ be a set which does not contain an infinite sequence $\{x_j\}$ whose caps $S(x_j)$ are disjoint. There exist finitely many points $\{x_j\} \subset E$ and a constant $\eta \geq 1$, depending only on n, such that the caps $S(x_j)$ are disjoint and*

$$E \subset \bigcup_j T(\eta S(x_j)).$$

Here $\eta S(x_j)$ is the cap with the same center as $S(x_j)$ expanded by the factor η.

To establish (1.5), by Theorem 2.2 it is enough to show that

$$\left(\int_{B^n} |f|^q \, d\mu\right)^{1/q} \le C_2\left(\int_{S^{n-1}} f^*(s)^p \, d\sigma(s)\right)^{1/p}. \tag{4.3}$$

Define $g(x) = |f(x)|^{p/2}$. Notice that $g(x) \le g^*(s)$ for all $s \in S(x)$. Hence

$$g(x) \le \left(\frac{1}{|S(x)|}\int_{S(x)} g^*(s)\, d\sigma(s)\right).$$

Define

$$\tilde{g}(x) = \sup\left(\frac{1}{|S(y)|}\int_{S(y)} g^*(s)\, d\sigma(s)\right)$$

where the supremum is over all $S(y)$ such that $S(x) \subset S(y)$. Evidently, $g(x) \le \tilde{g}(x)$ for $x \in B^n$. Because of this (1.5) will follow from (4.3) when we show that

$$\left(\int_{B^n} \tilde{g}^{2q/p} \, d\mu\right)^{p/2q} \le C_3\left(\int_{S^{n-1}} g^*(s)^2 \, d\sigma(s)\right)^{1/2}. \tag{4.4}$$

In other words we need to show that the operator $T : g^* \to \tilde{g}$, from $L^2(S^{n-1}, d\sigma)$ to $L^{2q/p}(B^n, d\mu)$ is of strong-type $(2, 2q/p)$. Clearly T is of strong-type (∞, ∞). Hence (4.4) will follow from the Marcinkiewicz interpolation theorem (see [**9**]) if we show that T is of weak-type $(1, q/p)$.

Following [**5**] and [**4**] we define the following sets for $\varepsilon > 0$:

$$A_\lambda^\varepsilon = \left\{x \in B^n \mid \int_{S(x)} g^*(s)\, d\sigma(s) > \lambda(\varepsilon + |S(x)|)\right\},$$

$$B_\lambda^\varepsilon = \left\{x \in B^n \mid S(x) \subset S(y) \text{ for some } y \in A_\lambda^\varepsilon\right\},$$

$$F_\lambda = \{x \in B^n \mid \tilde{g}(x) > \lambda\}.$$

Notice that $\mu(F_\lambda) = \lim_{\varepsilon\to 0} \mu(B_\lambda^\varepsilon)$.

We may assume that $||g||_{H_2} < \infty$. If $\{x_j\} \subset A_\lambda^\varepsilon$ with $S(x_j)$ disjoint, then using Hölder's inequality and Theorem 2.2,

$$\begin{aligned}\lambda \sum_j (\varepsilon + |S(x_j)|) &\leq \sum_j \int_{S(x_j)} g^*(s)\, d\sigma(s) \\ &\leq \int_{S^{n-1}} g^*(s)\, d\sigma(s) \\ &\leq C(n,p) \biggl(\int_{S^{n-1}} g^*(s)^2\, d\sigma(s) \biggr)^{1/2} \\ &\leq C_1 ||g||_{H^2}.\end{aligned}$$

Thus any such set $\{x_j\}$ is finite. Hence applying Lemma 4.2 with $E = A_\lambda^\varepsilon$ and assuming $F_\lambda \neq \varnothing$, we obtain a finite set of points $\{x_j\} \subset A_\lambda^\varepsilon$ such that

$$A_\lambda^\varepsilon \subset \bigcup_j T(\eta S(x_j))$$

and the $S(x_j)$ are disjoint. It follows that $B_\lambda^\varepsilon \subset \cup_j T(\eta S(x_j))$. From (1.4) we get

$$\begin{aligned}\mu(B_\lambda^\varepsilon) &\leq \sum_j \mu(T(\eta(S(x_j)))) \\ &\leq C_0 \sum_j |\eta S(x_j)|^{q/p} \\ &\leq C_1 \biggl(\sum_j |S(x_j)| \biggr)^{q/p} \\ &\leq C_1 \biggl(\sum_j \lambda^{-1} \int_{S(x_j)} g^*(s)\, d\sigma(s) \biggr)^{q/p} \\ &\leq C_1 \lambda^{-q/p} \biggl(\int_{S^{n-1}} g^*(s)\, d\sigma(s) \biggr)^{q/p}.\end{aligned}$$

Thus T is of weak-type $(1, q/p)$ and the proof is complete. □

5.1. An example when $q < p$. Finally we give an example to show that Theorem 1.3 does not hold when $q < p$.

For $\gamma = (n-1)q/p$, define a measure $\mu = \sum 2^{-k\gamma}\delta_{x_k}$ with $x_k = (0,\dots,0,1-2^{-k})$, $k = 0,1,2,\dots$. Then μ is a t-Carleson measure with $t = q/p$. Define, for $-(n-1)/p < \beta < -(n-1)/2p$,

$$f_\beta(x) = (x-e_n)|x-e_n|^{\beta-1}$$

where $e_n = (0,0,\dots,0,1)$. Each f_β is K-quasiconformal in B^n with $K = \max((n-1)/p, 2p/(n-1))^{n-1}$. Then

$$\begin{aligned}||f_\beta||_{H^p}^p &= \int_{S^{n-1}} |s-e_n|^{p\beta}\,d\sigma(s)\\ &= C_1\int_0^\pi \sin^{p\beta}(\varphi/2)\sin^{n-2}\varphi\,d\varphi\\ &= C_2\int_0^{\pi/2}\sin^{(p\beta+n-2)}\varphi\cos^{n-2}\varphi\,d\varphi.\end{aligned}$$

If $n=2$, then

$$\begin{aligned}||f_\beta||_{H^p}^p &\le C_2\int_0^{\pi/2}(2\varphi/\pi)^{p\beta}\,d\varphi\\ &= \frac{C_3}{p\beta+1}.\end{aligned}$$

If $n \neq 2$, then

$$\begin{aligned}||f_\beta||_{H^p}^p &\le C_2\int_0^{\pi/2}\sin^{(p\beta+n-2)}\varphi\cos\varphi\,d\varphi\\ &\le C_2\int_0^1 u^{(p\beta+n-2)}\,du\\ &= \frac{C_4}{p\beta+n-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int |f_\beta|^q\,d\mu &= \sum_{k=0}^\infty |f_\beta(x_k)|^q 2^{-k\gamma}\\ &= \sum_{k=0}^\infty 2^{-k(q\beta+\gamma)}\\ &= (1-2^{-q(\beta+(n-1)/p)})^{-1}.\end{aligned}$$

Combining the above results we obtain

$$\begin{aligned}\lim_{\beta\to-(n-1)/p^+}\left(\frac{\int_{B^n}|f_\beta|^q\,d\mu}{||f_\beta||^q_{H^p}}\right)^p &\geq \lim_{\beta\to-(n-1)/p^+}\frac{C_5(p\beta+n-1)^q}{\{1-2^{-q(\beta+(n-1)/p)}\}^p}\\ &\geq \lim_{\beta\to-(n-1)/p^+} C_6(1-2^{-q(\beta+(n-1)/p)})^{q-p}=\infty\end{aligned}$$

assuming that $q < p$.

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