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THE RIESZ INTEGRAL AND AN $L^p - L^q$ ESTIMATE FOR THE CAUCHY PROBLEM OF THE WAVE OPERATOR

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ABSTRACT. In 1949, M. Riesz [**3**] generalized the Riemann-Liouville integral of one-variable to high dimensional Euclidean spaces and obtained a powerful method now known as the Riesz integral for studying wave operators. In this paper we apply the Riesz integral to get the global space-time estimate

$$||u||_q \leq C\{||w||_p + t^{(1-n)/(n+1)}(||g||_p + ||\nabla f||_p)\}$$

where $1/q = 1/p - 2/(n+1)$, $1/p + 1/q = 1$, and u is the solution of the Cauchy problem $\Box u(x,t) = w(x,t)$ in $\mathbf{R}_+^{n+1}$, $u(x,0) = f(x)$, and $\partial_t u(x,0) = g(x)$.

1. The Riesz distribution. For $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, we denote $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. Let $\mathbf{R}^{n+1} = \{(x,t) : x \in \mathbf{R}^n, t \in \mathbf{R}\}$, and define

$$\rho^\lambda = \begin{cases} (t^2 - |x|^2)^{\lambda/2} & \text{if } t \geq |x| \\ 0 & \text{otherwise.} \end{cases}$$

For $\operatorname{Re}\lambda > -2$, ρ^λ is a locally integrable function on $\mathbf{R}^{n+1}$ and so defines a distribution

$$\langle \rho^\lambda, \phi \rangle = \int_{\mathbf{R}^{n+1}} \rho^\lambda \phi(x,t)\,dx\,dt$$

for $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$. In spherical coordinates, the above integral can be written as

$$\langle \rho^\lambda, \phi \rangle = \int_0^\infty \int_0^t (t^2 - r^2)^{\lambda/2} r^{n-1} \bar{\phi}(r,t)\,dr\,dt$$

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where $\bar{\phi}(r,t) = \int_{S^{n-1}} \phi(r\omega,t)\,d\sigma(\omega)$ is a C^∞-function in (r^2,t) with compact support. By the change of variable $r = \sqrt{s}t$

$$\langle \rho^\lambda, \phi\rangle = \int_0^\infty t^{\lambda+n}\Phi_\lambda(t)\,dt \tag{1.1}$$

where

$$\Phi_\lambda(t) = \frac{1}{2}\int_0^1 (1-s)^{\lambda/2}s^{(n-2)/2}\bar{\phi}(\sqrt{s}t,t)\,ds$$

is a holomorphic function in λ for $\operatorname{Re}\lambda > -2$ and can be extended by analytic continuation to a meromorphic function in $\lambda \in \mathbf{C}$ with poles at $\lambda = -2, -4, -6, \dots$. For $\lambda \neq -2, -4, -6, \dots$, $\Phi_\lambda(t)$ is a C^∞-function in t with compact support.

Now let n be odd, $\lambda = -n-2k$. Then

$$\begin{aligned}\Phi^{(2k-1)}_{-n-2k}(0) &= \frac{1}{2}\int_0^1 (1-s)^{(-n-2k)/2}s^{(n-2)/2}\left[\frac{\partial^{2k-1}}{\partial t^{2k-1}}\bar{\phi}(\sqrt{s}t,t)\right]_{t=0} ds\\ &= \frac{1}{2}\int_0^1 (1-s)^{(-n-2k)/2}s^{(n-2)/2}\\ &\qquad\cdot\left\{\sum_{j=0}^{2k-1}\binom{2k-1}{j}s^{j/2}\partial_1^j\bar{\phi}(0,0)\partial_2^{2k-1-j}\bar{\phi}(0,0)\right\}ds.\end{aligned}$$

For j even,

$$\int_0^1 (1-s)^{(-n-2k)/2}s^{(n+j-2)/2}\,ds = \frac{\Gamma(\frac{-n-2k+2}{2})\Gamma(\frac{n+j}{2})}{\Gamma(\frac{-2k+j+2}{2})} = 0.$$

But in another case, for j odd, $\partial_1^j\bar{\phi}(0,0) = 0$. In other words, $\Phi_{-n-2k}(t)$ has zero derivative of order $2k-1$ at $t=0$, and hence

$$\operatorname*{Res}_{\lambda=-n-2k}\langle\rho^\lambda,\phi\rangle = \frac{\Phi^{(2k-1)}_{-n-2k}(0)}{(2k-1)!} = 0$$

by (1.1) and $\operatorname*{Res}_{a=-j} t_+^a = \big((-1)^{j-1}/(j-1)!\big)\cdot\delta^{(j-1)}$ [**2**, p. 68]. Therefore, ρ^λ is a holomorphic function in λ for $\operatorname{Re}\lambda > -2$ and can be extended by analytic continuation to a meromorphic function in $\lambda \in \mathbf{C}$ with poles at

$$\begin{cases} \text{(i)} & \lambda = -2, -4, -6, \dots \\ \text{(ii)} & \lambda = -n-1, -n-3, -n-5, \dots . \end{cases} \tag{1.2}$$

For n even, ρ^λ has simple poles at these points; for n odd, it has simple poles at $\lambda = -2, -4, \dots, -n+1$ and double poles at $\lambda = -n-1, -n-3, \dots$.

Let the wave operator be denoted by

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

For $\operatorname{Re}\lambda > -2$, $\Box\rho^{\lambda+2} = (\lambda+2)(\lambda+n+1)\rho^\lambda$, and so by iteration for $k = 1, 2, 3, \dots,$

$$\Box^k \rho^{\lambda+2k} = (\lambda+2)(\lambda+4)\cdots(\lambda+2k)(\lambda+n+1) \cdot (\lambda+n+3)\cdots(\lambda+n+2k-1)\rho^\lambda;$$

that is,

$$\langle \rho^\lambda, \phi\rangle = \frac{\langle \Box^k \rho^{\lambda+2k}, \phi\rangle}{(\lambda+2)(\lambda+4)\cdots(\lambda+2k)(\lambda+n+1)(\lambda+n+3)\cdots(\lambda+n+2k-1)}. \tag{1.3}$$

By analytic continuation (1.3) holds also for $\lambda \in \mathbf{C}$ except at the singularities of (1.2).

The distribution ρ^λ can be normalized. Its construction was first given by M. Riesz [**3**].

Definition. The *Riesz distribution* is defined by

$$Z_\alpha = \frac{\rho^{\alpha-n-1}}{2^{\alpha-1}\pi^{(n-1)/2}\Gamma(\alpha/2)\Gamma((\alpha-n+1)/2)}.$$

The constant $H(\alpha, n) = 2^{\alpha-1}\pi^{(n-1)/2}\Gamma(\alpha/2)\Gamma((\alpha-n+1)/2)$ is so determined that

$$\langle Z_\alpha, e^{-t}\rangle = 1. \tag{1.4}$$

We note that $\Gamma(\alpha/2)$ has simple poles at $\alpha = 0, -2, -4, \dots,$ (i.e., $\alpha - n - 1 = -n-1, -n-3, -n-5, \dots$) and $\Gamma((\alpha-n+1)/2)$ has simple poles at $\alpha - n + 1 = 0, -2, -4, \dots,$ (i.e., $\alpha - n - 1 = -2, -4, -6, \dots$). Hence Z_α is an entire function of $\alpha \in \mathbf{C}$ and satisfies

$$\begin{aligned} \Box Z_\alpha &= Z_{\alpha-2} \\ \Box^k Z_\alpha &= Z_{\alpha-2k}, \qquad k = 0, 1, 2, \dots . \end{aligned} \tag{1.5}$$

By calculating the residues of $\langle \rho^\lambda, \phi \rangle$, we have

$$\begin{aligned} Z_0 &= \delta \\ Z_{-2k} &= \Box^k \delta, \qquad k = 0, 1, 2, \dots . \end{aligned} \tag{1.6}$$

Moreover, the support of Z_α, for all complex $\alpha \in \mathbf{C}$, is contained in the forward cone $C = \{(x,t) \in \mathbf{R}^{n+1} : t \geq |x|\}$.

Combining (1.5) and (1.6), we obtain easily

$$\Box^k Z_{2k} = \delta, \qquad k = 0, 1, 2, \dots . \tag{1.7}$$

In particular, $\Box Z_2 = \delta$, so Z_2 is a fundamental solution of wave operator (cf. [**2**, Section 6.2]).

The convolution property of Z_α is given by

Theorem 1.8.

$$Z_\alpha * Z_\beta = Z_{\alpha+\beta}.$$

Note. Since $\operatorname{Supp}(Z_\alpha)$ and $\operatorname{Supp}(Z_\beta)$ are contained in the cone $C = \{(x,t) \in \mathbf{R}^{n+1} : t \geq |x|\}$, $\operatorname{Supp}(Z_\alpha * Z_\beta)$ is concentrated in the compact set $C \cap G$, where G is the reflection of C and translated by some vector on $\mathbf{R}^{n+1}$. That implies the convolution $Z_\alpha * Z_\beta$ exists.

Proof. It suffices to verify for $\operatorname{Re}\alpha$, $\operatorname{Re}\beta$ large enough. For $(x,t) \in \mathbf{R}^{n+1}$, let

$$\begin{aligned} T = \int_D &(\tau^2 - \xi_1^2 \cdots - \xi_n^2)^{(\alpha-n-1)/2} \\ &\cdot \big((t-\tau)^2 - (x_1-\xi_1)^2 \cdots - (x_n-\xi_n)^2\big)^{(\beta-n-1)/2} \, d\xi \, d\tau \end{aligned}$$

where $D = \{(\xi, \tau) \in \mathbf{R}^{n+1} : \tau \geq |\xi|, t - \tau \geq |x - \xi|\}$. By a rotation of the space axes followed by a two-dimensional Lorentz transformation,

$$T = \int_{D_1} (\tau^2 - \xi_1^2 \cdots - \xi_n^2)^{(\alpha-n-1)/2} ((s-\tau)^2 - \xi_1^2 - \cdots - \xi_n^2)^{(\beta-n-1)/2} \, d\xi \, d\tau$$

where $s = \sqrt{t^2 - |x|^2}$, $D_1 = \{(\xi, \tau) \in \mathbf{R}^{n+1} : \tau \geq |\xi|, s - \tau \geq |\xi|\}$. Thus, we have

$$T = \Omega_n \int_{D_2} (\tau^2 - \eta^2)^{(\alpha-n-1)/2} ((s-\tau)^2 - \eta^2)^{(\beta-n-1)/2} \eta^{n-1} \, d\eta \, d\tau$$

where $D_2 = \{(\eta, \tau) \in \mathbf{R}^2 : 0 \leq \tau - \eta \leq s, 0 \leq \tau + \eta \leq s\}$ and Ω_n is the hypersurface area of the unit sphere in $\mathbf{R}^n$. By the transformation $\tau + \eta = u$ and $\tau - \eta = v$,

$$\begin{aligned} T = \frac{\Omega_n}{2} \int_0^s \int_0^s u^{(\alpha-n-1)/2} (s-u)^{(\beta-n-1)/2} v^{(\alpha-n-1)/2} \\ \cdot (s-v)^{(\beta-n-1)/2} \left(\frac{u-v}{2} \right)^{n-1} du \, dv \\ = s^{\alpha+\beta-n-1} B_n(\alpha, \beta) \end{aligned}$$

where

$$\begin{aligned} B_n(\alpha, \beta) = \frac{\Omega_n}{2^n} \int_0^1 \int_0^1 u^{(\alpha-n-1)/2} (1-u)^{(\beta-n-1)/2} v^{(\alpha-n-1)/2} \\ \cdot (1-v)^{(\beta-n-1)/2} (u-v)^{n-1} \, du \, dv \end{aligned}$$

depends only on α, β, and n. We put $e^{-t} = e^{-(t-\tau)-\tau}$ into (1.4), then

$$H(\alpha, n) H(\beta, n) = \langle T, e^{-t} \rangle = B_n(\alpha, \beta) H(\alpha + \beta, n)$$

and so $B_n(\alpha,\beta) = H(\alpha,n)H(\beta,n)/H(\alpha+\beta,n)$. For $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$,

$$\begin{aligned}
\langle Z_\alpha * Z_\beta, \phi\rangle &= \frac{1}{H(\alpha,n)H(\beta,n)} \int_{s\geq|y|} (s^2-|y|^2)^{(\alpha-n-1)/2}\,dy\,ds \\
&\quad\cdot \int_{t\geq|x|} (t^2-|x|^2)^{(\beta-n-1)/2}\phi(x+y,t+s)\,dx\,dt \\
&= \frac{1}{H(\alpha,n)H(\beta,n)} \int_{s\geq|y|} (s^2-|y|^2)^{(\alpha-n-1)/2}\,dy\,ds \\
&\quad\cdot \int_{t-s\geq|x-y|} \big((t-s)^2-|x-y|^2\big)^{(\beta-n-1)/2}\phi(x,t)\,dx\,dt \\
&= \frac{1}{H(\alpha,n)H(\beta,n)} \int_{t\geq|x|} \phi(x,t)\,dx\,dt \\
&\quad\cdot \int_{(y,s)\in\mathcal{D}} (s^2-|y|^2)^{(\alpha-n-1)/2} \\
&\qquad\qquad \cdot \big((t-s)^2-|x-y|^2\big)^{(\beta-n-1)/2}\,dy\,ds \\
&= \frac{1}{H(\alpha+\beta,n)} \int_{t\geq|x|} \phi(x,t)(t^2-|x|^2)^{(\alpha+\beta-n-1)/2}\,dx\,dt \\
&= \langle Z_{\alpha+\beta}, \phi\rangle.
\end{aligned}$$

2. The solution of the Cauchy problem and $L^p - L^q$ estimate. As mentioned in the previous section, a fundamental solution of the wave equation is given by

$$E(x,t) = Z_2(x,t) = \frac{\rho^{1-n}}{2\pi^{(n-1)/2}\Gamma((3-n)/2)}$$

which can be used to solve the Cauchy problem

$$\begin{cases} \Box u(x,t) = w(x,t) & \text{for } t\geq 0 \\ u(x,0) = f(x) \\ \partial_t u(x,0) = g(x) \end{cases} \tag{2.1}$$

where w is a function on $\mathbf{R}^{n+1}$ that vanishes for $t<0$, f and g are functions on $\mathbf{R}^n$, all of those are assumed to be sufficiently differentiable.

The Riesz distribution can be interpreted as an operator. For any $\alpha \in \mathbf{C}$, any $\phi \in \mathcal{D}(\mathbf{R}^{n+1})$ or $\mathcal{S}(\mathbf{R}^{n+1})$ with support contained in a translation of the upper half space $t \geq 0$, we define

$$I^\alpha \phi = Z_\alpha * \phi$$

which is known as the *Riesz integral.* Corresponding to (1.5), (1.6), (1.7), and Theorem 1.8, the Riesz integral I^α has the following properties:

$$\begin{aligned} \Box I^\alpha &= I^{\alpha-2}, & \Box^k I^\alpha &= I^{\alpha-2k} \\ I^0 &= \text{identity}, & I^{-2k} &= \Box^k \\ I^2\Box &= \Box I^2 = \text{identity}, & I^{2k}\Box^k &= \Box^k I^{2k} = \text{identity} \\ I^\alpha I^\beta &= I^{\alpha+\beta} \end{aligned}$$

for $k = 0, 1, 2, \ldots$.

For $(x,t) \in \mathbf{R}^{n+1}$, let $\Omega = \{(\xi,\tau) \in \mathbf{R}^{n+1} : t-\tau \geq |x-\xi|, 0 \leq \tau \leq t\}$, then $\partial\Omega = \{(\xi,\tau) \in \mathbf{R}^{n+1} : t-\tau = |x-\xi|, 0 \leq \tau \leq t\} \cup \{(\xi,0) \in \mathbf{R}^{n+1} : |x-\xi| \leq t\} \equiv B_1 \cup B_2$. We use Green's formula

$$\int_\Omega (u\Box v - v\Box u)\, dV = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial N} - v\frac{\partial u}{\partial N}\right) dS.$$

Let $v = v(\xi,\tau) = (1/H(\alpha+2,n))((t-\tau)^2 - |x-\xi|^2)^{(\alpha-n+1)/2}$. Then $\Box_{(\xi,\tau)} v(\xi,\tau) = (1/H(\alpha,n))((t-\tau)^2 - |x-\xi|^2)^{(\alpha-n-1)/2}$. Now $v = 0$, $\partial v/\partial N = 0$ on B_1. Hence

(2.2)

$$\begin{aligned} &\frac{1}{H(\alpha,n)} \int_{t-\tau\geq|x-\xi| \text{ and } \tau\geq 0} u(\xi,\tau)\big((t-\tau)^2 - |x-\xi|^2\big)^{(\alpha-n-1)/2}\, d\xi\, d\tau \\ &\qquad - \frac{1}{H(\alpha+2,n)} \int_{t-\tau\geq|x-\xi| \text{ and } \tau\geq 0} \Box u(\xi,\tau)\big((t-\tau)^2 \\ &\qquad\qquad \cdot -|x-\xi|^2\big)^{(\alpha-n+1)/2}\, d\xi\, d\tau \\ &\quad = \frac{1}{H(\alpha+2,n)} \int_{|x-\xi|\leq t} (t^2 - |x-\xi|^2)^{(\alpha-n+1)/2} \partial_\tau u(\xi,0)\, d\xi \\ &\qquad + \frac{\alpha-n+1}{H(\alpha+2,n)}\, t \int_{|x-\xi|\leq t} (t^2 - |x-\xi|^2)^{(\alpha-n-1)/2} u(\xi,0)\, d\xi. \end{aligned}$$

Let u be the desired solution of (2.1) that vanishes for $t < 0$, taking $\alpha = 0$ in (2.2),

(2.3)

$$\begin{aligned} u(x,t) &= I^2 w(x,t) + \frac{1}{H(2,n)} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} g(\xi)\, d\xi \\ &\quad + \frac{1-n}{H(2,n)} t \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(-1-n)/2} f(\xi)\, d\xi \\ &= I^2 w(x,t) + \frac{1}{H(2,n)} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} g(\xi)\, d\xi \\ &\quad + \frac{1}{H(2,n)} \frac{d}{dt} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} f(\xi)\, d\xi \\ &= (E *_{(x,t)} w)(x,t) + (E *_{(x)} g)(x,t) + \left(\frac{\partial E}{\partial t} *_{(x)} f\right)(x,t) \end{aligned}$$

where $*_{(x,t)}$ and $*_{(x)}$ denote the convolution with respect to variable (x,t) or x only.

Moreover, if n is even, $g \in C^{(n+2)/2}(\mathbf{R}^n)$, and $f \in C^{(n+4)/2}(\mathbf{R}^n)$, we can write

$$\begin{aligned} &\frac{1}{H(2,n)} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} g(\xi)\, d\xi \\ &\qquad = (2\pi)^{-n/2} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} t^{n-1} \int_{|\xi|\le 1} \frac{g(x+t\xi)}{\sqrt{1-|\xi|^2}}\, d\xi \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{H(2,n)} \frac{\partial}{\partial t} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} f(\xi)\, d\xi \\ &\qquad = (2\pi)^{-n/2} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-2)/2} t^{n-1} \int_{|\xi|\le 1} \frac{f(x+t\xi)}{\sqrt{1-|\xi|^2}}\, d\xi. \end{aligned}$$

If n is odd and $n \ge 3$, $g \in C^{(n+1)/2}(\mathbf{R}^n)$ and $f \in C^{(n+3)/2}(\mathbf{R}^n)$, then we have

$$\begin{aligned} &\frac{1}{H(2,n)} \int_{|x-\xi|\le t} (t^2 - |x-\xi|^2)^{(1-n)/2} g(\xi)\, d\xi \\ &\qquad = \frac{1}{2} (2\pi)^{(1-n)/2} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2} t^{n-2} \int_{|\xi|=1} g(x+t\xi)\, d\sigma(\xi) \end{aligned}$$

and

$$\begin{aligned}\frac{1}{H(2,n)}\frac{\partial}{\partial t}\int_{|x-\xi|\le t}(t^2-|x-\xi|^2)^{(1-n)/2}f(\xi)\,d\xi \\ = \frac{1}{2}(2\pi)^{(1-n)/2}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(n-3)/2}t^{n-1}\int_{|\xi|=1}f(x+t\xi)\,d\sigma(\xi).\end{aligned}$$

If also $w(x,t)$ belongs to $C^{[n/2]+1}(\mathbf{R}^{n+1})$ that vanishes for $t<0$, then (2.3) is the classical solution of the Cauchy problem (2.1).

For the case $n=0$, the Riesz integral has the form

$$I_1^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\,dt$$

which is also called *Riemann-Liouville integral*. Hardy-Littlewood [**1**] has proved that I_1^α is of type (p,q) with

$$1/q = 1/p-\alpha, \qquad 0<\alpha<1/p,\ p>1.$$

We expect the Riesz integral with $\alpha=2$

$$\begin{aligned}I^2\phi(x,t) &= Z_2 * \phi(x,t) \\ &= \frac{1}{H(2,n)}\int_{t-\tau\ge|x-\xi|}((t-\tau)^2-|x-\xi|^2)^{(1-n)/2}\phi(\xi,\tau)\,d\xi\,d\tau\end{aligned}$$

has similar property for $1/q = 1/p-2/(n+1)$ and $1/p+1/q=1$.

By Stein-Weiss [**4**, p. 171, Theorem 4.15],

$$\mathcal{F}((1-|x|^2)_+^\lambda)(\xi) = (2\pi)^{n/2}2^\lambda\Gamma(\lambda+1)|\xi|^{(-2\lambda-n)/2}J_{(2\lambda+n)/2}(|\xi|)$$

and so

$$\begin{aligned}\mathcal{F}_x((t^2-|x|^2)_+^{\lambda/2})(\xi) &= t^\lambda\mathcal{F}_x(\delta_{t^{-1}}(1-|x|^2)_+^{\lambda/2})(\xi) \\ &= t^{\lambda+n}\delta_t\mathcal{F}_x((1-|x|^2)_+^{\lambda/2})(\xi) \\ &= t^{\lambda+n}\delta_t\Big((2\pi)^{n/2}2^{\lambda/2}\Gamma(\lambda/2+1) \\ &\qquad\cdot|\xi|^{(-\lambda-n)/2}J_{(\lambda+n)/2}(|\xi|)\Big) \\ &= (2\pi)^{n/2}2^{\lambda/2}\Gamma(\lambda/2+1)\left|\frac{\xi}{t}\right|^{(-\lambda-n)/2}J_{(\lambda+n)/2}(|t\xi|)\end{aligned}$$

where $\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbf{R}^n} e^{-ix\xi}f(x)\,dx$ denotes the Fourier transform, $\mathcal{F}_x\phi(\xi,t)=\tilde{\phi}(\xi,t)$ the Fourier transform with respect to x, $\mathcal{F}_\xi^{-1}$ the inverse Fourier transform with respect to ξ, and $\delta_t f(x)=f(tx)$. Hence

$$\tilde{E}(\xi,t)=\frac{1}{2\pi^{(n-1)/2}\Gamma((3-n)/2)}\mathcal{F}_x((t^2-|x|^2)_+^{(1-n)/2})(\xi)=\frac{\sin|\xi|t}{|\xi|}$$

and so by (2.3)

$$\begin{aligned}\tilde{u}(\xi,t)&=\int_{-\infty}^{\infty}\tilde{E}(\xi,t-s)\tilde{w}(\xi,s)\,ds+\frac{\sin|\xi|t}{|\xi|}\hat{g}(\xi)+\cos|\xi|t\cdot\hat{f}(\xi)\\&=\int_0^t\frac{\sin|\xi|(t-s)}{|\xi|}\tilde{w}(\xi,s)\,ds+\frac{\sin|\xi|t}{|\xi|}\hat{g}(\xi)+\cos|\xi|t\cdot\hat{f}(\xi).\end{aligned}$$

Strichartz [**5**] has proved that

$$\begin{aligned}||\mathcal{F}_\xi^{-1}\left(\frac{\sin|\xi|}{|\xi|}\hat{g}(\xi)\right)||_q&\leqq C_1||g||_p\\||\mathcal{F}_\xi^{-1}(\cos|\xi|\cdot\hat{f}(\xi))||_q&\leqq C_2||\nabla f||_p\end{aligned}$$

for $1/p+1/q=1$ and $1/q=1/p-2/(n+1)$, $n\geqq 2$. Hence the solution $u(x,t)$ of problem (2.1) with $w=0$ satisfies that

$$\begin{aligned}||u||_q&\leqq||\mathcal{F}_\xi^{-1}\left(\frac{\sin|\xi|t}{|\xi|}\hat{g}(\xi)\right)||_q+||\mathcal{F}_\xi^{-1}\big(\cos|\xi|t\cdot\hat{f}(\xi)\big)||_q\\&=||t^n\mathcal{F}_\xi^{-1}\left(\frac{\sin|\xi|t}{|\xi|}t^{-n}\hat{g}(\xi)\right)||_q\\&\quad+||t^n\mathcal{F}_\xi^{-1}\big(\cos|\xi|t\cdot t^{-n}\hat{f}(\xi)\big)||_q\\&=||t^{n+1}\mathcal{F}_\xi^{-1}\delta_t\left(\frac{\sin|\xi|}{|\xi|}\cdot t^{-n}\delta_{t^{-1}}\hat{g}(\xi)\right)||_q\\&\quad+||t^n\mathcal{F}_\xi^{-1}\delta_t\big(\cos|\xi|\cdot t^{-n}\delta_{t^{-1}}\hat{f}(\xi)\big)||_q\\&=||t\delta_{t^{-1}}\mathcal{F}_\xi^{-1}\left(\frac{\sin|\xi|}{|\xi|}\widehat{\delta_t g}(\xi)\right)||_q\\&\quad+||\delta_{t^{-1}}\mathcal{F}_\xi^{-1}\big(\cos|\xi|\cdot\widehat{\delta_t f}(\xi)\big)||_q\\&=t^{1+(n/q)}||\mathcal{F}_\xi^{-1}\left(\frac{\sin|\xi|}{|\xi|}\widehat{\delta_t g}(\xi)\right)||_q\end{aligned}\tag{2.4}$$

$$
\begin{aligned}
&+ t^{n/q}||\mathcal{F}_\xi^{-1}\big(\cos|\xi|\cdot\widehat{\delta_t f}(\xi)\big)||_q \\
&\leqq C_1 t^{1+(n/q)}||\delta_t g||_p + C_2 t^{n/q}||\nabla(\delta_t f)||_p \\
&= C_1 t^{1+(n/q)-(n/p)}||g||_p + C_2 t^{1+(n/q)-(n/p)}||\nabla f||_p.
\end{aligned}
$$

On the other hand, let $f = g = 0$ in problem (2.1), by Jensen's inequality and (2.4)

$$
\begin{aligned}
||u(\cdot,t)||_q &= ||\mathcal{F}_\xi^{-1}\int_0^t \frac{\sin|\xi|(t-s)}{|\xi|}\tilde{w}(\xi,s)\,ds||_q \\
&\leqq \int_0^t ||\mathcal{F}_\xi^{-1}\bigg(\frac{\sin|\xi|(t-s)}{|\xi|}\tilde{w}(\xi,s)\bigg)||_q\,ds \\
&\leqq C_3\int_0^t (t-s)^{1+(n/q)-(n/p)}||w(\cdot,s)||_p\,dx.
\end{aligned}
$$

The last integral is exactly the Riemann-Liouville integral $I_1^{2/(n+1)}(||w(\cdot,t)||_p)$ when $1+n/q-n/p = 1-(2n)/(n+1) = (1-n)/(n+1)$. Now we take L^q-norm with respect to the time variable t, then

$$
||u||_q \le C_3||I_1^{2/(n+1)}(||w(\cdot,t)||_p)||_q \le C_4||w||_p \tag{2.5}
$$

since $I_1^{2/(n+1)}$ is of type (p,q) with $1/q = 1/p - 2/(n+1)$.

Combining (2.4) and (2.5) we obtain the global space-time estimate for the Cauchy problem (2.1)

$$
||u||_q \le C(||w||_p + t^{(1-n)/(n+1)}(||g||_p + ||\nabla f||_p))
$$

where $1/q = 1/p - 2/(n+1)$ and $1/p + 1/q = 1$.

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