

**PHASE PORTRAITS FOR QUADRATIC SYSTEMS
HAVING A FOCUS AND ONE ANTISADDLE**

JOAN C. ARTÉS AND JAUME LLIBRE

ABSTRACT. We determine all possible phase portraits for quadratic systems having a focus and one antisaddle modulus limit cycles and antisaddle behavior.

1. Introduction. In 1984 Cherkas and Gaiko [9] proved that if a quadratic system has a focus (or a center) at the origin and no other finite critical point except an antisaddle (an elementary critical point with index +1), it can be transformed by a linear change of variables into the form

$$(1) \quad \begin{aligned} x' &= \alpha x - y - \alpha x^2 + (a + \alpha\gamma)xy + (b - \gamma + c\alpha)y^2 = P(x, y), \\ y' &= x + \alpha y - x^2 + (\gamma - a\alpha)xy + (\alpha\gamma + c - b\alpha)y^2 = Q(x, y), \end{aligned}$$

where

$$(2) \quad b^2 - 4(a - 1)c < 0 \quad \text{and} \quad a > 1.$$

When $\alpha = 0$ and the three focal quantities at $(0, 0)$ are zero, the origin is a center. In general, the problem of separating a focus from a center, and once the focus is separated from the center, to discriminate the order and stability of the nonhyperbolic (weak) focus, is not easy. For quadratic systems this problem was solved partially by Kapteyn [17] and completely by Bautin [6] (see also Li Chengzhi [19]), by using the three independent focal quantities associated to a center of a quadratic system.

Also, for suitable values of the parameters, the antisaddle of system (1) can become a center. Since the quadratic systems having a center were classified by Vulpe [30] we do not consider them in this paper.

Our main result is the following one.

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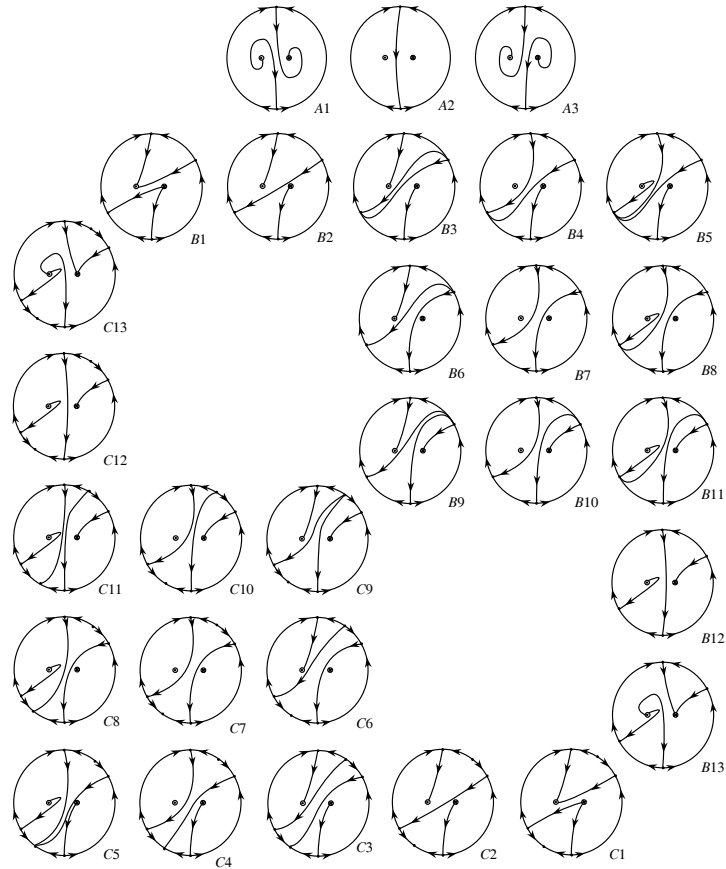


FIGURE 1. The 29 phase portraits on the Poincaré disc of the quadratic systems having one focus and one antisaddle different from a center. Here \circ denotes a focus surrounded by perhaps several limit cycles, and \bullet denotes either \circ or a node.

Theorem A. *The phase portrait of any quadratic system having one focus and one antisaddle different from a center is topologically equivalent to one of the 29 configurations of Figure 1 without taking into account limit cycles and the nature of the antisaddle. Moreover, each of the configurations of Figure 1 is realizable for a quadratic system.*

Theorem A is proved in the next section.

Other families of quadratic systems have been studied: bounded quadratic systems (see [14] and [10]), homogeneous quadratic systems (see [21], [13], and [29]), quadratic systems having a star nodal point [7], chordal quadratic systems [15], quadratic systems with a unique finite critical point [11], quadratic systems with a weak focus of third order [4], quadratic systems with four finite critical points and one invariant straight line (see [5] and [27]), Hamiltonian quadratic systems [1], gradient quadratic systems [2], For more information on quadratic systems and their applications, see the good bibliographical survey of J.W. Reyn [25] containing a summary of approximately 600 papers on quadratic systems.

2. Proof of Theorem A. Let

$$\begin{aligned}
 (3) \quad x' &= \sum_{\substack{i+j=0 \\ i,j \geq 0}}^2 a_{ij} x^i y^j = P(x, y), \\
 y' &= \sum_{\substack{i+j=0 \\ i,j \geq 0}}^2 b_{ij} x^i y^j = Q(x, y),
 \end{aligned}$$

be a quadratic system. If (x_0, y_0) is a critical point of (3), then x_0 is a root of the resultant of P and Q with respect to y , i.e., of $R(P, Q)(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$; and y_0 is a root of the resultant of P and Q with respect to x , i.e., of $R(P, Q)(y) = A'y^4 + B'y^3 + C'y^2 + D'y + E'$. We remark that

$$(4) \quad A = A' = \begin{vmatrix} a_{20} & a_{02} \\ b_{20} & b_{02} \end{vmatrix}^2 - \begin{vmatrix} a_{20} & a_{11} \\ b_{20} & b_{11} \end{vmatrix} \begin{vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{vmatrix}.$$

Proposition 1. *A quadratic system with a focus or a center and no other finite critical point except an antisaddle has $A < 0$.*

Proof. From [9] such a quadratic system can be written as system (1) satisfying conditions (2). The resultant of $P(x, y)$ and $Q(x, y)$ in (1)

with respect to the variable x is $Ay^4 + B'y^3 + C'y^2$ where A is given by (4). Notice that the coefficient of y and the independent term of the above resultant are zero because $(0, 0)$ and $(0, 1)$ are critical points of system (1). After some easy calculations from (2), we have

$$\begin{aligned} C' &= -(\alpha^2 + 1)^2(a - 1) < 0, \\ B'^2 - 4AC' &= a^2(\alpha^2 + 1)^4[b^2 - 4(a - 1)c] < 0. \end{aligned}$$

Hence, $A < 0$. \square

As usual, we compactify the quadratic system to a system on the Poincaré sphere, obtaining two copies of the phase portrait, one on the northern hemisphere and the other on the southern hemisphere. The equator S^1 of the Poincaré sphere is invariant for the compactified system, and the critical points of the compactified system on S^1 are called *infinite critical points* of the quadratic system; for more details see, for instance, [16]. We represent the phase portrait of any quadratic system on the closed northern hemisphere of the Poincaré sphere, also called the *Poincaré disc*.

In order to obtain the analytic expression of the compactified system, the sphere S^2 is considered as a differential manifold. We consider six local charts given by $U_k = \{y \in S^2 : y_k > 0\}$, $V_k = \{y \in S^2 : y_k < 0\}$ for $k = 1, 2, 3$. The corresponding coordinate maps $F_k : U_k \rightarrow \mathbf{R}^2$ and $G_k : V_k \rightarrow \mathbf{R}^2$, are defined by $F_k(y) = G_k(y) = (y_m y_k^{-1}, y_n y_k^{-1})$ for $m < n$ and $m, n \neq k$. We shall denote by $z = (z_1, z_2)$ the value of $F_k(y)$ or $G_k(y)$ for any k , so that z represents different things according to the local chart that we are considering. Note that the points of S^1 in any local chart have $z_2 = 0$.

The compactified system in the local chart (U_1, F_1) is given by

$$(5) \quad \begin{aligned} z'_1 &= z_2^2 \left[-z_1 P \left(\frac{1}{z_2}, \frac{z_1}{z_2} \right) + Q \left(\frac{1}{z_2}, \frac{z_1}{z_2} \right) \right], \\ z'_2 &= -z_2^3 P \left(\frac{1}{z_2}, \frac{z_1}{z_2} \right). \end{aligned}$$

The expression for (U_2, F_2) is

$$(6) \quad \begin{aligned} z'_1 &= z_2^2 \left[P\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) - z_1 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right) \right], \\ z'_2 &= -z_2^3 Q\left(\frac{z_1}{z_2}, \frac{1}{z_2}\right), \end{aligned}$$

and for (U_3, F_3) is

$$z'_1 = P(z_1, z_2), \quad z'_2 = Q(z_1, z_2).$$

For $k = 1, 2, 3$, the expression in (V_k, G_k) is the same as that in (U_k, F_k) multiplied by -1 .

In this paper we say that the phase portraits of two quadratic systems are *topologically equivalent* if there exists a homeomorphism of the Poincaré sphere carrying orbits of one phase portrait onto orbits of the other one, preserving sense but not necessarily parameterization.

After proving the next proposition, we saw that it follows from Theorem 12 of Reyn [24] but for the sake of completeness we will prove it.

Proposition 2. *The infinite critical points of any quadratic system with $A \neq 0$ have nonzero eigenvalues associated to the eigenvector transversal to infinity.*

Proof. Let $(z_1, 0)$ be an infinite critical point of (3) in the local chart U_1 . Then z_1 is a root of the polynomial

$$F(z) = Q_2(1, z) - zP_2(1, z),$$

where P_2 and Q_2 are the homogeneous parts of P and Q of degree 2 (see (5)). Therefore,

$$(7) \quad F(z) = -a_{02}z^3 - (a_{11} - b_{02})z^2 - (a_{20} - b_{11})z + b_{20}.$$

From (5) the eigenvalue associated to the eigenvector transversal to infinity at $(z_1, 0)$ is

$$-P_2(1, z_1) = -(a_{02}z_1^2 + a_{11}z_1 + a_{20}).$$

So, from (7), this eigenvalue will be zero if and only if z_1 satisfies

$$(8) \quad \begin{aligned} a_{02}z^2 + a_{11}z + a_{20} &= 0, \\ b_{02}z^2 + b_{11}z + b_{20} &= 0. \end{aligned}$$

We claim that if system (8) has a solution then A (defined in (4)) is zero. Therefore, from the claim, the proposition follows for the infinite critical points in U_1 .

Now we shall prove the claim. First, if $a_{02} = b_{02} = 0$, then $A = 0$ (see (4)). Suppose that $a_{02} = 0$ and $b_{02} \neq 0$, the case $a_{02} \neq 0$ and $b_{02} = 0$ follows in a similar way. Clearly $A = 0$ if $a_{11} = a_{20} = 0$. Therefore, in order that system (8) has a solution $z_1 = \alpha$, we must assume that $a_{11} \neq 0$. Then system (8) becomes

$$a_{11}(z - \alpha) = 0, \quad b_{02}(z^2 - (\alpha + \beta)z + \alpha\beta) = 0,$$

with $\alpha = -a_{20}/a_{11}$, $b_{11} = -b_{02}(\alpha + \beta)$ and $b_{20} = b_{02}\alpha\beta$. Therefore

$$A = \begin{vmatrix} -a_{11}\alpha & 0 \\ b_{02}\alpha\beta & b_{02} \end{vmatrix}^2 - \begin{vmatrix} -a_{11}\alpha & a_{11} \\ b_{02}\alpha\beta & -b_{02}(\alpha + \beta) \end{vmatrix} \begin{vmatrix} a_{11} & 0 \\ -b_{02}(\alpha + \beta) & b_{02} \end{vmatrix} = 0.$$

Hence, we can assume that $a_{02} \neq 0$ and $b_{02} \neq 0$, and consequently, system (8) goes over to

$$\begin{aligned} a_{02}(z - \alpha)(z - \beta) &= 0, \\ b_{02}(z - \alpha)(z - \gamma) &= 0, \end{aligned}$$

with $a_{11} = -a_{02}(\alpha + \beta)$, $a_{20} = a_{02}\alpha\beta$, $b_{11} = -b_{02}(\alpha + \gamma)$ and $b_{20} = b_{02}\alpha\gamma$. Therefore

$$A = \begin{vmatrix} a_{02}\alpha\beta & a_{02} \\ b_{02}\alpha\gamma & b_{02} \end{vmatrix}^2 - \begin{vmatrix} a_{02}\alpha\beta & -a_{02}(\alpha + \beta) \\ b_{02}\alpha\gamma & -b_{02}(\alpha + \gamma) \end{vmatrix} \begin{vmatrix} -a_{02}(\alpha + \beta) & a_{02} \\ -b_{02}(\alpha + \gamma) & b_{02} \end{vmatrix} = 0.$$

In short, we have proved the claim.

To end the proof of the proposition we need to show that if the origin of the local chart U_2 is an infinite critical point then its eigenvalue associated to the eigenvector transversal to infinity is nonzero.

Let $(z_1, 0)$ be an infinite critical point of (3) in the local chart U_2 . Then z_1 is a root of the polynomial

$$G(z) = P_2(z, 1) - zQ_2(z, 1),$$

(see (6)); i.e.,

$$G(z) = -b_{20}z^3 - (b_{11} - a_{20})z^2 - (b_{02} - a_{11})z + a_{02}.$$

From (6) the eigenvalue associated to the eigenvector transversal to infinity at $(z_1, 0)$ is

$$-Q_2(z_1, 1) = -(b_{20}z_1^2 + b_{11}z_1 + b_{02}).$$

So, from $G(z_1) = 0$, this eigenvalue will be zero if and only if z_1 satisfies

$$(9) \quad \begin{aligned} a_{20}z^2 + a_{11}z + a_{02} &= 0, \\ b_{20}z^2 + b_{11}z + b_{02} &= 0. \end{aligned}$$

Now we need to show that if $z_1 = 0$ is a solution of system (9), then $A = 0$. Since the solution 0 implies $a_{02} = b_{02} = 0$, from (4) it follows immediately that $A = 0$. \square

A critical point (x_0, y_0) is called *elementary* if

$$\det \Big|_{(x_0, y_0)} = \frac{\partial P}{\partial x}(x_0, y_0) \frac{\partial Q}{\partial y}(x_0, y_0) - \frac{\partial P}{\partial y}(x_0, y_0) \frac{\partial Q}{\partial x}(x_0, y_0) \neq 0.$$

It is well-known that an elementary critical point must be either a saddle, a node, a focus or a center. The critical point is called *semi-elementary* if $\det \Big|_{(x_0, y_0)} = 0$ and

$$\text{tr} \Big|_{(x_0, y_0)} = \frac{\partial P}{\partial x}(x_0, y_0) + \frac{\partial Q}{\partial y}(x_0, y_0) \neq 0.$$

The semi-elementary critical points can only be either a saddle, a node or a saddle-node. For more details on elementary and semi-elementary critical points, see [3].

From the well-known Poincaré-Hopf theorem (see, for instance, [18]), we have that if the compactified system on the Poincaré sphere S^2

associated to system (3) has finitely many critical points, then the sum of the indices of all its critical points is equal to 2. Now we will apply this result to our quadratic system (1).

First we will show that the compactified system associated to system (1) has finitely many critical points. We know that system (1) has only two finite critical points. From (5) the infinite critical points $(z_1, 0)$ of (1) in the local chart U_1 must be roots of the polynomial

$$\begin{aligned} F(z) &= Q_2(1, z) - zP_2(1, z) \\ &= -(-\gamma + b + c\alpha)z^3 + (-a + c - b\alpha)z^2 \\ &\quad + (\alpha + \gamma - a\alpha)z - 1. \end{aligned}$$

Since $F(z)$ is not identically zero it follows that there are at most three infinite critical points on U_1 . Therefore, from (5) and (6) system (1) has at most three pairs of diametrically opposite infinite critical points. Notice that if system (1) has an infinite critical point, then its diametrically opposite point on the Poincaré sphere is also critical and has the same local phase portrait reparameterized in converse sense. In particular, an infinite critical point and its diametrically opposite have the same index.

Since both finite critical points of system (1) have index 1, the sum of the indices of the critical points on the northern and southern open hemispheres for the compactified system associated to (1) is 4. As the sum of the indices of all its critical points is 2, it follows that the sum of the indices of all its infinite critical points is -2 .

From Propositions 1 and 2 we get that all infinite critical points of system (1) are elementary or semi-elementary, and since the sum of their indices is -2 , we only have the following three possibilities for the infinite critical points of system (1):

- (A) a pair of diametrically opposite saddles,
- (B) one pair of diametrically opposite saddles and one pair of diametrically opposite saddle-nodes,
- (C) two pairs of diametrically opposite saddles and one pair of diametrically opposite nodes.

Notice that from Coll [12] (see also [26]) it is not possible to have a quadratic system with one pair of diametrically opposite saddles and two pairs of diametrically opposite saddle-nodes at infinity.

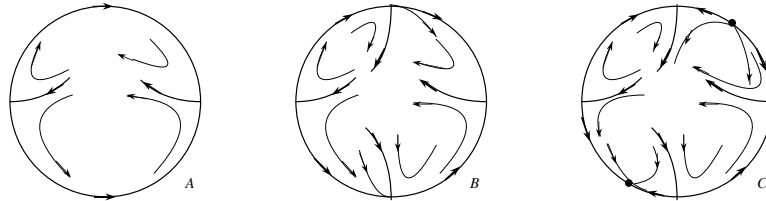


FIGURE 2. Local behavior at infinity; 2 separatrices for Case A and 4 for Cases B and C.

The closed northern hemisphere of the Poincaré sphere is called the *Poincaré disk*. As usual we draw the phase portrait of a polynomial system on the Poincaré disk.

The local behavior of the flow of system (1) in a neighborhood of infinity is given in Figure 2 according with the Case A, B or C. We remark that the saddle-nodes at infinity with non-zero eigenvalue associated to the eigenvector transversal to infinity must have local phase portrait as in Figure 2(B), for more details see Theorem 65 of [3].

From the local behavior of the flow of system (1) in a neighborhood of infinity, we will prove that only the 29 phase portraits (modulus limit cycles) given in Figure 1 are realizable for system (1); 3 in Case A and 13 for each of the Cases B and C. For more details, see [20] and [22].

In Case A we have two separatrices of the two saddles at infinity contained in the interior of the Poincaré disc, one stable γ^s and one unstable γ^u , and two finite critical points with index +1 (nodes or foci). We must try to find the α -limit of γ^s and the ω -limit of γ^u . The α -limit of γ^s can only be the focus, the antisaddle or the saddle point at infinity with separatrix γ^u , thus $\gamma^s = \gamma^u$ in the latter case. Each one of these options forces the ω -limit of γ^u , giving the phase portraits A.1, A.2 and A.3.

In Case B there are two pairs of infinite critical points, the pair s and s' of saddles and the pair sn and sn' of saddle-nodes. They define four separatrices contained in the interior of the Poincaré disc, two

stable $\gamma^{s_1}, \gamma^{s_2}$ and two unstable $\gamma^{u_1}, \gamma^{u_2}$. Moreover, we have two finite critical points with index +1 (nodes or foci). The pair of saddle-nodes at infinity defines two parabolic sectors contained in the interior of the Poincaré disc, one stable and the other unstable. If we follow the infinity S^1 in counterclockwise direction, we find the above four separatrices with the following ordering $\gamma^{s_1}, \gamma^{s_2}, \gamma^{u_1}$ and γ^{u_2} . The separatrices γ^{s_1} and γ^{u_1} are defined by the pair of saddle-nodes at infinity, and the separatrices γ^{s_2} and γ^{u_2} by the pair of saddles at infinity. The attractor parabolic sector has γ^{s_1} as a separatrix and the repeller parabolic sector has γ^{u_1} as a separatrix. We must find the α -limit of γ^{s_1} and γ^{s_2} and the ω -limit of γ^{u_1} and γ^{u_2} .

From Lemma 11.5 of [31] it follows for quadratic systems that a noninvariant straight line under the flow passing through a finite critical point and ending at a pair of infinite critical points cannot contain any other finite critical point. Since one of the two finite critical points of system (1) is a focus, it follows that there must exist a straight line M ending at the pair of infinite saddles which separates the plane into two open half-planes, each one containing one finite critical point. In what follows we denote by r and l the two finite critical points. Moreover, we may assume that l is the focus and r is the antisaddle (the other possibility would follow by doing an axial symmetry with respect to the straight line M). We say that an orbit has α - or ω -limit R (respectively L) when its α - or ω - limit is either the point r (respectively l) or a limit cycle surrounding it.

Suppose that R is the α -limit of γ^{s_2} . Then γ^{s_1} may have up to five different α -limits, namely,

- (1) R ,
- (2) sn' and $\gamma^{s_1} = \gamma^{u_1}$,
- (3) sn' and $\gamma^{s_1} \neq \gamma^{u_1}$,
- (4) s' and $\gamma^{s_1} = \gamma^{u_2}$,
- (5) L .

Once its α -limit is decided, both unstable separatrices γ^{u_1} and γ^{u_2} have forced their ω -limits. These five different possibilities give the phase portraits B.1, B.2, B.3, B.4 and B.5.

Suppose that sn' is the α -limit of γ^{s_2} and $\gamma^{s_2} = \gamma^{u_1}$. Since the sum of the indices of the critical points inside the graphic formed by

$\gamma^{s_2} = \gamma^{u_1}$, s , sn' and the orbit at infinity going from s to sn' , is +1 (see for instance [23, p. 279]), r is the unique critical point inside such a graphic. Then γ^{s_1} may have up to three different α -limits, namely

- (1) sn' and $\gamma^{s_1} \neq \gamma^{u_1}$,
- (2) s' and $\gamma^{s_1} = \gamma^{u_2}$,
- (3) L .

These three different possibilities give the phase portraits B.6, B.7 and B.8.

Assume that sn' is the α -limit of γ^{s_2} and $\gamma^{s_2} \neq \gamma^{u_1}$. By similar arguments to the ones of the previous case, r is the unique critical point inside the region bounded by γ^{s_2} , s , sn' and the orbit at infinity going from s to sn' . Then γ^{s_1} may have up to three different α -limits, namely

- (1) sn' and $\gamma^{s_1} \neq \gamma^{u_1}$,
- (2) s' and $\gamma^{s_1} = \gamma^{u_2}$,
- (3) L .

These three different possibilities give the phase portraits B.9, B.10 and B.11.

Assume that s' is the α -limit of γ^{s_2} . Then $\gamma^{s_2} = \gamma^{u_2}$ is an invariant straight line by the flow (see [28]). Consequently, we get the phase portrait B.12.

Suppose that L is the α -limit of γ^{s_2} . Then γ^{s_1} may have up to five different α -limits, namely,

- (1) R ,
- (2) sn' and $\gamma^{s_1} = \gamma^{u_1}$,
- (3) sn' and $\gamma^{s_1} \neq \gamma^{u_1}$,
- (4) s' and $\gamma^{s_1} = \gamma^{u_2}$,
- (5) L .

However, we will see that the former four cases are not possible. From the proof of Lemma 11.5 of [31], it follows for quadratic systems that a noninvariant straight line under the flow ending in two infinite critical points, can only have one contact point (i.e., the vector field at that point is tangent to the straight line). If we draw the phase portraits in

the first four cases, we get that on the straight line M (going from s to s' and separating the points r and l), there are at least two contact points, a contradiction. Then, the possibility (5) forces the phase portrait B.13.

Case C can be argued exactly in the same way as Case B, just substituting the behavior of the nodal sectors at infinity in Case B by the nodal points in Case C.

We can realize all 29 cases. First we realize the 3 phase portraits of Case A, and all 13 of Case B. Either by a perturbation of these systems or by proceeding in a similar way, we can realize the 13 phase portraits of Case C.

Let $\gamma = b + c\alpha$, $a = 2$, $b = 0$ and $c = 1$. Then system (1) has phase portrait

- A.1. if $\alpha = -0.1$,
- A.2. if $\alpha = 0$,
- A.3. if $\alpha = 0.1$.

Let $a = 3$, $b = 1$, $\gamma = b + c\alpha$ and c be the convenient real solution of

$$\alpha^2 c^2 - 2(a\alpha^2 - b\alpha - \alpha^2 - 2)c + (a\alpha - b)^2 - 2a\alpha^2 - 2b\alpha + \alpha^2 = 0.$$

Then system (1) has phase portrait

- B.1. if $\alpha = -0.1$,
- B.2. if $\alpha = 0$,
- B.3. if $\alpha = 0.1$,
- B.4. if $\alpha \approx 0.134$,
- B.5. if $\alpha = 0.2$.

Similarly, let $a = 3$, $b = 1$, $\gamma = b + c\alpha$ and $c = a + b\alpha$. Then system (1) has phase portrait

- B.13. if $\alpha = -0.1$,
- B.12. if $\alpha = 0$,
- B.11. if $\alpha = 0.1$,
- B.10. if $\alpha \approx 0.2534$,
- B.9. if $\alpha = 0.3$.

In the first group, we have put a saddle on the point $(0, 0)$ of the local chart U_2 while, in the second group we have put a saddle-node there. If we put in both cases a saddle, or a saddle-node on the point $(0, 0)$ of the local cart U_2 , then the coefficients become more complicated. However, as it is possible to have both cases in this same position, then we deduce, by continuity arguments that there must exist Cases B.6 and B.8, the former by continuity between the Cases B.3 and B.9, and the latter by continuity between the Cases B.5 and B.11.

Finally, Case B.7 appears in the following way. Shen Bo-qian [8] proves that the system

$$\begin{aligned}x' &= \alpha_1(xy - 1) + \beta_1 + \beta_2x + \beta_3x^2, \\y' &= \alpha_2(xy - 1) - \beta_3 - \beta_2y - \beta_1y^2,\end{aligned}$$

has the hyperbola $xy = 1$ as an integral curve. Moreover, if $\beta_2^2 - 4\beta_1\beta_3 < 0$, then there are only two foci and no other finite critical points. Then, for example, if $\alpha_2 = \beta_1 = \beta_3 = 2$, $\alpha_1 = -3$ and $\beta_2 = 0$, we have exactly one infinite saddle and one infinite saddle-node with nonzero eigenvalue associated to the eigenvector transversal to infinity. So Case B.7 is realizable. This case is not written as system (1) but a change of variables can bring it to such a natural form.

Finally, we remark that reversing the orientation of the orbits, the phase portraits A.1 and A.3, and C.i and C.14- i for $i = 1, \dots, 6$, are topologically equivalent.

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DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT AUTÒNOMA DE BARCELONA,
08193 BELLATERRA, BARCELONA, SPAIN.