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COMBINATORIAL PROPERTIES OF INTERVALS IN FINITE SOLVABLE GROUPS

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1. Introduction. For a finite solvable group G and a subgroup $H \leq G$ we want to gather information about combinatorial properties of the interval $[H,G] := \{U \mid H \leq U \leq G\}$. We do this by means of algebraic and topological combinatorics. In particular, we investigate the shellability and, hence the Cohen-Macaulay property of $[H,G]$. As an introduction to this theory, we refer the reader to the paper of Björner [**1**] or the book of Stanley [**12**]. An application of these techniques yields insight in the group theoretic questions about the number of maximal subgroups in $[H,G]$ which lie in a given conjugacy class. Moreover, this enables us to compute the orders of the normalizers of the subgroups, for which $[H,G]$ is a complemented lattice (i.e., for all $U \in [H,G]$ there is a W in $[H,G]$ such that $U \cap W = H$ and the subgroups U and W generate G).

We denote by $[U]$ the conjugacy class $\{U^g \mid g \in G\}$ of the subgroup U of G. The combinatorial invariant used to gain knowledge about $|[H,G] \cap [M]|$ for a maximal subgroup M of G is the Möbius number [**11**] $\mu([H,G])$ of $[H,G]$. It will turn out that in the case $\mu([H,G]) = 0$ the Möbius number cannot provide enough information about the number of maximal subgroups. Therefore, we will give the final results about numbers of maximal subgroups only for intervals $[H,G]$, for which $\mu([H,G]) \neq 0$. Since in finite solvable groups the condition $\mu([H,G]) = 0$ is equivalent to the fact that the interval $[H,G]$ is a complemented lattice [**5, 9**], we call these groups C-subgroups of G [**15**]. We denote by $\kappa(G)$ the partially ordered set (poset for short) of all C-subgroups of the group G and by $[H,G]_\kappa$ the poset $(\kappa(G) \cap [H,G]) \cup \{H\}$. By the definition of the Möbius number [**11**] it is clear that $[H,G]$ and $[H,G]_\kappa$ have the same Möbius number. In addition, we show that if H is a C-subgroup, then the interval $[H,G]_\kappa$ is a lexicographically shellable poset

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and hence Cohen-Macaulay. By giving the shelling of $[H,G]_\kappa$ we are able to compute the Möbius number in terms of numbers of maximal subgroups. This gives some combinatorial insight into the well-known result of Gaschütz [**2**]. Up to now we have gotten a lot of information about the structure of $[H,G]$, but we need additional knowledge for the computation of numbers of maximal subgroups.

To do this, we pass to orbit posets. We impose a partial order on the set $[H,G]/G$ of conjugacy classes $[U]$ of the subgroups U in $[H,G]$ as follows

$$[U] \le [V] \Leftrightarrow \exists\, g \in G : U^g \le V.$$

The poset $[H,G]_\kappa/G$ is analogously defined. Our next step is the calculation of the Möbius number $\mu([H,G]_\kappa/G)$. We show that the shelling of $[H,G]_\kappa$ induces a shelling of $[H,G]_\kappa/G$. This is a surprising fact since, in general, shellability does not convey from a poset to its orbit poset [**17**]. We shall give some more general criteria which force the orbit poset of a lexicographically shellable poset to be lexicographically shellable. Again, we use the shelling to compute the Möbius number of $[H,G]_\kappa/G$ in terms of numbers of conjugacy classes of maximal subgroups. Since $\mu([H,G]_\kappa/G) = \mu([H,G]/G)$ [**15**, Theorem 5.11] we now have control over the Möbius number of $[H,G]/G$. Calculating the Möbius number of a poset by a shelling means counting certain maximal chains in that poset. Since the maximal chains counted in $\mu([H,G]/G)$ are the orbits of the chains counted in $\mu([H,G])$ we can relate both numbers by the length of the orbits. Hence, we obtain the following formula

$$|G : N_G(H)| \cdot \mu([H,G]) = |HG' : H| \cdot \mu([H,G]/G)$$

for an arbitrary subgroup H of G. Here $N_G(H)$ denotes the normalizer of H in G. This generalizes the result of Hawkes, Isaacs and Özaydin [**6**, Theorem 7.2] for the case $H = 1$.

Now we specialize again to C-subgroups H of a solvable group G. For these groups the previous formula gives the desired relation for the numbers of maximal subgroups in the intervals $[H,G]$. Additionally, we can extract some information about particular conjugacy classes. Furthermore, the formula allows the computation of the order of the normalizer $N_G(H)$ of a C-subgroup H by means of numbers of maximal subgroups containing H. As an application we give the number of p-Sylow groups Q if Q is a C-subgroup.

For the rest of the paper all groups are finite and solvable.

2. Shellability of the poset of C-subgroups. In the following we list some of the basic properties of the C-subgroups introduced before. For the rest of this section we fix a chief series

$$\mathcal{R} : 1 = N_0 < N_1 < \cdots < N_{l-1} < N_l < \cdots < N_k = G$$

of G. By I we denote the set of indices i of chief factors N_i/N_{i-1}. Every maximal subgroup M complements a unique chief factor N_i/N_{i-1} in the chief series $\mathcal{R}$ (i.e., $M \cap N_i = N_{i-1}$ and $MN_i = G$). If M complements the minimal normal subgroup N_1 in the given chief series, then

$$\begin{aligned}\mathcal{R}_M : 1 = K_0 < K_1 = N_2 \cap M < \cdots < K_{l-1} \\ = N_l \cap M < K_l = N_{l+1} \cap M < \cdots < K_{k-1} = M\end{aligned}$$

is a chief series of M. Similarly we know that

$$\begin{aligned}\mathcal{R}^N : 1 = L_0 < L_1 = N_2/N_1 < \cdots < L_{l-1} \\ = N_l/N_1 < L_l = N_{l+1}/N_1 < \cdots < L_{k-1} = G/N\end{aligned}$$

is a chief series of G/N_1.

The next proposition gives a group theoretical characterization of C-subgroups.

Proposition 2.1 [**15**, Proposition 3.2], [**10**, Hilfssatz 1.3]. *A subgroup H of G is a C-subgroup if and only if there is a subset $J \subseteq I$ and complements M_j of the chief factors N_j/N_{j-1} for $j \in J$ such that $H = \cap_{j \in J} M_j$. The set J is uniquely determined by H and $\mathcal{R}$. Moreover, $|J|$ is independent of the choice of the chief series $\mathcal{R}$.*

We call a representation of a C-subgroup H as in the preceding proposition a complement representation of H. Since the index set J only depends on $\mathcal{R}$ and H, we write $J = \mathcal{I}(\mathcal{R}, H)$. The set J assigns a type (depending on $\mathcal{R}$) to each C-subgroup H.

Now we will introduce two "cover"-relations which will play an important role in our context. The first one can be defined for general posets. For a poset P we say that an element y of P covers an element

x of P if $x < y$ and the interval $[x, y] = \{z \mid x \leq z \leq y\}$ contains only x and y. The second "cover"-relation relates subgroups of a solvable group and chief factors. We say that a subgroup H of a solvable group covers (respectively, avoids) the chief factor N_i/N_{i-1} if $HN_{i-1} \geq N_i$ (respectively) $H \cap N_i \leq N_{i-1}$). In particular, it is well known that if $H = \cap_{j \in J} M_j$, $J = \mathcal{I}(\mathcal{R}, H)$ is a complement representation of the C-subgroup H, then H avoids all chief factor N_i/N_{i-1} for $i \in J$ and covers all others. We speak in this context of the cover-avoidance property of C-subgroups. Both "cover"-relations are standard notations but they are completely unrelated. In the sequel it will be clear from the context which "cover"-relation we mean.

As a consequence of Proposition 2.1, we are able to prove some facts about maximal chains in the poset $[H, G]_\kappa$.

Corollary 2.2. *The length* (*i.e., cardinality*-Λ) *of a maximal chain in* $[H, G]_\kappa$ *equals the number of chief factors which have a complement in* $[H, G]$; $+\Lambda$ *if* H *is not a* C*-subgroup. Hence all maximal chains in* $[H, G]_\kappa$ *have the same length.*

Proof. Let $H = H_0 < \cdots < H_t < G$ be a maximal chain in $[H, G]_\kappa$. It is obvious that if H_j avoids a chief factor then H_{j-1} avoids the chief factor too. Hence, the remarks about the cover-avoidance property of C-subgroups preceding this corollary show that the assignment $H \mapsto \mathcal{I}(\mathcal{R}, H)$ is monotone. Now the assertion follows from Proposition 2.1. $\square$

For an arbitrary poset P we will denote the set of maximal chains in P by $\mathcal{C}(P)$. Assume we have labeled the edges of the Hasse diagram of a poset P by a labeling λ_0 with natural numbers. Then this induces a mapping λ from the set of maximal chains into the set of tuples of natural numbers by taking the maximal chain $C : x_0 < x_1 < \cdots < x_t$ to the tuple $\lambda(C) = (\lambda_0(x_0 < x_1), \ldots, \lambda_0(x_{t-1} < x_t))$. Now the lexicographic order on the tuples induces a partial order on the maximal chains. We say that for $C, C' \in \mathcal{C}(P)$ the chain C precedes C' if and only if $\lambda(C)$ precedes $\lambda(C')$ in the lexicographic order. We are interested in special properties of such labelings λ.

A poset P with least element $\hat{0}$ and greatest element $\hat{1}$ is called lexicographically shellable (edge-lexicographically shellable) if

(i) Every maximal chain in P has the same length.

(ii) There is a labeling λ_0 of the edges of the Hasse diagram of P by natural numbers such that for each interval $[x, y]$ of P:

(E1) There is a unique maximal chain $x = x_0 < \cdots < x_t = y$ such that

$$\lambda_0(x_0 < x_1) < \lambda_0(x_1 < x_2) < \cdots < \lambda_0(x_{t-1} < x_t).$$

We call such a chain an ascending chain.

(E2) The unique ascending chain is the least element in any order on $\mathcal{C}([x, y])$ induced by the lexicographic order on the labeled maximal chains.

A labeling of the edges of the Hasse diagram of a poset P is called L-labeling [**1**, Definition 2.1] if it satisfies the conditions (E1) and (E2) of the previous definition.

Theorem 2.3. *If H is a C-subgroup H of G, then the poset $[H, G]_\kappa$ is lexicographically shellable.*

Proof. From Corollary 2.2 we know that all maximal chains in $[H, G]_\kappa$ have the same length. Hence it remains to establish an L-labeling for $[H, G]_\kappa$.

At first we define a labeling λ_0^G of the two-element chains $U < V$, where V covers U. We set $\lambda_0^G(U < V) = j$ for the unique index j in $\mathcal{I}(\mathcal{R}, U) - \mathcal{I}(\mathcal{R}, V)$. As usual, we extend λ_0^G to a mapping λ^G from $\mathcal{C}([H, G]_\kappa)$ to I^c where c is the number of chief factors which have a complement in $[H, G]$. We set $\lambda^G(H_1 < H_2 < \cdots < H_c) := (\lambda_0^G(H_1 < H_2), \lambda_0^G(H_2 < H_3), \ldots, \lambda_0^G(H_{c-1} < H_c))$. Now we have to verify that this mapping is an L-labeling and hence induces a lexicographic shelling. We choose an arbitrary interval $[U, V]^\kappa = [H, G]_\kappa \cap [U, V]$ in $[H, G]_\kappa$. Let $J = \mathcal{I}(\mathcal{R}, U) - \mathcal{I}(\mathcal{R}, V)$ be the set of indices of the chief factors covered by V but not by U. For $j \in J$ we choose maximal subgroups M_j which complement N_j/N_{j-1} and contain U. If $j_1 < j_2 < \cdots < j_t$ is the ordering of the set J, then we set $H_i := V \cap \cap_{r=1}^{t-i} M_{j_r}$ for $i = 0, \ldots, t$. The chain $U = H_0 < H_1 < \cdots < H_t = V$ is a chain in $\mathcal{C}([U, V]^\kappa)$ by Proposition 2.1. From the choice of the maximal subgroups M_j we infer

$$\lambda_0^G(H_0 < H_1) = j_1 < \lambda_0^G(H_1 < H_2) = j_2 < \cdots < \lambda_0^G(H_{t-1} < H_t) = j_t.$$

Hence, $H_0 < H_1 < \cdots < H_t$ is a maximal ascending chain. By the fact that all maximal chains in $[U, V]^\kappa$ have a label set in J^t we see that for all maximal ascending λ^G-chains $U = A_0 < A_1 < \cdots < A_t = V$ in $[U, V]^\kappa$, we have $\lambda_0^G(A_r < A_{r+1}) = j_{r+1}$. In order to verify condition (E1) we will prove by induction on the group order $|G|$ that $H_0 < \cdots < H_t$ is the unique ascending chain. Before we can complete the proof we remark:

(A) If A_i and A_{i+1} are contained in a complement M of the minimal normal subgroup N_1, then A_i and A_{i+1} are C-subgroups of M [**5**, 1.3]. Furthermore, A_{i+1} covers A_i as a C-subgroup of M and for the chief series $\mathcal{R}$ and $\mathcal{R}_M$ the equation $\lambda_0^G(A_i < A_{i+1}) = \lambda_0^M(A_i < A_{i+1}) + 1$ holds.

(B) If the minimal normal subgroup N_1 is contained in A_i and A_{i+1}, then A_i/N_1 and A_{i+1}/N_1 are C-subgroups of G/N_1 [**5**, 1.3]. Furthermore, A_{i+1} covers A_i as a C-subgroup of G/N_1 and for the chief series $\mathcal{R}$ and $\mathcal{R}^N$ the equation $\lambda_0^G(A_i < A_{i+1}) = \lambda_0^{G/N_1}(A_i/N_1 < A_{i+1}/N_1) + 1$ holds.

Now by (A) and (B) we deduce that ascending λ^G-chains in G which consist of subgroups of M correspond to the ascending λ^M-chains in M. Analogously, we see that ascending λ^G-chains in G which consist of subgroups containing N_1 correspond to the ascending λ^{G/N_1}-chains in G/N_1. These two conclusions provide the induction bases for the proof of the claim that the chain $H_1 < \cdots < H_t$ is the unique ascending λ^G-chain in $[U, V]^\kappa$.

(a) If V is contained in a complement M of the minimal normal subgroup $N = N_1$, then the assertion follows from the fact that $\kappa(M) \cong \kappa(G) \cap [1, M]$ [**5**, 1.5].

(b) If U contains the minimal normal subgroup N, then the assertion follows from $\kappa(G/N) \cong [N, G]_\kappa(-\{N\})$.

(c) If V is not contained in a complement of the minimal normal subgroup N, then V covers the minimal subgroup N (see remarks after Proposition 2.1). This implies $N \leq V$. If N is not contained in U, then NU is a C-subgroup containing U [**5**, 1.4]. In these cases we have

$\lambda_0(U < NU) = 1$ and NU is the only C-subgroup covering U with this label. Hence, $H_1 = NU$ and the assertion follows by induction from the case (b).

As mentioned before, all maximal chains in $[U, V]^\kappa$ have the same label set. Hence, we deduce that the unique ascending chain in the interval is the least in the lexicographic ordering. This proves condition (E2) for L-labelings and completes the proof. □

As mentioned before, given an (L)-labeling of a poset P the lexicographic order on the labels of the maximal chains induces a partial order on the maximal chain in $\mathcal{C}(P)$. By the theorem of Björner [**1**, Theorem 2.3] we know that any linear extension of this order will induce a so-called shelling order (see [**1**]). We call the L-labeling λ^G of the maximal chains in $[H, G]_\kappa$, introduced in the preceding theorem, the standard L-labeling for $[H, G]$ with respect to $\mathcal{R}$ (note that H has to be a C-subgroup). We will call any shelling order (i.e., order on the maximal chains, satisfying certain conditions [**1**]), which is constructed by extending the lexicographic order on the labels to a linear order on the maximal chains, a standard shelling of $[H, G]_\kappa$ with respect to $\mathcal{R}$.

We should mention that, for an arbitrary subgroup H of G, the poset $[H, G]_\kappa$ is shellable, too, but the proof is much more involved. See also [**14**] for results on particular cases of this fact.

As an immediate corollary we obtain the shellability of the subgroup lattice $\Lambda(G)$ of a supersolvable group G in the case when $\Lambda(G)$ is complemented. This is a result of Björner [**1**, Theorem 3.2] who proves it more generally for all supersolvable groups G. Before we can give the corollary we have to state an easy group theoretical lemma.

Lemma 2.4. *If G is a supersolvable group and if $\Lambda(G)$ is a complemented lattice, then $\Lambda(G) = \kappa(G)$.*

Proof. Let G be a minimal counterexample and let H be a subgroup of G such that $[H, G]$ is not complemented. Furthermore, we may assume that for all subgroups $V > H$ of G the interval $[V, G]$ is complemented. If H contains a normal subgroup N, then the contradiction follows by induction applied to H/N and G/N [**5**, 1.3]. If H is contained in a complement M of the minimal normal subgroup N, then the contradiction follows again by induction applied to H as a subgroup of M [**5**, 1.3]. Let $\mathcal{M}$ be the set of maximal subgroups of G containing H. In the case $H = \Phi(G,H) := \cap_{M\in\mathcal{M}} M$ either H is contained in a complement of the minimal normal subgroup N_1 or N_1 is contained in every maximal subgroup in M (i.e., H contains N_1). Both alternatives contradict the assumptions. Therefore, $H \neq \Phi(G,H) := \cap_{M\in\mathcal{M}} M$ and N_1 is contained in every $M \in \mathcal{M}$. By the maximality of H it follows that $HN_1 = \Phi(G,H)$ and $\Phi(G,H)$ is the only subgroup covering H. Let U be any subgroup of G covering $\Phi(G,H)$; then the interval $[H,U]$ is a chain of three subgroups and hence not complemented. Now we use the classification of supersolvable groups with complemented subgroup lattice [**13**, Theorem 24]. From this we easily infer that the property of having a complemented subgroup lattice is inherited by subgroups in supersolvable groups. Now by the minimality of G we deduce that $G = U$. Let $\overline{N} \neq N_1$ be another minimal normal subgroup. Since H is core free (i.e., $\cap_{g\in G} H^g = 1$) we know that $\overline{N}H \neq N_1H$ is a subgroup covering H which contradicts the assumption that $[H,G]$ is a chain. Therefore, N_1 is the only minimal normal subgroup of G. Hence, $p = |N_1|$ is the largest prime dividing $|G|$ and the maximal normal p-subgroup $O_p(G)$ is a p-Sylow subgroup of G. Since the subgroup lattice of G is complemented, $O_p(G)$ is an elementary abelian p-subgroup (use [**13**, Theorem 24] again). The group G acts as a supersolvable matrix group on $O_p(G)$. Hence, G acts not faithful, of course, as a group of upper triangular matrices. If there is an entry above the diagonal in any of the matrices, then the p-Sylow subgroup of G is not abelian, which contradicts the structures of $O_p(G)$. Hence, G acts as a group of diagonal matrices on $O_p(G)$. Therefore, all subgroups of $O_p(G)$ are normal. Since N_1 is the only minimal normal subgroup of G we have $O_p(G) = N_1$. Now N_1 is also a p-Sylow subgroup of G which shows that $\gcd(|G/N_1|, |N_1|) = 1$. Hence, the Schur-Zassenhaus theorem applies and shows that all complements of N_1 in G (respectively in $M = HN_1$) are conjugate. Taking such a complement M_1 in G we obtain a complement $M \cap M_1$ of N_1 in M. Since H is another complement of N_1 in M we have shown that H and $M \cap M_1$ are conjugate. But this implies that H is contained in a complement of N_1 in G. Again, this contradicts the assumption that $[H,G]$ is a chain and completes the proof. □

Corollary 2.5 [**1**]. *If G is a supersolvable group and if $\Lambda(G)$ is a complemented lattice, then $\Lambda(G)$ is lexicographically shellable.*

Proof. This follows from Theorem 2.3 and Lemma 2.4. □

Another application of Theorem 2.3 will be the computation of the Möbius number of an interval $[U, G]$ in $\Lambda(G)$. Here we use tools developed by Stanley [**12**]. Let P be a shellable poset, and let $C_1 < \cdots < C_t$ be a shelling of P. For all $l \leq t$, we write $\overline{C_l}$ for the simplicial complex generated by $C_1, \ldots, C_l$. By $R(C_l)$ we denote the set $\{x \in C_l \mid C_l - \{x\} \in \overline{C_{l-1}}\}$ and by h we denote the cardinality of the maximal chains in P. The principal result [**12**] we need says that for a shellable poset P and for a shelling $C_1 < \cdots < C_t$ we have

$$\mu(P) = (-1)^{h+1} \cdot |\{C_l \mid R(C_l) = C_l\}|.$$

Before we can apply this fact to our situation we need the following proposition. We call a maximal chain in a poset P with an L-labeling λ descending, if the labels are (weakly) decreasing from the bottom to the top edges.

Proposition 2.6 [**12**, Theorem 3.13.2]. *Let P be a lexicographically shellable poset of rank t. Let $\lambda : \mathcal{C}(P) \to \mathbf{N}^t$ be an L-labeling. If $C_1 < \cdots < C_l$ is a shelling order of the maximal chains, which is constructed by extending the lexicographic order on the labels $\lambda(\mathcal{C}(P))$ to a linear order of the maximal chains. Then*

$$R(C_i) = C_i \Leftrightarrow C_i \text{ is a descending } \lambda\text{-chain.}$$

As an immediate consequence we retrieve an old result of Gaschütz [**2**, Chapter 5].

Theorem 2.7 [**2**, Chapter 5]. *Let H be a subgroup of G.*

(i) *If H is not a C-subgroup of G, then $\mu([H, G]) = 0$.*

(ii) *If H is a C-subgroup of G, then*

$$\mu([H, G]) = (-1)^t \cdot \prod_{j \in \mathcal{I}(\mathcal{R}, H)} m_j.$$

By t we denote the cardinality of $\mathcal{I}(\mathcal{R}, H)$ and by m_j the number of complements of N_j/N_{j-1} containing H.

Proof. If the interval $[H, G]$ is not complemented, then it is well known that $\mu([H, G]) = 0$. Hence, we can restrict our attention to complemented intervals $[H, G]$. By the definition of the Möbius number it is obvious that $\mu([H, G]) = \mu([H, G]_\kappa)$. Now by Theorem 2.3 we know that $[H, G]_\kappa$ is lexicographically shellable. Let λ^G be the standard L-labeling of $[H, G]_\kappa$ given in Theorem 2.3. By Theorem 2.3 the labeling λ^G satisfies the assumptions of Proposition 2.6. Hence, $\mu([H, G])$ equals $(-1)^t$ times the number of descending maximal λ^G-chains in $[H, G]_\kappa$. Let J denote the set $\mathcal{I}(\mathcal{R}, H)$ and let $j_1 < \cdots < j_t$ be the usual ordering of J. Then we claim that the descending λ^G-chains are of the form

$$\bigcap_{j \in J} M_j < \bigcap_{j \in J - \{j_t\}} M_j < \cdots < M_{j_1} \cap M_{j_2} < M_{j_1}$$

where M_j is a complement of N_j/N_{j-1}. The fact that each of these chains is descending and the fact that all descending chains are of this form follows again from the construction of the labeling and the cover-avoidance property of the C-subgroups (see remark after Proposition 2.1). Hence, there is a surjective map from sequences $M_{j_t}, M_{j_{t-1}}, \dots, M_{j_2}, M_{j_1}$ of complements M_j of the chief factors N_j/N_{j-1} for $j \in J$ to the descending chains. We will prove that this map is actually injective. We will now proceed by induction on $|G|$. We may assume that H is core free. Therefore, there exists a maximal subgroup containing H which complements the minimal normal subgroup $N = N_1$. In particular, $j_1 = 1$. Assume $M_{j_t}, M_{j_{t-1}}, \dots, M_{j_2}, M_{j_1}$ and $M'_{j_t}, M'_{j_{t-1}}, \dots, M'^{j_2}, M'^{j_1}$ are two sequences which map to the same maximal chain. Obviously, $M_{j_1} = M'_{j_1}$. Since $j_1 = 1$ we have [**5**, 1.3]

$$\kappa(M_{j_1}) = \kappa(M'_{j_1}) = \{H \in \kappa(G) \mid H \leq M_{j_1}\}.$$

Now by induction we conclude that $M_{j_i} \cap M_{j_1} = M'_{j_i} \cap M_{j_1}$ for $2 \leq i \leq t$. But $N(M_{j_i} \cap M_{j_1}) = M_{j_i}$ and $N(M'_{j_i} \cap M_{j_1}) = M'_{j_i}$ implies $M_{j_i} = M'_{j_i}$ also for $2 \leq i \leq t$. Since the number of these sequences of maximal subgroups is $\prod_{j \in J} m_j$ the assertion follows from Proposition 2.6. $\square$

The reader may easily deduce from the general version of Proposition 2.6, as stated in Stanley's book, a formula for the Möbius numbers of the rank selected subposets of $[H, G]_\kappa$.

3. Shelling orbit posets. In this section we shall work in a more general setting. If a group G acts on the poset P, then we say that P is a G-poset. The orbit poset P/G is the partially ordered set on the set of orbits $[x] := \{x^g \mid g \in G\}$ ordered by

$$[x] \leq [y] :\Leftrightarrow \exists\, g \in G : x^g \leq y.$$

It is well known that for a shellable poset P the orbit poset P/G is not necessarily shellable. A crucial example is the lattice $\prod_n$ of set partitions of an n-element set with the symmetric group $G = S_n$ as a group of automorphisms. The orbit poset $\prod_n / S_n$ is the poset of integer partitions P_n. For small n it is known that P_n is shellable. But Ziegler [**17**, (3)] shows that for $n \geq 19$ the lattice P_n is no longer shellable. Here we give a criterion for a shellable G-poset P which forces that P/G is shellable too. Let P be a G-poset. For a maximal chain $C = (x_1 < \cdots < x_t) \in \mathcal{C}(P)$ we denote by C/G the chain $([x_1] < \cdots < [x_t]) \in \mathcal{C}(P/G)$.

Proposition 3.1. *Let P be a lexicographically shellable G-poset. Let $\lambda : \mathcal{C}(P) \to J^t$ be an L-labeling. Assume further that P and λ satisfy the following conditions*:

(i) *For all chains $C \in \mathcal{C}(P)$ and $g \in G$ we have $\lambda(C^g) = \lambda(C)$.*

(ii) *For all chains $C = (x_0 < \cdots < x_t)$ and all elements $g_0, \ldots, g_t \in G$, for which the elements $x_0^{g_0} < \cdots < x_t^{g_t}$ form another chain there is a $g \in G$ such that $x_i^{g_i} = x_i^g$ for $i = 0, \ldots, t$.*

Then

$$\lambda/G : \begin{cases} \mathcal{C}(P/G) \to J^t \\ C/G \mapsto \lambda(C) \end{cases}$$

is an L-labeling. In particular, P/G is lexicographically shellable.

Proof. Let $[[x], [y]]$ be an arbitrary interval in P. We may assume that $x \leq y$. Let $[x] = [x_0] < [x_1] < \cdots < [x_t] = [y]$ be a maximal chain in $[[x], [y]]$. From the definition of the ordering on P/G we infer that there are elements $g_0, \dots, g_t$ such that $x = x_0^{g_0} < x_1^{g_1} < \cdots < x_t^{g_t} = y$ is a maximal chain in $[x, y]$. By assumption (ii) there is an element $g \in G$ with the property $x_i^{g_i} = x_i^g$ for all $i = 0, \dots, t$. Hence, we infer from (i) and the definition of λ/G that

$$\lambda/G([x]=[x_0]<[x_1]<\cdots<[x_t]=[y])=\lambda(x=x_0^{g_0}<x_1^{g_1}<\cdots<x_t^{g_t}).$$

Let $x = v_0 < v_1 < \cdots < v_t = y$ be the unique maximal ascending λ-chain in $[x, y]$. Then, by the definition of λ/G, it is trivial that $[x] = [v_0] < [v_1] < \cdots < [v_t] = [y]$ is an ascending λ/G-chain in P/G. If $[x] = [w_0] < [w_1] < \cdots < [w_t] = [y]$ is any other maximal ascending λ/G-chain in $[[x], [y]]$, then, by assumption (ii) we conclude that there is an element $g \in G$ such that $x = w_0^g < w_1^g < \cdots < w_t^g = y$. Furthermore, condition (i) implies that $x = w_0^g < w_1^g < \cdots < w_t^g = y$ is a maximal ascending λ-chain in $[x, y]$. Now λ is an L-labeling. Therefore, condition (E1) proves that $v_i = w_i$ for all $i = 0, \dots, t$. This shows that $[x] = [v_0] < [v_1] < \cdots < [v_t] = [y]$ is the unique ascending λ/G-chain in $[[x], [y]]$.

Analogously to the first part of the proof, one concludes that the unique maximal ascending λ-chain in $[x, y]$ is the least in the order induced on $\mathcal{C}([[x], [y]])$ by the lexicographic order of the labels. This implies (E2), and we conclude that the poset P/G is lexicographically shellable. □

We would like to mention that there is an analog of Proposition 3.1 for general shellable posets. It replaces the first condition by the existence of a shelling order of the maximal chains such that:

The maximal chains are ordered orbit by orbit. The ordering of the chains in any orbit is arbitrary.

The proof under that assumption is an easy modification of the proof given for Proposition 3.1.

We will call an L-labeling which fulfills the conditions of Proposition 3.1 an admissible L-labeling. Here we shall briefly return to the theorem given by Ziegler [**17**] which shows that $P_{19} = \prod_{19}/S_{19}$ is not shellable. Since all $\prod_n$ are supersolvable lattices they are lexicographically shellable. On the other hand, no chain in $\prod_n$ is invariant under S_n. Therefore, no L-labeling can be G-equivariant (note that in this case the unique ascending chain has to be fixed by the G-operation). For $n \geq 5$ condition (ii) of Proposition 3.1 fails too. But at least for $n < 11$ the poset P_n is still shellable. This shows that there should be a chance of weakening the conditions of Proposition 3.1.

The next theorem deals with intervals in solvable groups.

Theorem 3.2. *Let G be a group and H a C-subgroup of G. Then the poset $[H, G]_\kappa / N_G(H)$ is lexicographically shellable. Furthermore, the interval*

$$[H, G]/G = \{[U] \in \kappa(G)/G \mid [H] \leq [U] \leq [G]\}$$

is also lexicographically shellable.

Proof. Let λ^G be the standard L-labeling of $[H, G]_\kappa$ with respect to a fixed chief series $\mathcal{R}$. Since the labeling depends only on the cover-avoidance property with respect to the chief series $\mathcal{R}$ it is $N_G(H)$-equivariant. The action of $N_G(H)$ on $[H, G]_\kappa$ also fulfills the second condition of Theorem 3.1 [**16**, Proposition 4.7]. Hence, $[H, G]_\kappa / N_G(H)$ is lexicographically shellable and λ^G/G gives the corresponding L-labeling. Also, by [**16**, Proposition 4.7] we conclude that $[H, G]_\kappa / N_G(H) \cong \{[U] \in \kappa(G)/G \mid [H] \leq [U] \leq [G]\}$. Thus, the second assertion follows immediately. □

As a corollary, we obtain a result on subgroup lattices of supersolvable groups.

Corollary 3.3. *Let G be a supersolvable group such that $\Lambda(G)$ is complemented. Then $\Lambda(G)/G$ is lexicographically shellable.*

Proof. The assertion follows from Theorem 3.2 and Lemma 2.4. □

We would like to mention that we do not know any supersolvable group G such that $\Lambda(G)/G$ is not shellable. Here again, all $\Lambda(G)$ are supersolvable lattices. One uses a chief series as an M-chain for the definition of the lexicographic shelling. Since a chief series is obviously fixed under the action of G, the L-labeling becomes G-equivariant. But the second condition on Proposition 3.1 fails even for some p-groups. However, the poset $\Lambda(G)/G$ proved to be shellable for all examples we have tested.

Now we return to the analysis of Möbius numbers of posets.

Lemma 3.4. *Let P be a lexicographically shellable G-poset. Let λ be an admissible L-labeling. Then for the lexicographic ordering $C_1/G < \cdots < C_l/G$ of $\mathcal{C}(P/G)$ according to λ/G, we have*

$$\mathcal{R}(C_i/G) = C_i/G \Leftrightarrow \mathcal{R}(C_i) = C_i.$$

Proof. By construction of λ/G this follows from Proposition 2.6. □

Now we can prove an analog of Theorem 2.7.

Theorem 3.5 [**15**, Theorem 6.5]. *Let H be a subgroup of G.*

(i) *If H is not a C-subgroup of G, then $\mu([H,G]/G) = 0$.*

(ii) *If H is a C-subgroup of G, then*

$$\mu([H,G]/G) = (-1)^t \cdot \prod_{j \in \mathcal{I}(\mathcal{R},H)} c_j.$$

By t we denote the cardinality of $\mathcal{I}(\mathcal{R},H)$ and by c_j the number of conjugacy classes of complements of N_j/N_{j-1} which have a representative containing H.

Proof. If the interval $[H,G]$ is not complemented, then it is known [**15**, Theorem 6.5] that $\mu([H,G]/G)$ is 0. We fix a set J of indices of complemented chief factors N_j/N_{j-1}. For every $j \in J$ let M_j be a complement of the chief factor N_j/N_{j-1}. Then the conjugacy class of the intersection $\cap_{j\in J} M_j$ determines the conjugacy classes $[M_j]$. Hence, from Theorem 3.2 and Lemma 3.4 we infer that the number of descending maximal λ-chains in $[H,G]_\kappa/N_G(H)$ equals the product

$\prod_{j \in \mathcal{I}(\mathcal{R},H)} c_j$. The rest of the assertion follows from Proposition 2.6 and [**15**, Theorem 5.11]. □

Finally we give an easy lemma which will be applied to $[H, G]_\kappa$ in Section 5.

Lemma 3.6. *Let P be a lexicographically shellable G-poset with an admissible L-labeling λ. If all descending chains $C \in \mathcal{C}(P/G)$ have stabilizers of the same order s, then $\mu(P) = \mu(P/G) \cdot |G|/s$.*

Proof. This follows immediately from Proposition 3.1 and Lemma 3.4. □

4. Complements of chief factors. Here we want to sum up the most obvious relations between the numbers of complements of different chief factors. Thereby, we obtain some simplifications of the formulas in Theorems 2.7 and 3.5. The tool for this simplification is given in Proposition 4.4. We leave an explicit reformulation of both theorems to the reader.

By the type of a chief factor we denote the isomorphy class of the chief factor regarded as a G-module. For a G-module ω we write $C_G(\omega)$ for the centralizer of ω in G. We will later also use the notation $C_G(H)$ to denote the centralizer in G of a group H on which G acts by conjugation.

Lemma 4.1. *Let M_1, M_2 be maximal subgroups of the same type ω which are not conjugate. Let $\{M_1, M_2, M_3, \dots, M_n\}$ be a maximal set of pairwise non-conjugate maximal subgroups of type ω which contain $M_1 \cap M_2$. Then, for all $1 \le i \le n$, there is a chief series $\mathcal{R}_i$ through $C_G(\omega)$ and $C_G(\omega) \cap M_1 \cap M_2$ such that:*

(i) *For $j \neq i$, the series*

$$C_G(\omega) = N_l^i > C_G(\omega) \cap M_i = N_{l-1}^i > C_G(\omega) \cap M_i \cap M_j = N_{l-2}^i$$

is a section of $\mathcal{R}_i$ and the number l is independent of i.

(ii) *The conjugacy class $[M_i]$ is the set of all complements of N_l^i/N_{l-1}^i.*

(iii) *The conjugacy classes* $[M_j]$ *for* $j \neq i$ *are the complements of* N^i_{l-1}/N^i_{l-2}.

Proof. By the symmetry of the assumptions we may assume that $i = 1$. Hence, we drop the superscript 1 in the notation of the normal subgroups in a chief series $\mathcal{R}_1$. It is well known that the subgroup chain

$$C_G(\omega) > C_G(\omega) \cap M_1 > C_G(\omega) \cap M_1 \cap M_2$$

is a chain of normal subgroups which cannot be refined. This shows that there is a chief series $\mathcal{R}_1$ which satisfies (i). Obviously, N_l/N_{l-1} is complemented by M_1. Since we have $C_G(N_l/N_{l-1}) = C_G(\omega) = N_l$ all complements of N_l/N_{l-1} are conjugate. Therefore, (ii) is satisfied. Since the complements of N_{l-1}/N_{l-2} contain $M_1 \cap M_2$, the third condition follows. □

Now we need the following trivial remark.

Remark 4.2. Let $x_1, \ldots, x_n$, $n \geq 3$ be strictly positive real numbers which satisfy the following system of equations:

$$\begin{gathered} x_1x_2 + x_1x_3 + \cdots + x_1x_{n-2} + x_1x_{n-1} + x_1x_n = \\ x_2x_1 + x_2x_3 + \cdots + x_2x_{n-2} + x_2x_{n-1} + x_2x_n = \\ \cdots = \\ \cdots = \\ \cdots = \\ x_{n-1}x_1 + x_{n-1}x_2 + x_{n-1}x_3 + \cdots + x_{n-1}x_{n-2} + x_{n-1}x_n = \\ x_nx_1 + x_nx_2 + x_nx_3 + \cdots + x_nx_{n-2} + x_nx_{n-1}. \end{gathered}$$

Then we have $x_1 = x_2 = \cdots = x_n$.

Lemma 4.3. *Let* H *be a* C*-subgroup of* G. *Let* N_s/N_{s-1} *and* N_t/N_{t-1} *be two isomorphic (as* G*-modules) chief factors of* G *which have complements containing* H. *Then the number of complements of* N_s/N_{s-1} *containing* H *equals the number of complements of* N_t/N_{t-1} *containing* H.

Proof. We fix a complements M_1 of N_s/N_{s-1} and a complement M_2 of N_t/N_{t-1}. Let $[M_1], [M_2], [M_3], \ldots, [M_n]$ be the different conjugacy classes of maximal subgroups which contain $M_1 \cap M_2$. For $1 \leq i \leq n$ we choose chief series $\mathcal{R}_i$ satisfying the conditions of Lemma 4.1. We denote by x_i the number of elements of the conjugacy class $[M_i]$ which contain H. Now we take a complement representation $H = \cap_{j \in J} Z_j$ relative to the chief series $\mathcal{R}_1$. By construction, the index set J is the same for complement representations of H with respect to the different chief series $\mathcal{R}_i$. Hence, if m_j^i denotes the number of complements of the j-th chief factor in the chief series $\mathcal{R}_i$, then we have

$$m_j^i = m_j^{i'} \qquad \text{for } j \neq l, l-1 \tag{1}$$

$$m_l^i = x_i \tag{2}$$

$$m_{l-1}^i = \sum_{t \neq i} x_t. \tag{3}$$

From Theorem 2.7 we deduce the following system of equations. Here k denotes the length of a chief series of G

$$(-1)^{k-1} \cdot \prod_{j \in J} m_j^i = (-1)^{k-1} \cdot \prod_{j \in J} m_j^{i'}. \tag{4}$$

Canceling the factors in (4) by using equation (1), we obtain equation (5).

$$m_l^i \cdot m_{l-1}^i = m_l^{i'} \cdot m_{l-1}^{i'}. \tag{5}$$

Applying equations (2) and (3) we derive the equation (6).

$$x_i \cdot \sum_{t \neq i} x_t = x_{i'} \cdot \sum_{t \neq i'} x_t. \tag{6}$$

But this gives rise to the system of equations in Remark 4.2. Moreover, it is well known that $n \geq 3$ holds in this case. Hence, we obtain the asserted identity $x_1 = x_2 = \cdots = x_n$. □

Proposition 4.4. *Let H be a C-subgroup of G. Let J_ω be the set of indices j of the chief factors N_j/N_{j-1} which are of type ω and have complements containing H. For $j \in J_\omega$ we denote by m_j the number of* *complements of* N_j/N_{j-1} *in one conjugacy class. By* c_j *we denote the number of conjugacy classes of complements of* N_j/N_{j-1} *which have a representative containing* H*. Then the following equations hold*

$$\prod_{j\in J_\omega} m_j = m_j^{|J_\omega|} \qquad \text{for any } j \in J_\omega$$
$$\prod_{j\in J_\omega} c_j = p^{e_\omega}.$$

Here p *is the prime dividing* $|N_j/N_{j-1}|$ *for* $j \in J_\omega$*. The number* e_ω *depends on* $|J_\omega|$ *and the* G*-module type* ω*.*

Proof. The first equation follows immediately from Proposition 4.3. The second equation follows from [**4**, Theorem 5.5] and [**3**, Chapter 3.2]. □

5. The Möbius number of intervals in solvable groups. In this section we want to relate the Möbius number of the interval $[H,G]$ to the Möbius number of the poset $[H,G]/G$. We denote by G' the commutator subgroup of G. We recall that a chief factor N_i/N_{i-1} is called central if $C_G(N_i/N_{i-1}) = G$. Otherwise, the chief factor is called eccentric. If J is a set of indices of chief factors, we denote by $J_e \subseteq J$ the set of indices of eccentric chief factors and by $J_c \subseteq J$ the set of indices of central chief factors in J.

Lemma 5.1. *Let* G *be a group and let* H *be a* C*-subgroup of* G*. Let*

$$H = H_1 < H_2 < \cdots < H_k = G$$

be a maximal descending λ*-chain in* $[H,G]_\kappa$ *for the standard* L*-labeling. Then the stabilizer of a maximal descending* λ*-chain in* G *has order* $|G : G'H|$*.*

Proof. We proceed by induction on k. If $k = 1$, then $H = G$ and the assertion is trivial. Now let k be greater than 1. We may assume that H is core free. Since the chain under consideration is descending, the maximal subgroup H_{k-1} complements the minimal normal subgroup N_1. Every element in G which normalizes the chain obviously normalizes H_{k-1}. Hence, we have to consider two cases:

(i) If N_1 is a central chief factor, then $G \cong N_1 \times H_{k-1}$. By induction [**5**, 1.3], the order of the normalizer in H_{k-1} of the chain $H = H_1 < \cdots < H_{k-1}$ equals $|H_{k-1} : H'_{k-1}H|$. Since N_1 centralizes the chain, the normalizer of the chain in G has order $|N_1| \cdot |H_{k-1} : H'_{k-1}H|$. Now the assertion follows from $H'_{k-1} = G'$.

(ii) If N_1 is an eccentric chief factor, then $N_G(H_{k-1}) = H_{k-1}$. Now we apply the induction hypothesis to the chain $H = H_1 < \cdots < H_{k-1}$ of C-subgroups [**5**, 1.3] of H_{k-1}. We obtain $|H_{k-1} : H'_{k-1}H|$ as the order of its normalizer. Now the assertion follows from $|H'_{k-1}| = |G'|/|N_1|$. □

The last lemma enables us to generalize Theorem 7.2 of [**6**] to arbitrary intervals $[H, G]$ of G.

Theorem 5.2. *Let H be an arbitrary subgroup of G; then*

$$|G : N_G(H)| \cdot \mu([H, G]) = |G'H : H| \cdot \mu([H, G]/G).$$

Proof. From Theorem 2.7 and Theorem 3.5 we deduce immediately that the equation holds if H is not a C-subgroup. Let s be the order of the stabilizer of a maximal descending λ-chain in $[H, G]$ within the group $N_G(H)/H$. We use the standard L-labeling with respect to a fixed chief series. From Lemma 3.6, we infer the equation

$$\mu([H, G]_\kappa / N_G(H)) \cdot \frac{|N_G(H)/H|}{s} = \mu([H, G]_\kappa). \tag{1}$$

The length of the orbit of a maximal chain in $[H, G]$ under the action of G is $|G : N_G(H)|$ times the length of the orbit under the action of $N_G(H)/H$. From this observation and Lemma 5.1 the equation

$$|G : N_G(H)| \cdot |N_G(H) : H| \cdot \frac{1}{s} = \frac{|G|}{|G : G'H|} \tag{2}$$

follows. Using the facts that $\mu([H, G]) = \mu([H, G]_\kappa)$ and $\mu([H, G]/G) = \mu([H, G]_\kappa/G)$ [**15**, Theorem 5.11] some trivial manipulations of (1) and (2) prove the result. □

Corollary 5.3. *Let H be a C-subgroup of the group G, and let $J = \mathcal{I}(\mathcal{R}, H)$. Let $\mathcal{M}_j$ be the set of complements of the chief factor N_j/N_{j-1}. If m_j denotes the number of elements of $\mathcal{M}_j$ which contain H and c_j denotes the number of conjugacy classes of subgroups contained in $\mathcal{M}_j$, then*

$$|G : N_G(H)| \cdot \prod_{j\in J} m_j = |G'H : H| \cdot \prod_{j \in J} c_j.$$

Proof. This follows trivially from Theorem 2.7, Theorem 3.5 and Theorem 5.2. □

Lemma 5.4. *Let H be a C-subgroup of G, and let J be the index set $\mathcal{I}(\mathcal{R}, H)$. Then the following identity holds*:

$$|G' \cap H| = \frac{|G'|}{\prod_{j \in J_e \cap J} |N_j/N_{j-1}|}.$$

Proof. Since complements of central chief factors contain G' we have $G' \cap H = G' \cap \bigcap_{j \in J_e} M_j$. Hence, we may assume $J = J_e$. Furthermore, we assume that H contains no normal subgroup. This implies that H itself is contained in a complement of the minimal normal subgroup. This shows that 1 is an index in J. Now $G' \cap M_1 = M_1'$ and H is a C-subgroup of M_1. Therefore, by induction on $|J|$ we have

$$|M' \cap H| = \frac{|M_1'|}{\prod_{j \in J - \{1\}} |(N_j \cap M_1)/(N_{j-1} \cap M_1)|}.$$

As in the proof of Theorem 3.1, one shows $|M_1' \cap H| = |G' \cap H|/|N_1|$ and $|M_1'| = |G'|/|N_1|$. Now the desired equation follows. □

Lemma 5.5. *Let H be a C-subgroup of the group G, and let J be the index set $\mathcal{I}(\mathcal{R}, H)$. We choose an index i which is strictly less than all indices in J and a complement M_i of the chief factor N_i/N_{i-1}. Then $N_G(H \cap M_i) \leq N_G(H)$. If l_i denotes the number of complements of N_i/N_{i-1} which contain $H \cap M_i$ and lie in one conjugacy class, then*

$$|N_G(H) : N_G(H \cap M_i)| \cdot l_i = |N_i/N_{i-1}|.$$

Proof. By the choice of i we have $(H \cap M_i)N_i = H$. Hence, if an element $g \in G$ normalizes the group $H \cap M_i$, then it also normalizes H. This gives the desired inclusion $N_G(H \cap M_i) \leq N_G(H)$.

For $j \in J$ we denote by m'_j the number of complements of the chief factor N_j/N_{j-1} which contain H. Analogously, for $j \in J \cup \{i\}$ we write m_j for the number of complements of the chief factor N_j/N_{j-1} which contain $H \cap M_i$. Since every complement of a chief factor N_j/N_{j-1} with index in J contains N_i, we have $m'_j = m_j$ for $j \in J$. If we apply Theorem 2.7, Theorem 3.5 and Theorem 5.2 to H and $H \cap M_j$, we get the following two equations:

$$|G : N_G(H)| \cdot \prod_{j \in J} m_j = |G'H : H| \cdot \prod_{j \in J} c_j \tag{1}$$

$$|G : N_G(H \cap M_i)| \cdot \prod_{j \in J \cap \{i\}} m_j = |G'(H \cap M_i) : H \cap M_i| \cdot \prod_{j \in J \cap \{i\}} c_j. \tag{2}$$

From Lemma 4.3 we derive the identity $l_i = m_i/c_i$. From Lemma 5.4 we deduce that

$$\frac{|G'H : H|}{|G'(H \cap M_i) : H \cap M_i|} = \begin{cases} 1 & \text{if } N_i/N_{i-1} \text{ is an eccentric chief factor} \\ |N_i/N_{i-1}| & \text{if } N_i/N_{i-1} \text{ is a central chief factor.} \end{cases}$$

Now the result follows by taking the quotients of the corresponding sides of the equations (1) and (2). □

Proposition 5.6. *Let H be a C-subgroup of the group G and let $J = \mathcal{I}(\mathcal{R}, H)$. If, for $j \in J$, we count by l_j the complements of the chief factor N_j/N_{j-1} which contain H and lie in one conjugacy class, then*

$$|N_G(H) : H| = \prod_{j \in J_e} l_j \cdot \prod_{j \in J_c} |N_j/N_{j-1}|.$$

Proof. If $J = J_c$, then H is normal in G and therefore we have $|N_G(H) : H| = |G : H|$. Since the order of an arbitrary C-subgroup H with complement representation $H = \cap_{j \in J} M_j$ is $|G|/(\prod_{j \in J} |N_j/N_{j-1}|)$ [**5**], the result follows in this case.

Now we proceed by induction on $|J_e|$. For $|J_e| = 0$ the result is already proved. Now for the case $|J_e| \geq 1$ we choose a chief series $\mathcal{R}$ which passes through the commutator subgroup G' of G. We may assume that H contains no normal subgroup. Hence, H itself is contained in a complement M_1 of the minimal normal subgroup N_1. By the choice of the chief series and since $|J_e| \geq 1$, the chief factor N_1/N_0 is eccentric. Now we apply the induction hypothesis to HN_1 which is a C-subgroup of G [**5**, 1.3]. This shows

$$|N_G(HN_1) : HN_1| = \prod_{j \in J_e - \{1\}} l_j \cdot \prod_{j \in J_c} |N_j/N_{j-1}|.$$

Since $H = HN_1 \cap M$ we can use Proposition 5.5 to complete the proof. □

As a corollary, we obtain a result about the number of p-Sylow subgroups of a solvable group, in which all p'-chief factors are complemented. It is an easy observation that a p-Sylow subgroup Q is a C-subgroup if and only if all p'-chief factors are complemented.

Corollary 5.7. *Let G be a group, let p be prime dividing $|G|$ such that all p'-chief factors of G are complemented, and let Q be a p-Sylow subgroup of G. If we denote by $J(p')$ the set of indices of p'-chief factors and by l_j the number of complements of the chief factor N_j/N_{j-1} which contain Q, then there are*

$$\frac{|G : Q|}{\prod_{j \in J(p')_e} l_j \cdot \prod_{j \in J(p')_c} |N_j/N_{j-1}|}$$

p-Sylow subgroups.

Proof. It is an elementary fact that the number of p-Sylow subgroups Q is equal to the index of the normalizer $N_G(Q)$. Hence we set $H = Q$ in the equation of Proposition 5.7 and multiply both sides with $|G : N_G(Q)|$. Now the result follows immediately. □

This section has pointed out the importance of $N_G(H)$ for the combinatorial invariants of the intervals $[H,G]$. Therefore, we devote the rest of the paper to a short remark on this subject.

We show that the factor group $N_G(H)/H$ is a semidirect product even if H is not a C-subgroup. We are grateful to the referee for providing a generalization of our original formulation of the following remark.

Remark 5.8. Let H be a subgroup and let N be a minimal normal subgroup of G. If H is contained in a complement M of N, then

$$N_G(H) = C_N(H)\cdot N_M(H) = HC_N(H)\cdot N_M(H) \tag{1}$$

and

$$N_M(H)\cap HC_N(H) = H. \tag{2}$$

In particular, $N_G(H)/H$ is the semidirect product of $HC_N(H)$ and $N_M(H)$.

Proof. Since equation (2) is trivial, we only care about equation (1). Clearly the inclusion $C_N(H)N_M(H)\le N_G(H)$ holds. Conversely, assume that we are given $g\in N_G(H)$; then $g=nm$ for some $n\in N$ and $m\in M$. For all $h\in H$ we have $m^{-1}n^{-1}hnm\in H$. This implies $n^{-1}hnh^{-1}\in M\cap N=1$. So $n\in C_N(H)$ which implies $m\in N_M(H)$. Now we infer the desired inclusion $N_G(H)\le C_N(H)N_M(H)$. Clearly, $N_M(H)$ normalizes $C_N(H)$. □

Hence the operation of $N_M(G)$ on $C_N(H)$ appears to be of interest. But, in general, one cannot expect that $C_N(G)$ will be a completely reducible $N_G(H)$-module [**7**].

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