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CHARACTERISTIC 4 WITT RINGS THAT ARE THE PRODUCT OF GROUP RINGS

TIMOTHY P. KELLER

ABSTRACT. Necessary and sufficient conditions for a Witt ring of characteristic four to be the Witt product of group rings are presented. The proof is similar to a result of M. Marshall for Witt rings of characteristic two.

The definition of a Witt ring is that of [**5**]. Following [**3**], G will denote the group of units associated with the Witt ring R, and $q : G \times G \to R$ will denote the quaternionic map. For $x \in G$, $D\langle 1, x\rangle$ denotes the value group of the Pfister form $\langle 1, x\rangle$; if one takes the quaternionic map as the primitive concept one may define $D\langle 1, x\rangle = \{y \in G \mid q(y, -x) = 0\}$. For $S \subseteq G$, $\mathrm{gr}\,(S)$ denotes the group generated by the set S.

Motivation for the main result comes from [**4**], where Marshall proves:

Theorem 1. *Suppose R is a Witt ring of characteristic two. Then, in the category of Witt rings, R is a product of n group rings if and only if there exists an element $a \neq 1$ in G satisfying*:

(i) $\mathrm{rad}\,(a) = D\langle 1, a\rangle$ *has* 2^n *elements and*

(ii) $D\langle 1, b\rangle D\langle 1, ab\rangle = G$ *holds for all* $b \in \mathrm{rad}\,(a)$.

The main result of this paper is an analogy of Theorem 1 for Witt rings of characteristic four. Making the correct analogy depends on the simple observation that $1 = -1$ for rings of characteristic two, and so the element a of Theorem 1 is not equal to -1; but -1 can serve as this special element when $\mathrm{char}\, R = 4$.

The main result:

Theorem 2. *Suppose R is a nondegenerate Witt ring of characteristic* 4. *Then in the category of Witt rings, R is a product of n group rings if and only if*:

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(i) $|D(2)| = 2^n$ *and* $D(2) \subseteq D\langle 1, \omega\rangle$ *for all* $\omega \in D(2)$.

(ii) $D\langle 1, \omega\rangle D\langle 1, -\omega\rangle = G$ *for all* $\omega \in D(2)$.

Remarks. 1. The assumption of nondegeneracy is not severe. If R is degenerate one may decompose R as a Witt product $R' \times R''$ where R' is nondegenerate and R'' is totally degenerate. If the conditions: $D(2) \subset D\langle 1, \omega\rangle$ for all $\omega \in D(2)$ and $D\langle 1, \omega\rangle D\langle 1, -\omega\rangle = G$ for all $\omega \in D(2)$ hold for G they also hold for G'; hence R' is a product of group rings. Moreover, a totally degenerate Witt ring is isomorphic to a product of $\mathbf{Z}_4$ and group rings $\mathbf{Z}_2[c_2]$ where c_2 is cyclic of order two.

2. The condition $D(2) \subseteq D\langle 1, \omega\rangle$ for all $\omega \in D(2)$ always holds when $|D(2)| = 4$.

The necessity of conditions (i) and (ii) is easily established. If R is the product of n group rings $R_1, \dots, R_n$ then $G = G_1 \times \cdots \times G_n$ and $D_G(2) = \{\pm 1, \pm 1, \dots, \pm 1\}$; and conditions (i) and (ii) are easily verified.

To establish the sufficiency requires proving a number of lemmas, analogous to Lemmas 9–16 of [**4**]. Sign changes must be watched carefully in the characteristic four case. In particular, the identity $q(\omega, -t) = q(\omega, t)$ holds if and only if $\omega \in D(2)$. This observation is frequently employed in the following proofs.

Lemma 1. *Suppose* $\omega \in D(2)$, $x \in G$. *Further suppose* $x = x_1 x_2$ *is a decomposition of* x *with* $x_1 \in D\langle 1, \omega\rangle$ *and* $x_2 \in D\langle 1, -\omega\rangle$. *Then* $D\langle 1, x\rangle \cap D(2)$ *is a subgroup of* $D\langle 1, x_i\rangle \cap D(2)$, $i = 1, 2$.

Proof. Let $\alpha \in D\langle 1, x\rangle \cap D(2)$. Then $q(x, -\alpha\omega) = q(x, -\omega) = q(x_2, -\omega) = q(x_2, -1)$. By the linkage property there is a $\delta \in D(2)$ so that

$$q(x_2, -1) = q(\delta x_2, -1) = q(\delta x_2, -\alpha\omega) = q(x, -\alpha\omega).$$

From that last equality it follows that $q(\delta x_1, -\alpha\omega) = 0$. Hypothesis (i) implies $q(\delta, -\alpha\omega) = 0$, hence also $q(x_1, -\alpha\omega) = 0$. But $x_1 \in D\langle 1, \omega\rangle$ implies $q(x_1, -\omega) = 0$ and so $q(x_1, \alpha) = 0$. Since $q(-x, \alpha) = q(x, \alpha) = 0$, $q(x_2, \alpha) = 0$ also. □

Lemma 2. *Suppose that, in addition to the hypotheses of Lemma* 1, $q(x,\omega) \neq 0$ *and* $q(x,-\omega) \neq 0$. *Then* $x_i \notin D(2)$ *for* $i = 1,2$ *and the inclusions of Lemma* 9 *are proper.*

Proof. If $x_1 \in D(2)$, then $x_1 \in D\langle 1,-\omega\rangle$ by (i); but then $x = x_2x_1 \in D\langle 1,-\omega\rangle$, that is $q(x,\omega) = 0$, a contradiction. Similarly, $x_2 \in D(2)$ would lead to the contradiction $q(x,-\omega) = 0$.

That $x_1 \in D\langle 1,\omega\rangle$ implies $-x_1 \in D\langle 1,\omega\rangle$ by (i); so $-\omega \in D\langle 1,x_1\rangle$. However, $-\omega \notin D\langle 1,x\rangle$, so $D\langle 1,x\rangle \cap D(2)$ is a proper subgroup of $D\langle 1,x_1\rangle \cap D(2)$. Similarly $\omega \in D\langle 1,x_2\rangle$ but $\omega \notin D\langle 1,x\rangle$ so $D\langle 1,x\rangle \cap D(2)$ is a proper subgroup of $D\langle 1,x_2\rangle \cap D(2)$. □

Analogously to [**4**] an element $x \in G\backslash D(2)$ is defined to be *maximal* if the group $D\langle 1,x\rangle \cap D(2)$ is maximal with respect to inclusion. On the set of all maximal elements define an equivalence relation by $x \sim y$ if $D\langle 1,x\rangle \cap D(2) = D\langle 1,y\rangle \cap D(2)$. The observations of Marshall are still valid and useful:

(1) $x \in G\backslash D(2)$ is maximal if and only if $D\langle 1,x\rangle \cap D(2)$ has index 2 in $D(2)$. Suppose $D\langle 1,x\rangle \cap D(2)$ has index z in $D(2)$. Then for $z \in D(2)$ either $z \in D\langle 1,x\rangle \cap D(2)$ or $-z \in D\langle 1,x\rangle \cap D(2)$.

Suppose there is an element y so that

$$D\langle 1,x\rangle \cap D(2) \subseteq D\langle 1,y\rangle \cap D(2) \subseteq D\langle 1,-xy\rangle \cap D(2).$$

Also suppose by way of contradiction there is a z in $D\langle 1,y\rangle \cap D(2)$ so that z is not in $D\langle 1,x\rangle \cap D(2)$; and since $-z$ and z are both elements of $D\langle 1,-xy\rangle \cap D(2)$ it follows that $-1 \in D\langle 1,-xy\rangle \cap D(2)$. But then $xy \in D(2)$ and $D\langle 1,x\rangle = D\langle 1,y\rangle$ after all.

If $D\langle 1,x\rangle \cap D(2)$ is maximal in $D(2)$, then for $z \in D(2)$ not in $D\langle 1,x\rangle \cap D(2)D\langle 1,xz\rangle \cap D(2) = D\langle 1,x\rangle \cap D(2)$. But $D\langle 1,x\rangle \cap D\langle 1,xz\rangle \subseteq D\langle 1,-z\rangle$, so $D\langle 1,x\rangle \cap D(2) = D\langle 1,-z\rangle \cap D(2)$. Hence $-z \in D\langle 1,x\rangle$, and $D\langle 1,x\rangle \cap D(2)$ has index z in $D(2)$.

(2) Each element $x \in G\backslash D(2)$ is a product of maximal elements.

(3) If $\omega \in D(2)$ and x is maximal ωx is maximal and $\omega x \sim x$. This follows from hypothesis (i).

(4) Suppose x and y are maximal and $x \sim y$. Then $D\langle 1,y\rangle \cap D(2) \cap D\langle 1,y\rangle = D\langle 1,x\rangle \cap D(2) \cap D\langle 1,y\rangle$; hence $D\langle 1,y\rangle \cap D(2) \subseteq$

$D\langle 1, xy\rangle \cap D(2)$. So either $xy \in D(2)$ (and $D\langle 1, xy\rangle \cap D(2) = D(2)$) or $xy \sim x$.

Let x be maximal and define $\Delta = \{y \mid y \text{ is maximal and } y \sim x\}\cdot D(2)$. From (3) and (4) it follows that Δ is a subgroup of G.

Lemma 3. *If $t_1, \dots, t_s$ are pairwise inequivalent maximal elements, then*

(1) *$t_1, \dots, t_s$ are linearly independent modulo $D(2)$. (Here $G\backslash D(2)$ is viewed as a $\mathbf{Z}/2\mathbf{Z}$ vector space.)*

(2) *$D\langle 1, t_1 \dots t_s\rangle \cap D(2) = \cap_i D\langle 1, t_i\rangle \cap D(2)$.*

Proof. The proof of (1) is by induction on the number of t_i, N, that are linearly independent. For any index i, $t_i \notin D(2)$ so for $N = 1$ one has a set of linearly independent elements. Now by induction suppose $t_1 \dots t_N \notin D(2)$, and consider $\delta = t_1 t_2 \dots t_N t_{N+1}$.

By way of contradiction, suppose $\delta \in D(2)$. Repeatedly using observation (4), one has the product $t_1 t_2 \dots t_N \sim t_1$. Now t_1 and t_{N+1} are inequivalent, hence $t_1 t_2 \dots t_N$ and t_{N+1} are inequivalent. From Lemmas 1 and 2 the inequivalence of $t_1 t_2 \dots t_N$ and t_{N+1} implies the existence of a $\beta \in D(2)$ so that

$$t_{N+1} \in D\langle 1, \beta\rangle \quad \text{and} \quad t_1 \dots t_N \in D\langle 1, -\beta\rangle.$$

Now hypothesis (i) and $\delta \in D(2)$ imply $\delta t_1 \dots t_N \in D\langle 1, -\beta\rangle$. But $\delta t_1 \dots t_N = t_{N+1}$; and $t_{N+1} \in D\langle 1, \beta\rangle \cap D\langle 1, -\beta\rangle$ implies $t_{N+1} \in D(2)$, a contradiction. Hence $\delta \notin D(2)$. And so, by induction, $N = s$.

For (2) one first notes that $\cap_i D\langle 1, t_i\rangle \cap D(2) \subseteq D\langle 1, t_1 \dots t_s\rangle \cap D(2)$ is always true. To obtain the reverse containment suppose, by way of contradiction, there is a $\beta \in D(2)$ so that $\beta \in D\langle 1, t_1 \dots t_s\rangle$; but $\beta \notin D\langle 1, t_i\rangle$ for some i. Without loss of generality suppose $\beta \notin D\langle 1, t_1\rangle$. If $\beta \notin D\langle 1, t_1\rangle$, then $t_1 \in D\langle 1, \beta\rangle$. Now since t_1 and $t_2 \dots t_s$ are inequivalent $t_2 \dots t_s \in D\langle 1, -\beta\rangle$. That $\beta \in D\langle 1, t_1 \dots t_s\rangle$ implies $t_1 \dots t_s \in D\langle 1, -\beta\rangle$; and then $t_1 = (t_1 \dots t_s)(t_2 \dots t_s) \in D\langle 1, -\beta\rangle$. But $t_1 \in D\langle 1, \beta\rangle \cap D\langle 1, -\beta\rangle$ implies $t_1 \in D(2)$, contradicting the maximality of t_1. □

The next lemma is a technical result that will be used in the proofs of the lemmas that follow.

Lemma 4. *Suppose $x \in D\langle 1, \beta\rangle$ and $y \in D\langle 1, -\beta\rangle$ for some $\beta \in D(2)$. Then there is a $\gamma \in D(2)$ so that $q(-\gamma x, \gamma y) = 0$.*

Proof. That $q(y, \beta) = 0$ and $q(x, -\beta) = 0$ imply $q(xy, \beta) = q(x, \beta) = q(x, -1)$.

Thus by the linkage property there is a $\delta \in D(2)$ such that $q(x, -1) = q(\delta x, -1) = q(\delta x, xy) = q(xy, \beta)$. This last equality implies $q(xy, \delta\beta x) = 0$. Then since $q(\delta x, xy) = q(xy, \beta)$. This last equality implies $q(xy, \delta\beta x) = 0$. Then since $q(-\delta\beta x, \delta\beta x) = 0$, one has $q(-\delta\beta y, \delta\beta x) = 0$. Take $\gamma = -\delta\beta$ and the proof is complete. □

Lemma 5. *Suppose $t_1, t_2, \dots, t_s$ are pairwise inequivalent maximal elements. Then the group $\cap_i D\langle 1, t_i\rangle \cap D(2)$ has index 2^s in $D(2)$. Moreover, $s \le n$.*

Proof. When $s = 1$ the truth of the conclusion follows from Lemmas 1 and 2. So assume $s \ge 2$. Let H be the group $\cap_i D\langle 1, t_i\rangle \cap D(2)$; by induction on s assume H has index 2^s in $D(2)$ and let t be a maximal element with $H \subseteq D\langle 1, t\rangle$. One must show $t \sim t_i$ for some index i, $1 \le i \le s$. By way of contradiction, suppose t is not equivalent to any t_i.

Since H has index 2^s in $D(2)$ there are elements $\beta_1, \dots, \beta_s$ in $D(2)$ that generate $D(2)$ modulo H. For every i, $D\langle 1, t_i\rangle \cap D(2)$ has index 2 in $D(2)$, hence one may choose these elements so that $q(t_i, -\beta_i) = 0$ and $q(t_j, \beta_i) = 0$ for all indices $j \ne i$.

Now $D\langle 1, t\rangle \cap D(2)$ also has index two in $D(2)$; hence, for any $\omega \in D(2)$ one has either $q(t, \omega) = 0$ or $q(t, -\omega) = 0$. If $q(t, \beta_i) = 0$ for all i, then $D(2) \subseteq D\langle 1, t\rangle$; in particular, $-1 \in D\langle 1, t\rangle$ and $t \in D(2)$, a contradiction. Therefore $q(t, -\beta_i) = 0$ for some index i. Without loss of generality assume this index is s.

Then one has $q(tt_s, -\beta_s) = 0$ and $q(t_1 t_2 \cdots t_{s-1}, \beta_s) = 0$. Applying Lemma 4 with $x = t_1 t_2 \cdots t_{s-1}$ and $y = tt_s$ there is a $\gamma \in D(2)$ so that $q(-\gamma x, \gamma y) = 0$. Replacing t_1 with γt_1 and t_s with γt_s one may assume

$\gamma = 1$ and $q(-x, y) = 0$.

Using Lemma 4 again, but now applied to elements x and $\beta_i y$, one concludes that for each index i, $1 \leq i \leq s-1$, there is an $\alpha_i \in D(2)$ so that $q(-\alpha_i x, \alpha_i \beta_i y) = 0$.

Expanding the last equality gives

(1) $q(-\alpha_i x, \beta_i) = q(-\alpha_i x, \alpha_i y)$ and

(2) $q(-x, \beta_i) = q(-\alpha_i x, \alpha_i y)$.

From $q(-x, y) = 0$ it follows that $q(xy, -x) = 0$. After expanding one has

(3) $q(xy, -\alpha_i x) = q(xy, \alpha_i)$

Now $q(-\alpha_i x, \alpha_i y) = q(xy, -\alpha_i x)$. So together (1), (2) and (3) imply

(4) $q(\beta_i, -x) = q(xy, \alpha_i)$.

From $q(\beta_i, t_j) = 0$ for $j \neq i$ and $q(t_i, -\beta_i) = 0$ it follows that

(5) $q(\beta_i, x) = q(-1, t_i)$.

Now $q(\beta_i, x) = q(\beta_i, -x)$ so from (4) and (5) it follows that $q(\alpha_i, xy) = q(-1, t_i)$.

Write α_i as a linear combination $\alpha_i = \prod_j \beta_j^{e_j} \delta$; $\delta \in H$ and $e_j \in \{0, 1\}$ for all j.

Then $q(\alpha_i, xy) = q(-1, t_1^{e_1} \dots t_s^{e_s} t^f)$ where $f \in \{0, 1\}$. This last equality and the equality $q(\alpha_i, xy) = q(-1, t_i)$ imply $q(-1, t_1^{e_1} t_2^{e_2} \dots t_{i-1}^{e_{i-1}} t_{i+1}^{e_{i+1}} \dots t_s^{e_s} t^f) = 0$.

By Lemma 3 the elements $t_1, t_2, \dots, t_s$ are linearly independent modulo $D(2)$, hence $e_i = 1$, $e_j = 0$ for $j \neq i$, and $f = 0$. Then α_i may be written as $\delta\beta_i$ and from (4) it follows that $q(\beta_i, -x) = q(\delta\beta_i, xy) = q(\beta_i, xy)$. Hence also $q(\beta_i, -y) = 0$, that is $q(\beta_i, -tt_s) = q(\beta_i, tt_s) = 0$. Now $q(\beta_i, tt_s) = 0$ and $q(\beta_i, t_s) = 0$ imply $q(\beta_i, t) = 0$, $1 \leq i \leq s-1$.

Now note that $\mathrm{gr}\,(\beta_1, \dots, \beta_{s-1}, H) \subseteq D(2) \cap D\langle 1, t\rangle$, and both groups have index 2 in $D(2)$; hence $D\langle 1, t\rangle \cap D(2) = \mathrm{gr}\,(\beta_1, \dots, \beta_{s-1}, H)$. Similarly $D\langle 1, t_s\rangle \cap D(2) = \mathrm{gr}\,(\beta_1, \dots, \beta_{s-1}, H)$. By definition $t \sim t_s$, contradicting the assumption that t was inequivalent to any t_i. □

If s is maximal, then $t_1, \dots, t_s$ is a basis for G modulo $D(2)$. Thus if $\beta \in H$, $q(\beta, t_i) = 0$ for $1 \leq i \leq s$ implies $q(\beta, t) = 0$ for all $t \in G$. By assumption R is nondegenerate, so this last identity implies $H = \{1\}$ and so $s = n$.

Lemma 6. *If t and u are inequivalent maximal elements, then exactly one of the following holds*: $q(t,u) = 0$, $q(-t,u) = 0$, $q(t,-u) = 0$, *and* $q(-t,-u) = 0$.

Proof. Since t and u are inequivalent there is a $\beta \in D(2)$ such that $t \in D\langle 1, -\beta\rangle$ and $u \in D\langle 1, \beta\rangle$. Then by Lemma 4 there is a $\gamma \in D(2)$ such that $q(-\gamma t, \gamma u) = 0$. Expanding this equality gives

(A) $q(-t, \gamma u) = q(-u, \gamma u)$.

(B) $q(u, -\gamma t) = q(t, -\gamma t)$.

Since t and u are maximal, t is an element of one of $D\langle 1, \gamma\rangle$ and $D\langle 1, -\gamma\rangle$; u is an element of one of $D\langle 1, \gamma\rangle$ and $D\langle 1, -\gamma\rangle$. This means there are four possibilities

(1) $q(\gamma, t) = 0$, $q(\gamma, u) = 0$

(2) $q(\gamma, t) = 0$, $q(-\gamma, u) = 0$

(3) $q(-\gamma, t) = 0$, $q(\gamma, u) = 0$

(4) $q(-\gamma, t) = 0$, $q(-\gamma, u) = 0$.

Now if (1) holds, $q(\gamma, ut) = 0$.

Identity (B) implies $q(-\gamma t, ut) = 0$. Hence, $q(-t, ut) = q(-t, u) = 0$.

If (2) holds $q(-\gamma, u) = 0$ implies $q(t, u) = q(-\gamma t, u)$, and then (B) implies $q(t, u) = q(t, -\gamma t) = q(u, -\gamma t)$. That $q(\gamma, t) = 0$ implies $q(-\gamma t, t) = 0$, hence $q(t, u) = 0$.

If (1) holds one notes that $q(\gamma, u) = q(\gamma, -u) = q(\gamma u, -u) = 0$. By (A), $q(-t, \gamma u) = 0$. Then $q(-t, -\gamma) = q(-t, -u)$. Now $q(-t, -\gamma) = q(t, -\gamma) = 0$; hence, $q(-t, -u) = 0$.

Now if (4) holds, then $q(-\gamma, ut) = 0$; and (A) implies that $q(ut, \gamma u) = 0$. Hence, $q(ut, -u) = 0$, and since $q(u, -u) = 0$ one has $q(t, -u) = 0$.

Since neither t nor u is an element of $D(2)$, it follows that the four possibilities given in the statement of the lemma are mutually exclusive. $\square$

Let $\Delta_1, \Delta_2, \dots, \Delta_n$ be the groups Δ_i defined by $\Delta_i = \{y \mid y \text{ is maximal and } y \sim t_i\} D(2)$, where $\{t_1, t_2, \dots, t_n\}$ is a maximal set of maximal inequivalent elements. Following [**4**], define a maximal element t_i of Δ_i to be Δ_j-*compatible* if either $q(t_i, t_j) = 0$ or $q(t_i, -t_j) = 0$ for all indices $j \neq i$.

Lemma 7. *For each pair of indices (i, j) with $i \neq j$ either t_i is Δ_j-compatible or $-t_i$ is Δ_j compatible.*

Proof. If the conclusion does not hold there exist $t_j, u_j \in \Delta_j$ so $q(t_i, t_j) = 0$ and $q(-t_i, u_j) = 0$. Consider the element $t_j u_j \in \Delta_j$. By observation (4) following Lemma 2, either $t_j u_j$ is maximal or $t_j u_j \in D(2)$.

Case 1. *$t_j u_j$ is maximal.* By Lemma 6 one of

$$q(t_i, t_j u_j), q(-t_i, t_j u_j), q(t_i, -t_j u_j), q(-t_i, -t_j u_j)$$

is zero. Now if $q(t_i, t_j u_j) = 0$, then $q(t_i, t_j) = 0$ implies $q(t_i, u_j) = 0$. But $q(t_i, u_j) = 0$ and $q(-t_i, u_j) = 0$ imply $u_j \in D(2)$, contradicting the maximality of u_j. If one of the other three possibilities maintains, one similarly obtains $u_j \in D(2)$ or $t_j \in D(2)$, contradicting maximality.

Case 2. $t_j u_j \in D(2)$. Since t_i is maximal, either $q(t_i, t_j u_j) = 0$ or $q(-t_i, t_j u_j) = 0$. If $q(t_i, t_j u_j) = 0$, then $q(t_i, t_j) = 0$ implies $q(t_i, u_j) = 0$; and then $q(-t_i, u_j) = 0$ implies $u_j \in D(2)$, contradicting the maximality of u_j. If $q(-t_i, t_j u_j) = 0$, then $q(-t_i, u_j) = 0$ implies $q(-t_i, t_j) = 0$; and then in turn $q(t_i, t_j) = 0$ implies $t_j \in D(2)$, contradicting the maximality of t_j. $\square$

For the remainder of the discussion let $\beta_1, \beta_2, \dots, \beta_n$ be a basis for $D(2)$ as described in Lemma 5, and let $\{t_1, \dots, t_n\}$ be a corresponding set of maximal elements. Recall $q(t_i, -\beta_i) = 0$ for all i and $q(t_i, \beta_j) = 0$ for all (i, j) with $i \neq j$.

Lemma 8. *Suppose that $u = \alpha t_i$ for some $\alpha \in D(2)$. Then either u is Δ_j compatible for all $j \neq i$ or $\beta_i u$ is Δ_j-compatible for all $j \neq i$.*

Proof. Write $u = \beta_1^{e_1}\beta_2^{e_2}\cdots\beta_n^{e_n}t_i$, $e_i \in \{0,1\}$ for $1 \le i \le n$. Then $q(u,t_j) = 0$ for all $j \neq i$. Since $q(\beta_j, t_i) = 0$ for all $j \neq i$, this implies that $q(\beta_i^{e_i}t_i, t_j) = 0$ for all $j \neq i$; further, since $q(t_i, \beta_i) = 0$, one may conclude $q((-1)^{e_i}t_i, t_j) = 0$ for all $j \neq i$.

One must have $e_i = 0$ if t_i is Δ_j-compatible, and one must have $e_i = 1$ if $-t_i$ is Δ_j-compatible for all $j \neq i$. There are no restrictions on e_j for $j \neq i$, and hence the result.

Define $S_i = \{t \in \Delta_i \mid t$ is maximal and t is Δ_j-compatible for all $j \neq i\}$; let $G_i = \mathrm{gr}\,(\beta_i, S_i)$. □

Proof of Theorem 2. Consider the groups $G_1, \dots, G_n$.

Since $\{\beta_1, \dots, \beta_n\}$ is a basis for $D(2)$, the definition of the groups G_i implies

$$G_1G_2\dots G_n = D(2)S_1S_2\dots S_n.$$

By Lemma 7, $\Delta_i = \{\pm 1\}S_i$. Hence $G_1G_2\dots G_n = D(2)\Delta_1\Delta_2\dots\Delta_n = G$.

The elements $\{\beta_1, \dots, \beta_n\}$ are linearly independent, as are the elements $\{t_1, \dots, t_n\}$; hence G is the direct product of the groups $G_1, G_2, \dots, G_n$.

Now consider $g_i = \beta_i^e t$ for $t \in S_i$ and $g_j = \beta_j^f u$ for $u \in S_j$, $i \neq j$ and $e, f \in \{0,1\}$. By definition of S_i and S_j, $q(u,t) = 0$. But the identities $q(\beta_i, t_j) = 0$, $q(\beta_j, t_i) = 0$ imply $q(\beta_i t_i, t_j) = q(t_i, t_j) = q(t_i, \beta_j t_j) = q(\beta_i t_i, \beta_j t_j)$.

Hence $q(g_i, g_j) = 0$ for all choices of e and f. Thus the decomposition $G = G_1 \times G_2 \times \cdots \times G_n$ is orthogonal.

The quaternionic map on G induces a quaternionic map on each G_i with associated Witt ring R_i. Note $D\langle 1, \beta_i\rangle \cap G_i = \{1, \beta_i\}$, that is, β_i is a rigid element of G_i. It follows that each R_i is a group ring. By Theorem 3.4 of [**2**], R is isomorphic to $R_1 \times R_2 \times \cdots \times R_n$. □

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Department of Mathematics, Southeast Missouri State University, Cape Girardeau, Missouri 63701