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A SPLITTING CRITERION FOR A CLASS OF MIXED MODULES

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1. Introduction. This paper deals with mixed modules over discrete valuation domains. In particular, the results hold for local mixed abelian groups.

We study a class of mixed modules $\mathcal{H}$ with the property that the torsion submodule is a direct sum of cyclics and the quotient modulo the torsion is divisible of arbitrary rank. We show that the property of splitting of an arbitrary mixed module can be decided if a relatively small homomorphic image of a submodule of the mixed module splits in $\mathcal{H}$. To help formulate a necessary and sufficient splitting criterion for the modules in the class $\mathcal{H}$ we describe the modules by generators and relations. We then introduce the concept of a small relation array and show that mixed modules in $\mathcal{H}$ with this property split. The result of Baer and Fomin on the splitting of modules in $\mathcal{H}$ with bounded torsion submodules becomes an easy corollary. We conclude with two examples, one with a small relation array and the other a nonsmall relation array.

Let R denote a *discrete valuation domain*, i.e., a local principal ideal domain with prime p. All modules are always understood to be R-modules. A module G is said to *split* if its torsion submodule $\mathbf{t}G$ is a direct summand.

Recall that a *basic submodule* is a pure submodule which is a direct sum of cyclic modules and has divisible quotient. By [**1**, Section 32] every module G contains a basic submodule B. This basic submodule B is the direct sum of its torsion submodule $\mathbf{t}B$ and a free submodule F. If L is any free pure submodule of the module G, then by [**1**, Section 32, Exercise 7] there is a free pure submodule F containing L such that $G/(\mathbf{t}G \oplus F)$ is torsion-free divisible. Such a pure free module F with torsion-free divisible quotient $G/(\mathbf{t}G \oplus F)$ is called *relatively maximal pure free* in G. A pure free submodule is relatively maximal if and

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only if $(F + \mathbf{t}G)/\mathbf{t}G$ is basic in $G/\mathbf{t}G$.

2. Splitting reduction. Let $\mathcal{H}$ be the class of mixed modules H of arbitrary torsion-free rank whose torsion submodules are direct sums of cyclics such that $H/\mathbf{t}H$ is divisible. In this section we reduce the splitting of an R-module G to that of a member of the class $\mathcal{H}$ which is a homomorphic image of a submodule of G.

Theorem 1. *Every mixed module G contains submodules $F \subset H$, where F is relatively maximal free pure in G, $G = \mathbf{t}G+H$ and $H/F \in \mathcal{H}$ of torsion-free rank equal to that of G/F. Moreover, G splits if H/F splits.*

Proof. Let B be a basic submodule of $\mathbf{t}G$. Then $\mathbf{t}G/B$ is a summand of G/B, i.e., $G/B = \mathbf{t}G/B \oplus H/B$. Thus, $G = H + \mathbf{t}G$. Moreover, $H/B \cong (G/B)/(\mathbf{t}G/B) \cong G/\mathbf{t}G$ is torsion-free and hence $\mathbf{t}H = B$. Thus G contains a submodule H whose torsion submodule is a direct sum of cyclics and such that $G = \mathbf{t}G+H$. By [**1**, Section 32, Exercise 7] H contains a pure free submodule F such that $H/(F \oplus \mathbf{t}H)$ is torsion-free divisible. It is straightforward to show that F is relatively maximal pure free in G. Furthermore, $(H/F)/((\mathbf{t}H \oplus F)/F) \cong H/(\mathbf{t}H \oplus F)$ is torsion-free implies that $\mathbf{t}(H/F) = (\mathbf{t}H \oplus F)/F \cong \mathbf{t}H$ is a direct sum of cyclics. Since $H/(\mathbf{t}H \oplus F)$ is torsion-free divisible, this further implies that $(H/F)/\mathbf{t}(H/F)$ is torsion-free divisible, i.e., $H/F \in \mathcal{H}$. The choice of H immediately implies that the torsion-free ranks of G/F and H/F are equal. If H/F splits, then $H/F = (\mathbf{t}H \oplus F)/F \oplus W/F$, since $\mathbf{t}(H/F) = (\mathbf{t}H \oplus F)/F$. Thus, $H = \mathbf{t}H \oplus W$ splits and $G = \mathbf{t}G + H = \mathbf{t}G \oplus W$ splits also. $\square$

Let G be a mixed module with submodules $F \subset H$. The quotient H/F is called an *$\mathcal{H}$-section* of G if F is relatively maximal free pure in G, $G = \mathbf{t}G + H$ and $H/F \in \mathcal{H}$. By Theorem 1 every mixed module has an $\mathcal{H}$-section.

The next corollary shows that for an arbitrary mixed module over a discrete valuation domain the "essential mixed structure" is contained in an $\mathcal{H}$-section.

Corollary 2. *A mixed module splits if and only if it has an $\mathcal{H}$-section which splits.*

Proof. If G splits, then $G = \mathbf{t}G \oplus H$ with H a torsion-free module. H contains a free pure submodule F such that H/F is torsion-free divisible. Thus, H/F is trivially a split $\mathcal{H}$-section. Theorem 1 provides the converse. $\square$

3. Generators and relations. In this section we describe mixed modules G in the class $\mathcal{H}$ by generators and relations. Let G be such a mixed module of torsion-free rank d with torsion submodule $\mathbf{t}G = \oplus_{i\in\mathbf{N}} \oplus_{v\in I_i} Rx_i^v$ of isomorphism type $\lambda = (s_i \mid i \in \mathbf{N})$, where $s_i = |I_i|$ and $\operatorname{ann} x_i^v = p^i R$ for $i \in \mathbf{N}$, $v \in I_i$. Then the quotient $G/\mathbf{t}G$ is a vector space over K of dimension d where K is the quotient field of R. Let I be a set of cardinality d; then the subset

$$B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\} \subset G$$

is called a *basic generating system* of G, if

(1) $\{x_i^v \mid i \in \mathbf{N}, v \in I_i\}$ is a basis of $\mathbf{t}G$ with $\operatorname{ann} x_i^v = p^i R$ for all $i \in \mathbf{N}$, $v \in I_i$,

(2) $G/\mathbf{t}G = \oplus_{k\in I} K\bar{a}_0^k$,

where $\bar{a}_{i-1}^k = a_{i-1}^k + \mathbf{t}G$ and $p\bar{a}_i^k = \bar{a}_{i-1}^k$ for all $i \in \mathbf{N}$, $k \in I$. The definition implies that the elements a_0^k, $k \in I$, are independent. It is easy to see that

$$G = \langle x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\rangle.$$

A basic generating system intrinsically defines a series of equations with coefficients in R that describe the relations among the generators. Modeling upon these relations we introduce the concept of an abstract array $\alpha = (\alpha_{i-1,j}^{k,u})$ with $i, j \in \mathbf{N}$, $u \in I_j$, $k \in I$ having entries in R and which is *row finite* in j and u, i.e., for a fixed pair (k, i) $\alpha_{i,j}^{k,u} \in p^j R$ for almost all pairs (j, u). Such arrays α are called *relation arrays.* If a torsion module $\mathbf{t}$ is a direct sum of cyclics and of isomorphism type $\lambda = (|I_i| \mid i \in \mathbf{N})$ and $d = |I|$, then we call the relation array α of *format* (λ, d). Note that all relation arrays of a mixed module in the class $\mathcal{H}$ have the same format.

The following results give descriptions of mixed modules in the class $\mathcal{H}$ by generators and relations.

Proposition 3. *Every mixed module G in the class $\mathcal{H}$ has a basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ and a relation array α relative to B given by*

$$pa_i^k = a_{i-1}^k + \sum_{j\in\mathbf{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u} x_j^u, \qquad i \in \mathbf{N}, k \in I. \tag{1}$$

The torsion elements $\sum_{j\in\mathbf{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u} x_j^u$ are uniquely determined by the choice of a_i^k.

Proof. It is clear that every mixed module $G \in \mathcal{H}$ has such a basic generating system B. Moreover, every basic generating system B of G determines a relation array. Since $p\bar{a}_i^k = \bar{a}_{i-1}^k$ for all $i \in \mathbf{N}$, $k \in I$ it follows that the torsion elements $\sum_{j\in\mathbf{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u} x_j^u$ are uniquely determined by the choice of the coset representatives a_i^k. □

Note that the coefficients $\alpha_{i-1,j}^{k,u}$ of α for the torsion elements $\sum_{j\in\mathbf{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u} x_j^u$ are uniquely determined up to the relevant order ideal, ann x_j^u.

Theorem 4. *Let $\mathbf{t}$ be a torsion module of isomorphism type λ. Every relation array α of format (λ, d) can be realized by a mixed module in the class $\mathcal{H}$ with a torsion submodule of isomorphism type λ and torsion-free rank d.*

Moreover, a mixed module in the class $\mathcal{H}$ with torsion submodule of isomorphism type λ and torsion-free rank d is determined up to isomorphism by a relation array relative to a basic generating system.

Proof. Let $\lambda = (s_i \mid i \in \mathbf{N})$ and I, I_i be sets such that $|I_i| = s_i$ for all $i \in \mathbf{N}$ and $d = |I|$. Let L be a free R-module on

$$\{t_i^v, w_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$$

and M the submodule of L generated by

$$\left\{p^i t_i^v, pw_i^k - w_{i-1}^k - \sum_{j\in\mathbf{N}}\sum_{u\in I_j} \bar{\alpha}_{i-1,j}^{k,u} t_j^u \mid i\in\mathbf{N}, v\in I_i, k\in I\right\},$$

where $\bar{\alpha}_{i,j}^{k,u} = \alpha_{i,j}^{k,u}$ for $\alpha_{i,j}^{k,u} \in R\backslash p^j R$ and $\bar{\alpha}_{i,j}^{k,u} = 0$ if $\alpha_{i,j}^{k,u} \in p^j R$. Let T and N denote the submodules of L given by $T = \langle t_i^v \mid i\in\mathbf{N}, v\in I_i\rangle$ and $N = \langle p^i t_i^v \mid i\in\mathbf{N}, v\in I_i\rangle$. It is easy to see that $T\cap M = N$; hence, $T/N = T(T\cap M) \cong (T+M)/M$ with basis $\{\bar{t}_i^v \mid i\in\mathbf{N}, v\in I_i\}$ where $\bar{x} = x + M$ for $x\in L$. Let $H = L/M$. We now show that $\mathbf{t}H = (T+M)/M$. One direction is trivial, namely, $(T+M)/M \subset \mathbf{t}H$.

For the reverse direction we write the elements $\bar{h}\in H$ in the form

$$\bar{h} = \bar{t} + \sum_{i\in\mathbf{N}}\sum_{k\in I} m_{i-1}^k \bar{w}_{i-1}^k,$$

where $\bar{t}\in (T+M)/M$, $m_{i-1}^k\in R$, $i\in\mathbf{N}$, $k\in I$, and $m_i^k = 0$ for almost all i and k. We may assume that for $i>0$ the m_i^k are units or 0 in view of

$$p\bar{w}_i^k = \bar{w}_{i-1}^k + \sum_{j\in\mathbf{N}}\sum_{u\in I_j} \bar{\alpha}_{i-1,j}^{k,u}\bar{t}_j^u, \qquad i\in\mathbf{N}, k\in I. \tag{2}$$

It suffices to show that if $\bar{h}$ is torsion and $\bar{h} = \sum_{i\in\mathbf{N}}\sum_{k\in I} m_{i-1}^k \bar{w}_{i-1}^k$ with $m_i^k \in R$, units or 0 for $i>0$, then $\bar{h} = 0$. Let $h = \sum_{i\in\mathbf{N}}\sum_{k\in I} m_{i-1}^k w_{i-1}^k \in \bar{h}$. Without loss of generality, we may assume $p\bar{h} = 0$, i.e., $ph\in M$. By renumbering, if necessary, there is a natural number $q, d_i^k\in R$ and $z\in N$ such that

$$p\left(\sum_{i=1}^{q+1}\sum_{k=1}^{q} m_{i-1}^k w_{i-1}^k\right) = z + \sum_{i=1}^{q}\sum_{k=1}^{q} d_i^k\left(pw_i^k - w_{i-1}^k - \sum_{j=1}^{q}\sum_{u=1}^{q}\bar{\alpha}_{i-1,j}^{k,u} t_j^u\right). \tag{3}$$

Equation (3) holds in the free module L, and so we may equate the coefficients of like terms. This gives the following:

$$pm_0^k = -d_1^k, \qquad pm_i^k = pd_i^k - d_{i+1}^k, \qquad 1\le i<q,\ 1\le k\le q,$$
$$pm_q^k = pd_q^k, \qquad 1\le k\le q.$$

Working down the equations in sequence we conclude that $d_i^k \in pR$ for all i and $1 \leq k \leq q$. Working the reverse direction and using the fact that $pd_q^k \in p^2R$ and m_i^k are units or 0 for $1 \leq i$, we obtain that $m_i^k = d_i^k = 0$ for $1 \leq i$; hence $m_0^k = 0$ for $1 \leq k \leq q$. Thus, $\bar{h} = 0$. This shows that $\mathbf{t}H = (T+M)/M$ as desired.

Moreover, $T + M = T \oplus \langle pw_i^k - w_{i-1}^k \mid i \in \mathbf{N}, k \in I\rangle$ and

$$\begin{aligned} H/\mathbf{t}H &\cong (L/M)/((T+M)/M) \cong L/(T+M) \\ &\cong \frac{\langle w_{i-1}^k \mid i \in \mathbf{N}, k \in I\rangle}{\langle pw_i^k - w_{i-1}^k \mid i \in \mathbf{N}, k \in I\rangle}. \end{aligned}$$

Hence $H/\mathbf{t}H$ is torsion-free and divisible of rank $|I| = d$.

From the above, it follows that $\{t_i^v + M \mid i \in \mathbf{N}, v \in I_i\}$ is a basis of $\mathbf{t}H$ and that $\{w_0^k + \mathbf{t}H \mid k \in I\}$ forms a basis of the vector space $H/\mathbf{t}H$. Moreover, by (2) and the fact that $H/\mathbf{t}H$ is torsion-free divisible, the set $B = \{\bar{t}_i^v, \bar{w}_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ is a basic generating system of H with relation array $(\bar{\alpha}_{i-1,j}^{k,u})$. Note that in the module H the relation array $(\bar{\alpha}_{i-1,j}^{k,u})$ can be replaced by $(\alpha_{i-1,j}^{k,u})$.

To show that H is determined up to isomorphism, let G be a mixed module in the class $\mathcal{H}$ with torsion submodule $\mathbf{t}G$ of isomorphism type λ, torsion-free dimension d and basic generating system $\widetilde{B} = \{\tilde{t}_i^v, \tilde{w}_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ and corresponding relation array α.

By $t_i^k \mapsto \tilde{t}_i^k$ and $w_{i-1}^k \mapsto \tilde{w}_{i-1}^k$ for all $i \in \mathbf{N}$, $v \in I_i$, $k \in I$, the free module L maps epimorphically to G. This map φ has a kernel containing M, because G has the relation array α. We show that $\ker\varphi = M$. Let $x \in \ker\varphi$. Without loss of generality, x can be written in the form

$$x = \sum_{k\in I} m_{i_k}^k w_{i_k}^k + \sum_{j\in\mathbf{N}}\sum_{u\in I_j} a_j^u t_j^u$$

with $a_j^k \in R\backslash p^jR$ or 0. Furthermore, we have $\varphi T \subset \mathbf{t}G$, hence

$$\begin{aligned} 0 &= \varphi(x + T + M) = \varphi x + \mathbf{t}G \\ &= \sum_{k\in I} m_{i_k}^k \tilde{w}_{i_k}^k + \mathbf{t}G \\ &= \sum_{k\in I} m_{i_k}^k p^{-i_k} \tilde{w}_0^k + \mathbf{t}G \in G/\mathbf{t}G. \end{aligned}$$

Because $G/\mathbf{t}G$ is a vector space, all $m_{i_k}^k$ are 0 by the linear independence of the $\tilde{w}_0^k + \mathbf{t}G$. Similarly, we get

$$0 = \varphi(x + M) = \sum_{j \in \mathbf{N}} \sum_{u \in I_j} a_j^u \tilde{t}_j^u$$

and by the independence of $\tilde{t}_j^u$ and $a_j^k \in R \backslash p^j R$ then $a_j^u = 0$, i.e., $x \in M$. □

4. A splitting criterion in the class $\mathcal{H}$. In this section we give a necessary and sufficient criterion for the splitting of modules in the class $\mathcal{H}$.

The proof of the next lemma is straightforward and is omitted.

Lemma 5. *Let $\{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ be a basic generating system of a module G in the class $\mathcal{H}$. Let $\mu \in K$ and l be a natural number such that $p^l \mu \in R$. Then*

$$p^l \mu a_{l+i}^k \in \mu(a_i^k + \mathbf{t}G) \in G/\mathbf{t}G.$$

The relation array $\alpha = (\alpha_{i-1,j}^{k,u})$ is said to be *small* if the array $(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s})_{s,j}^{k,u}$ is a relation array. The last array is not necessarily a relation array since the row finiteness in j and u is not guaranteed.

The concept of smallness was motivated by various papers on splitting and in particular the simple idea of a p-sequence, see [**2, 3, 4**]. For example, a rank one mixed module in $\mathcal{H}$ which is split has a height sequence, and thus a basic generating system with corresponding relation array, which requires relatively few nonzero coefficients for the torsion elements, i.e., a small relation array.

Theorem 6. *For a module G in the class $\mathcal{H}$ the following are equivalent*:

(1) *G splits*;

(2) *G has only small relation arrays*;

(3) *G has one small relation array.*

Proof. (1) $\Rightarrow$ (2). If G splits, then one may choose a basic generating system $A = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ with relation array 0. If $C = \{y_i^v, c_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ is another basic generating system of G, we must show that the relation array corresponding to C is small. First we form a basic generating system $B' = \{y_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ which replaces the torsion generators of A. Clearly the relation array corresponding to B' is also 0. Consider the automorphism of the vector space $G/\mathbf{t}G$ of dimension $d = |I|$, which maps $a_0^k + \mathbf{t}G \mapsto c_0^k + \mathbf{t}G$. This automorphism is described by an invertible row finite $d \times d$-matrix $D = (d_{kh})$ with entries in K via

$$c_0^k + \mathbf{t}G = \sum_{h \in I} d_{kh}(a_0^h + \mathbf{t}G)$$

for all $k \in I$. By Lemma 5 we choose new torsion-free generators which are in the same cosets modulo $\mathbf{t}G$ as c_i^k, namely,

$$b_i^k = \sum_{h \in I} p^{f(k)} d_{kh} a_{f(k)+i}^h,$$

for all $i \in \mathbf{N}$, $k \in I$, where $f : I \to \mathbf{N} \cup \{0\}$ is a function with the property that $p^{f(k)} d_{kh} \in R$ for all $h \in I$. Thus we have

$$\begin{aligned} p^i(c_i^k - b_i^k) &= p^i c_i^k - p^i b_i^k \\ &\equiv c_0^k - \sum_{h \in I} p^{f(k)+i} d_{kh} a_{f(k)+i}^h \pmod{\mathbf{t}G} \\ &\equiv \sum_{h \in I} d_{kh}(a_0^h - p^{f(k)+i} a_{f(k)+i}^h) = 0 \pmod{\mathbf{t}G}. \end{aligned}$$

Hence $c_i^k - b_i^k \in \mathbf{t}G$ for all $i \in \mathbf{N}$, $k \in I$. The relation array of G corresponding to the basic generating system $B = \{y_i^v, b_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ is also 0 since

$$\begin{aligned} pb_i^k &= \sum_{h \in I} p^{f(k)} d_{kh} p a_{f(k)+i}^h \\ &= \sum_{h \in I} p^{f(k)} d_{kh} a_{f(k)+i-1}^h = b_{i-1}^k \end{aligned}$$

for all $i \in \mathbf{N}$, $k \in I$. So, in order to compute the relation array relative to C, it remains to replace the b_i^k in the basic generating system B

within their cosets modulo $\mathbf{t}G$ by $c_i^k = b_i^k + t_i^k$. Write the torsion elements $t_i^k = c_i^k - b_i^k$ in terms of the generators $\{y_j^u \mid j \in \mathbf{N}, u \in I_j\}$ as follows $t_i^k = \sum_{j\geq 1}\sum_{u\in I_j} \tau_{i,j}^{k,u} y_j^u$. It is easy to check that the array $(\tau_{i,j}^{k,u})$ satisfies the definition of a relation array. Now

$$\begin{aligned} pc_i^k &= p(b_i^k + t_i^k) = b_{i-1}^k + pt_i^k \\ &= c_{i-1}^k - t_{i-1}^k + pt_i^k \\ &= c_{i-1}^k + \sum_{j\geq 1}\sum_{u\in I_j} \gamma_{i-1,j}^{k,u} y_j^u, \end{aligned}$$

where $\gamma_{i-1,j}^{k,u} \equiv p\tau_{i,j}^{k,u} - \tau_{i-1,j}^{k,u} \pmod{p^j}$. It suffices to show that $(\gamma_{i-1,j}^{k,u}) \equiv p(\tau_{i,j}^{k,u}) - (\tau_{i-1,j}^{k,u})$ is small. But

$$\begin{aligned} \sum_{i=s}^{s+j-1} \gamma_{i,j}^{k,u} p^{i-s} &\equiv \sum_{i=s+1}^{s+j} \tau_{i,j}^{k,u} p^{i-s} - \sum_{i=s}^{s+j-1} \tau_{i,j}^{k,u} p^{i-s} \\ &\equiv -\tau_{s,j}^{k,u} \pmod{p^j} \end{aligned}$$

and the fact that $(\tau_{i,j}^{k,u})$ is a relation array implies that the relation array $(\gamma_{i-1,j}^{k,u})$ is small.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1). Let $\{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ be a basic generating system of G with small relation array α. Let the pair (k,s) be fixed. By the row finiteness in u and j the elements

$$y_j^{k,s} = \sum_{u\in I_j}\left(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s}\right) x_j^u$$

are well defined. Note that $y_j^{k,s}$ is a sum over the columns of the array α and that for a fixed pair $(k,s) \in I \times \mathbf{N}$ the sum $\sum_{j\geq 1} y_j^{k,s}$ is not well defined in general. However, when the relation array α is small, i.e., $(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s})_{s,j}^{k,u}$ is a relation array, the sum makes sense. Hence, for a fixed pair (k,s) only finitely many elements $y_j^{k,s}$ are not 0 and the sum $\sum_{j\geq 1} y_j^{k,s}$ is defined. The above allows us to introduce new generators b_s^k, $(k,s) \in I \times \mathbf{N}$, where

$$b_s^k = a_s^k + \sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s}\Big) x_j^u. \tag{4}$$

Then we get by Proposition 3

$$\begin{aligned}
pb_s^k &= pa_s^k + p\sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s}\Big) x_j^u \\
&= a_{s-1}^k + \sum_{j\geq 1}\sum_{u\in I_j} \alpha_{s-1,j}^{k,u} x_j^u + \sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s+1}\Big) x_j^u \\
&= b_{s-1}^k - \sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s-1}^{s+j-2} \alpha_{i,j}^{k,u} p^{i-s+1}\Big) x_j^u + \sum_{j\geq 1}\sum_{u\in I_j} \alpha_{s-1,j}^{k,u} x_j^u \\
&\quad + \sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s+1}\Big) x_j^u \\
&= b_{s-1}^k - \sum_{j\geq 1}\sum_{u\in I_j}\Big(\sum_{i=s-1}^{s+j-2} \alpha_{i,j}^{k,u} p^{i-s+1} - \alpha_{s-1,j}^{k,u} \\
&\qquad\qquad - \sum_{i=s}^{s+j-1} \alpha_{i,j}^{k,u} p^{i-s+1}\Big) x_j^u \\
&= b_{s-1}^k + \sum_{j\geq 1}\sum_{u\in I_j} \alpha_{s+j-1,j}^{k,u} p^j x_j^u \\
&= b_{s-1}^k.
\end{aligned}$$

Thus, the relation array corresponding to the new basic generating system is 0 and G splits. □

It is well known that if the torsion submodule is bounded, then the module splits. For the class $\mathcal{H}$ this is an easy consequence of the above theorem since there must exist a natural number m such that $\alpha_{i-1,j}^{k,u} = 0$ for all $j \geq m$, hence the relation array is small and the module would split.

Examples. Let G and H be p-local mixed groups of torsion-free rank 1 with torsion subgroup $\mathbf{t} = \oplus_{i\in\mathbf{N}}\langle x_i\rangle \cong \oplus_{i\in\mathbf{N}}\mathbf{Z}(p^i)$, divisible quotients $G/\mathbf{t}G \cong H/\mathbf{t}H \cong \mathbf{Q}$ and basic generating systems $\{x_i, a_{i-1} \mid i \in \mathbf{N}\}$ where p^i is the order of x_i for all $i \in \mathbf{N}$. Let H have the relations

$$pa_{2i} = a_{2i-1}, \qquad pa_{2i-1} = a_{2i-2} + x_i,$$

and let G have the relations

$$pa_i = a_{i-1} + x_i$$

for all $i \in \mathbf{N}$. The only nonzero entries in the relation array of the group H are $\alpha^H_{2i-2,i} = 1$; for G the only nonzero entries are $\alpha^G_{i-1,i} = 1$. Thus, H has a small relation array and hence splits while G does not. The splitting of H can be seen directly by forming generators of the torsion-free direct summand as in formula (4).

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