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A QUASI INNER PRODUCT APPROACH FOR CONSTRUCTING SOLUTION REPRESENTATIONS OF CAUCHY PROBLEMS

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ABSTRACT. Integral representations for a wide class of Cauchy problems are developed by employing the method of quasi inner products. This approach reduces the construction of solutions to translations followed by integrations. Bounds on solutions corresponding to polynomial data are obtained and these permit proving series representation theorems in terms of special polynomial and other sets. Among the problems considered are ones associated with the Yukawa, the Helmholtz, the EPD and the GASPT equations. Some nonstandard and higher order equations are also considered.

1. Introduction. Function theoretic methods have played a significant role in the development of solutions of partial differential equations and in deducing their properties. These methods have a long and rich history and ongoing refinements continue to lead to new and important results for initial and boundary value problems. Major innovators in this subject include E.T. Whittaker [**23, 24**], S. Bergman [**2**] and I. Vekua [**20**]. The list of authors who have made contributions in this area is indeed extensive and the reader is further referred to the treatises of R.P. Gilbert [**17**] and K.W. Bauer and S. Ruscheweyh [**1**] for a broad coverage of the subject and for their extensive bibliographies.

At the heart of most of these approaches for solving partial differential equations is the Cauchy integral formula and various modifications of it. In this paper we employ one such version which permits the construction of integral and series representations for a wide class of Cauchy problems that involve, for example, the wave equation, the Laplace equation, the Euler-Poisson-Darboux (EPD) equation, the equation of generalized axially symmetric potential theory (GASPT), the Yukawa

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equation, and the Helmholtz equation. This version of the integral formula is the quasi inner product (qip) which was introduced in [**5**]. It was employed in [**6**] to construct solutions of a variety of Cauchy problems by carrying out complex transformations on solutions of generalized heat problems. In a number of the applications considered there, the complex transformations reduced to real transformations and these, in turn, defined transmutational type formulas which could be extended to apply to abstract generalizations of the underlying pair of problems (see [**3, 8, 9, 13** and **14**], for example, for background material on transmutations). In this paper we will be concerned with developing a more systematic approach for obtaining real transformations for hyperbolic problems and their complex analogues for elliptic problems. Further applications of qips may be found in the construction of ascent type formulas for the solutions of various "higher dimensional" hyperbolic and elliptic problems [**7**].

Our approach for constructing real transformations of the type mentioned above is to first regard the solution of a Cauchy problem as the result of a formal solution operator series acting on appropriate data associated with the problem. The question then becomes the one of assigning a precise meaning to this formalism. The approach taken in [**3**] was to relate the given formal operator series acting on the data, through appropriate integral transformations, to a simpler formal operator series acting on the same data and for which a precise meaning could be assigned. In that paper, the simpler formal operator series were taken to be real translation operators or heat solution operators. The transformations that related these pairs of formal operator series then defined transmutations among the solutions of the various hypergeometric type Cauchy problems. Those transmutations took on the form of Laplace transforms, inverse Laplace transforms and convolutions. In this paper we take a different approach for interpreting the original formal solution operator series by splitting it into two or more component series by means of the quasi inner product. In most of the cases considered, one or more of these component series will be taken to define real or complex translation operators. Then the formal solution operator series can be replaced by a quasi inner product operator which can be applied to the data to obtain a solution of the original Cauchy problem. At the heart of this approach is an operational property which permits moving factors and operators from one function to another. By using this approach, one can derive integral formulas for solutions of problems involving the equations mentioned in the first paragraph above as well as many more. Moreover, these formulas permit establishing relatively simple bounds on solutions corresponding to polynomial data, and these special solutions can be employed to establish series representation theorems for solutions of numerous Cauchy problems.

In Section 2 we present the basic background on quasi inner products. Included are the definition of the quasi inner product, its operational properties, and examples of quasi inner products that occur frequently in the applications. As will be seen later, these operational properties permit considerable flexibility in assigning solutions to problems (depending upon the analyticity properties of the underlying data). We also use the quasi inner product to rewrite special types of series that later appear frequently in the form of formal solution operator series for Cauchy problems. One consequence of this will be a representation formula for the solution of an initial value problem for a second order ordinary differential equation as a quasi inner product of solutions of a pair of first order problems. Section 3 will be concerned with a discussion of the two general types of Cauchy problems to be considered, namely generalized wave problems and generalized EPD problems. It will be seen that a common function can be constructed and employed for solving both of these types of problems. A brief discussion of translation operators and heat solution operators will also be included. The results of these two sections will be applied in Sections 4 to 6 to construct integral formulas for solutions for a representative variety of special cases of these problems. In Sections 4 and 5 we treat, respectively, the wave type and generalized EPD type problems, and some classical solution formulas will be rederived in a new way. It will be seen that appropriate factorizations of differential operators play a key role in the solution construction process. Section 6 will specifically treat problems associated with the Yukawa and Helmholtz type equations and solutions of these will be constructed corresponding to polynomial data. As will be seen, the solution formulas of Sections 4 to 6 permit one to obtain simple and useful bounds on solutions for polynomial data, and these will be employed in Section 7 to discuss series representations of solutions for representative types of problems considered, more specifically for those involving the Yukawa, Helmholtz and the EPD equations (see [**10**] for an alternative approach that makes use of bounds on solutions expressed in terms of Jacobi polynomials). Such series solutions play crucial roles in developing analytic function theories for the underlying equations (see [**11, 12, 16, 25** and **26**]). Finally, additional examples will be considered in Section 8 that illustrate a variety of aspects of the method of qips in the construction of solutions of higher order problems.

2. Quasi inner products. Let $f_1(z_1), f_2(z_2)$ and $f_3(z_3)$ denote three functions that are analytic in the z_j in disks D_j centered at the origin. Further, suppose that

$$f_j(z_j) = \sum_{n=0}^{\infty} a_n^j z_j^n$$

for $z_j \in D_j$, $j = 1, 2, 3$. We define the quasi inner product $f_1(z_1) \circ f_2(z_2)$ of $f_1(z_1)$ and $f_2(z_2)$ by the relation

(2.1)
$$f_1(z_1) \circ f_2(z_2) = (2\pi)^{-1} \int_0^{2\pi} f_1(z_1 e^{i\theta}) f_2(z_2 e^{-i\theta})\, d\theta = \sum_{n=0}^{\infty} a_n^1 a_n^2 z_1^n z_2^n.$$

We note that this product can be trivial if $f_1(z_1) \neq 0$ and $f_2(z_2) \neq 0$. For example, choose the function $f_1(z_1)$ to be even in z_1 and choose the function $f_2(z_2)$ to be odd in z_2 (also, see [**5**]). The product (2.1) is closely related to J. Hadamard's convolution for discussing the singularities of the function

$$\sum_{n=0}^{\infty} a_n^1 a_n^2 z^n$$

in terms of the singularities of $f_1(z)$ and $f_2(z)$ (the "multiplication of singularities" theorem) [**27**]. This can lead to some questions on the domain of validity of the operation $\circ$ and its properties when discussing differential operators. However, in our applications of (2.1) to differential equations, the underlying functions $f_j(z_j)$ will be *entire* in their arguments and the corresponding quasi inner products will be nontrivial. In particular, we will use the quasi inner product to express "entire" formal solution operators in terms of integrals that involve a product of exponential functions and operators acting on an underlying data function.

In some applications, one or more of the underlying functions entering the quasi inner product may depend upon two or more variables. For such cases, we use underscores, as in [**6**], to indicate the variables being singled out in the two functions to be used for forming the quasi inner product. Thus, for example, we write

(2.2)
$$f_1(\underline{z}_1, z_3) \circ f_2(z_2, \underline{z}_3, z_4) = (2\pi)^{-1} \int_0^{2\pi} f_1(z_1 e^{i\theta}, z_3) f_2(z_2, z_3 e^{-i\theta}, z_4)\, d\theta.$$

From (2.1) we observe that if z_j and $Z_j \in D_j$, $j = 1, 2, 3$, and if $z_1 z_2 = Z_1 Z_2$, then

(2.3)
$$\begin{aligned} &\text{(a)} \quad f_1(z_1) \circ f_2(z_2) = f_2(z_2) \circ f_1(z_1) \\ &\text{(b)} \quad f_1(z_1) \circ [f_2(z_2) + f_3(z_2)] = f_1(z_1) \circ f_2(z_2) + f_1(z_1) \circ f_3(z_2) \\ &\text{(c)} \quad f_1(z_1) \circ f_2(z_2) = f_1(Z_1) \circ f_2(Z_2) \end{aligned}$$

provided that $z_2 \in D_3$ in formula (b). The elementary property (2.3c) permits the moving of a factor from the argument of one of the functions to the argument of the other when forming qips. If $f_1(z_1)$ and $f_2(z_2)$ are entire in z_1 and z_2, then we can write

(2.4)
$$\begin{aligned} f_1(A^2 \underline{x}_1) \circ f_2(B^2 \underline{x}_2) &= (2\pi)^{-1} \int_0^{2\pi} f_1(A^2 x_1 e^{i\theta}) f_2(B^2 x_2 e^{-i\theta})\, d\theta \\ &= (2\pi)^{-1} \int_0^{2\pi} f_1(ABx_1 e^{i\theta}) f_2(ABx_2 e^{-i\theta})\, d\theta \end{aligned}$$

where A and B can denote scalar factors or commutative differential operators. Of particular interest for later sections, we have the identity

$$\begin{aligned} (2.5) \quad (2\pi)^{-1} \int_0^{2\pi} & e^{A^2 x_1 e^{i\theta}} \cdot e^{B^2 x_2 e^{-i\theta}}\, d\theta \\ &= (2\pi)^{-1} \int_0^{2\pi} e^{ABe^{i\theta}\sqrt{x_1 \cdot x_2}} e^{ABe^{-i\theta}\sqrt{x_1 \cdot x_2}}\, d\theta \\ &= (2\pi)^{-1} \int_0^{2\pi} e^{2AB\sqrt{x_1 \cdot x_2}(\cos\theta)}\, d\theta \end{aligned}$$

when x_1 and x_2 are nonnegative.

For the purpose of developing solutions for various second order Cauchy problems, we need to call upon quasi inner product representations for special functions defined by the series $\sum_{k=0}^{\infty} x^k/[(b)_k(c)_k]$ where the symbol $(b)_k$ is defined by $(b)_0 = 1$ and $(b)_k = b(b+1)\cdots(b+k-1)$ if $k \geq 1$. Our primary aim is to express these functions as integrals in which the free variables appear in exponential functions. These formulas will prove to be useful in later sections on partial differential equations when one or both of the free variables are replaced by appropriate operators. In the discussion to follow, we take $b \geq 1$ and $c > 1$. We consider other choices of b in Section 5.

Let

$$g_b(x) = \sum_{k=0}^{\infty} x^k/(b)_k.$$

Now $g_1(x) = e^x$ and it is an elementary exercise, using the Laplace transform, to show that $g_b(x) = (b-1)\int_0^1 \sigma^{b-2} e^{x(1-\sigma)}\,d\sigma$ if $b > 1$. Next, let $F_{b,c}(x) = \sum_{k=0}^{\infty} x^k/[(b)_k(c)_k]$ with $b > 1$ and $c > 1$ (this notation is suggestive of hypergeometric functions and $F_{b,c}(x) = {}_1F_2(1; b, c; x)$). Then, using the integral for the function $g_b(x)$, we have

(2.6)
$$\begin{aligned} F_{b,c}(xy) &= g_b(\underline{x}) \circ g_c(\underline{y}) \\ &= (b-1)(c-1) \\ &\quad \cdot \int_0^1\int_0^1 \sigma_1^{b-2}\sigma_2^{c-2}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{x(1-\sigma_1)e^{i\theta}} e^{y(1-\sigma_2)e^{-i\theta}}\,d\theta\right\} d\sigma_1\,d\sigma_2 \\ &= (b-1)(c-1) \\ &\quad \cdot \int_0^1\int_0^1 \sigma_1^{b-2}\sigma_2^{c-2}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\sqrt{xy(1-\sigma_1)(1-\sigma_2)}(\cos\theta)}\,d\theta\right\} d\sigma_1\,d\sigma_2 \end{aligned}$$

by (2.5). On the other hand, if $b > 1$ and $c = 1$, this set of relations must be replaced by

$$\begin{aligned} F(xy) &= g_b(\underline{x}) \circ g_1(\underline{y}) \\ &= (b-1)\int_0^1 \sigma_1^{b-2}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{x(1-\sigma_1)e^{i\theta}} e^{ye^{-i\theta}}\,d\theta\right\} d\sigma_1 \\ &= (b-1)\int_0^1 \sigma_1^{b-2}\left\{\frac{1}{2\pi}\int_0^{2\pi} e^{2\sqrt{xy(1-\sigma_1)}(\cos\theta)}\,d\theta\right\} d\sigma_1. \end{aligned} \tag{2.7}$$

It is useful, for some purposes, to view the quasi inner product formulas (2.6) and (2.7) as defining decompositions of the solutions of initial value problems for higher order differential equations in terms of solutions of a pair of lower order initial value problems. Conversely, the composing of solutions of lower order problems to build up to solutions of higher order problems has obvious connections with transmutations and provides an alternative approach for developing solutions of higher order problems. In order to present one such relation in the form of a theorem of the type for transmutations as were stated in [**3, 8** and **9**], we first need to write out the initial value problems satisfied by the functions $g_b(x)$ and $F_{b,c}(x)$ $(= F_{b,c}(x \cdot 1))$ above.

In the case of the function $g_b(x)$, it is relatively easy to show that it satisfies the nonhomogeneous initial value problem

$$xg_b'(x) + (b-1-x)g_b(x) = b-1, \qquad g_b(0) = 1. \tag{2.8}$$

This follows by multiplying the series for $g_b(x)$ by x^{b-1}, differentiating with respect to x and then rearranging terms appropriately. When $b = 1$, (2.9) reduces to the one for the exponential function. Using a repetition of this approach, one can also show that the function $F_{b,c}(x)$ is a solution of the nonhomogeneous initial value problem

(2.9)

$$x^2F''(x) + (b+c-1)xF'(x) + [(b-1)(c-1) - x]F(x) = (b-1)(c-1)$$
$$F(0) = 1, \qquad F'(0) = 1/(bc).$$

With these equations and initial conditions established, we have:

Theorem 2.1. *Let $u(x)$ and $v(x)$ denote, respectively, solutions of the initial value problems*

$$\begin{aligned} xu'(x) + (b-1-x)u(x) &= b-1, \qquad u(0) = 1 \\ xv'(x) + (c-1-x)v(x) &= c-1, \qquad v(0) = 1. \end{aligned} \tag{2.10}$$

Then the solution of the initial value problem

(2.11)

$$x^2w''(x) + (b+c-1)xw'(x) + [(b-1)(c-1) - x]w(x) = (b-1)(c-1),$$
$$w(0) = 1, \qquad w'(0) = 1/(bc)$$

is given by the formula

$$w(x) = \frac{1}{2\pi}\int_0^{2\pi} u(xe^{i\theta})v(e^{-i\theta})\,d\theta. \tag{2.12}$$

One can establish a variety of theorems of this form for expressing solutions of third and higher order problems in terms of lower order ones.

3. Background on Cauchy problems. Let $x = (x_1, x_2, \dots, x_n)$, and let $D = (D_1, D_2, \dots, D_n)$ in which $D_j\phi(x) = \partial\phi(x)/\partial x_j$. Further, let $P(D)$ denote a partial differential operator in the D_j with constant coefficients. When $n = 1$, we take $D_1 = D$ for simplicity. Usually we will select $P(D)$ to be a second order operator, and we generally restrict n to the values 1 and 2. In this section, we wish to consider initial value problems associated with classical partial differential equations having the forms

$$\text{(3.1)} \qquad \begin{aligned} &\text{(a)} \quad u_{tt}(x,t) = P(D)u(x,t) \\ &\text{(b)} \quad u_{tt}(x,t) + \frac{a}{t}u_t(x,t) = P(D)u(x,t), \quad a > 1. \end{aligned}$$

We defer the case of (3.1b) with $-1 < a < 1$ to Section 5. For problems associated with (3.1a), we take initial conditions to have the form $u(x,0) = 0$, $u_t(x,0) = \phi(x)$. A solution $U(x,t)$ of (3.1a) corresponding to the initial conditions $U(x,0) = \phi(x)$, $U_t(x,0) = 0$ can then be obtained as $U(x,t) = \partial u(x,t)/\partial t$. On the other hand, we associate initial conditions of the form $u(x,0) = \phi(x)$, $u_t(x,0) = 0$ with the equation (3.1b). For the purpose of solving such initial value problems, we will need to call upon evaluations of special operator series acting on data functions. The ones to be employed in this paper are given as follows:

(3.2)

$$\text{(a)} \quad \sum_{k=0}^{\infty} \frac{t^k D_j^k}{k!}\phi(x) = e^{tD_j}\phi(x) = \phi(x_1, \dots, x_{k-1}, x_k + t, x_{k+1}, \dots, x_n)$$

$$\text{(b)} \quad \sum_{k=0}^{\infty} \frac{t^k D_j^{2k}}{k!}\phi(x) = e^{tD_j^2}\phi(x) = h_j(x,t)$$

in which the function $h_j(x,t)$ denotes a solution of the initial value heat problem

$$\text{(3.3)} \qquad h_t(x,t) = D_j^2 h(x,t), \qquad h(x,0) = \phi(x).$$

The formula (3.2a) defines a translation on the component variable x_j in the function $\phi(x)$. If t is real, we identify the operator e^{tD_j} as a translator if $\phi(x)$ has continuous first derivatives (i.e., the function $u(x,t) = e^{tD}\phi(x) = \phi(x+t)$ satisfies the problem $u_t = u_x$, $u(x,0) = \phi(x)$ if $\phi(x) \in C^1$). If t is complex, this data function is taken to be analytic in the variables x_j. In (3.2b), we identify the operator e^{tDj^2} as assigning to $\phi(x)$ a solution to the problem (3.3). If t is positive, the function $\phi(x)$ can be taken to be bounded and continuous while if t is negative or complex, this function is taken to be entire of growth $(2,\tau)$ (see [**19**]). In the majority of constructions of solution functions, we make use of translation operators in order to avoid unnecessary analyticity requirements on $\phi(x)$.

A. *Wave problems.* We now consider the equation (3.1a) with associated initial conditions. Using power series methods, the solution of this problem can formally be expressed in the form

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} P^k(D)\cdot\phi(x) \\
&= t\sum_{k=0}^{\infty}\left(\frac{t^k}{(3/2)_k}\right)\left(\frac{t^kP^k(D)}{2^{2k}k!}\right)\phi(x) \\
&= \frac{t}{2\pi}\int_0^{2\pi} g_{3/2}(te^{i\theta})e^{te^{-i\theta}P(D)/4}\cdot\phi(x)\,d\theta \\
&= \frac{t}{2}\int_0^1 \sigma^{-1/2}K(x,t,\sigma;\phi)\,d\sigma
\end{aligned}
\tag{3.4}
$$

where

$$
K(x,t,\sigma;\phi) = \frac{1}{2\pi}\int_0^{2\pi}\{e^{t(1-\sigma)e^{i\theta}}e^{te^{-i\theta}P(D)/4}\}\phi(x)\,d\theta. \tag{3.5}
$$

The last member of (3.4) follows by using (2.7). Then, to obtain a solution function, one needs to construct the function $K(x,t,\sigma;\phi)$, and this clearly depends upon the specific choice for the operator $P(D)$ and the analytic behavior of $\phi(x)$. Particular examples of problems of type (3.1a) are considered in Section 4.

B. *Euler-Poisson-Darboux type problems.* Again, using series methods, we can show that the solution of (3.1b) with associated initial conditions can formally be expressed as

$$
\begin{aligned}
u(x,t) &= \sum_{k=0}^{\infty} \frac{t^{2k} P^k(D)}{2^{2k}((a+1)/2)_k k!}\phi(x) \\
&= \sum_{k=0}^{\infty} \left(\frac{t^k}{((1+a)/2)_k}\right) \cdot \left(\frac{t^k P^k(D)}{4^k k!}\right)\phi(x) \\
&= \frac{1}{2\pi}\int_0^{2\pi} g_{(a+1)/2}(te^{i\theta}) e^{te^{-i\theta}P(D)/4}\phi(x)\, d\theta \\
&= \frac{a-1}{2}\int_0^1 \sigma^{(a-3)/2} K(x,t,\sigma;\phi)\, d\sigma.
\end{aligned} \tag{3.6}
$$

Note that the term in parentheses in the last member of this is the same as the corresponding term in the last member of (3.4). Once this term has been computed for specific choices of $P(D)$ and $\phi(x)$, one can immediately write integral formulas for solutions for two different types of problems. This shows that the calculations used in Section 4 can be carried over to parts of Section 5.

4. Generalized "wave" type problems. We will now apply the results of these last two sections to construct integral formulas for solutions of (3.1a) with associated initial conditions corresponding to different choices of n and the operator $P(D)$. The more elementary examples among these are primarily used to indicate how the solution forms permit attaching simple bounds on some of the solutions. Analyticity requirements on the data function $\phi(x)$ will be noted. The integrals obtained for these solutions will be double integrals in which one of the variables of integration represents an angular measure. While these integrals take on a quite different form from the standard ones for the classical wave problems, changes in the variables of integration will lead to the more familiar results. The word "wave" in the title of this section has been selected as a catch all term even though $P(D)$ may be the negative of an elliptic operator. The reason for this choice of words is to avoid creating a new section with a change in the classification of the underlying equation. In examples A–E below, we employ real and complex translations to construct solution functions. In F, we rework example C by employing a solution of a heat equation corresponding to entire data.

A. $n = 1$, $P(D) = D^2$. With these selections, it follows from (3.5) that

$$
\begin{aligned}
K(x,t,\sigma;\phi) &= (2\pi)^{-1}\int_0^{2\pi} e^{t\sqrt{t-\sigma}e^{i\theta}D/2} e^{te^{-i\theta}\sqrt{1-\sigma}D/2}\phi(x)\,d\theta \\
&= (2\pi)^{-1}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\sigma})D}\phi(x)\,d\sigma \\
&= (2\pi)^{-1}\int_0^{2\pi} \phi(x+t\cos\theta\sqrt{1-\sigma})\,d\theta
\end{aligned}
\tag{4.1}
$$

where the last follows by (3.2a). Inserting this back into (3.4), we obtain

$$
u(x,t) = \frac{t}{4\pi}\int_0^1 \sigma^{-1/2}\bigg(\int_0^{2\pi}\phi(x+t\sqrt{1-\sigma}\cos\theta)\,d\theta\bigg)\,d\sigma \tag{4.2}
$$

provided that $\phi(x) \in C^1$. The reduction of this to the familiar d'Alembert integral will be deferred to subsection B below.

Now suppose that we select $\phi(x) = x^n$, and let $w_{2,n}(x,t)$ denote the special solution $u(x,t)$ corresponding to this ϕ. It follows that

$$
|\phi(x+t\cos\theta\sqrt{1-\sigma})| = |(x+t\cos\theta\sqrt{1-\sigma})^n| \le (|x|+|t|)^n. \tag{4.3}
$$

Thus, we find from (4.2) that $w_{2,n}(x,t)$ satisfies the following inequality:

$$
|w_{2,n}(x,t)| \le |t|(|x|+|t|)^n. \tag{4.4}
$$

These $w_{2,n}(x,t)$, $n = 0,1,2,\ldots$, constitute one of the two classes of wave polynomials [**10**].

B. $n = 2$, $P(D) = D_1^2 + D_2^2$. We first rewrite the operator $P(D)$ in the factored form $(D_1 + iD_2)(D_1 - iD_2)$. Upon introducing this into

the inside integral in (3.5), we find that
(4.5)

$$\begin{aligned} K(x,t,\sigma;\phi) &= (2\pi)^{-1}\int_0^{2\pi}\{e^{t(1-\sigma)e^{i\theta}}e^{te^{-i\theta}(D_1+iD_2)(D_1-iD_2)/4}\}\phi(x_1,x_2)\,d\theta \\ &= (2\pi)^{-1}\int_0^{2\pi}\{e^{t\sqrt{1-\sigma}e^{i\theta}(D_1+iD_2)/2}e^{t\sqrt{1-\sigma}e^{-i\theta}(D_1-iD_2)/2}\}\phi(x_1,x_2)\,d\theta \\ &= (2\pi)^{-1}\int_0^{2\pi}\{e^{(t\cos\theta\sqrt{1-\sigma})D_1}e^{-(t\sin\theta\sqrt{1-\sigma})D_2}\}\phi(x_1,x_2)\,d\theta \\ &= (2\pi)^{-1}\int_0^{2\pi}\phi(x_1+t\cos\theta\sqrt{1-\sigma},x_2-t\sin\theta\sqrt{1-\sigma})\,d\theta \end{aligned}$$

provided that $\phi(x_1,x_2)\in C^2$ in x_1 and x_2. It is not hard to show that the term $-t\sin\theta\sqrt{1-\sigma}$ can be replaced in the last integral above by $t\sin\theta\sqrt{1-\sigma}$. Inserting this last altered integral back into (3.4), we finally obtain

$$(4.6)\quad u(x,t)=\frac{t}{4\pi}\int_0^1\sigma^{-1/2}\bigg\{\int_0^{2\pi}\phi(x_1+t\cos\theta\sqrt{1-\sigma}, x_2+t\sin\theta\sqrt{1-\sigma})\,d\theta\bigg\}\,d\sigma.$$

To show that this reduces to the familiar Poisson integral representation, first introduce the change of variables $\sigma=1-\tau^2$. Then we obtain

$$u(x,t)=\frac{t}{2\pi}\int_0^1(1-\tau^2)^{-1/2}\bigg\{\int_0^{2\pi}\phi(x_1+t\tau\cos\theta,x_2+t\tau\sin\theta)\,d\theta\bigg\}\,d\tau.$$

With the further changes of variables $\xi=x_1+t\tau\cos\theta$, $\eta=x_2+t\tau\sin\theta$, the region of integration reduces to that of the disk $B(x_1,x_2;t)$, i.e., $(x_1-\xi)^2+(x_2)-\eta)^2\le t^2$. The formula $u(x,t)$ then becomes

$$(4.7)\qquad u(x,t)=\frac{1}{2\pi}\int_{B(x_1,x_2;t)}\frac{\phi(\xi,\eta)}{\sqrt{t^2-(x_1-\xi)^2-(x_2-\eta)^2}}\,d\xi\,d\eta.$$

With similar changes of variables, the function in (4.2) can be written in this same form with $\phi(\xi,\eta)$ replaced by $\phi(\xi)$. A partial integration of the resulting integral with respect to η then leads to the familiar d'Alembert formula.

The form of the solution function (4.6) permits us to establish the following result for an abstract generalization of this problem:

Theorem 4.1. *Let A_1 and A_2 be generators of continuous groups in a Banach space X with $A_1A_2 = A_2A_1$, and let $\phi \in \mathcal{D}(A_1^2) \cap \mathcal{D}(A_2^2)$. Then a solution of the abstract wave problem $v_{tt} = (A_1^2 + A_2^2)v$, $v(0) = 0$, $v_t(0) = \phi$ is given by the formula*

$$v(t) = \frac{t}{4\pi}\int_0^1 \sigma^{-1/2}\bigg\{\int_0^{2\pi} G_{A_1}(t\cos\theta\sqrt{1-\sigma})[G_{A_2}(t\sin\theta\sqrt{1-\sigma})\phi]\,d\theta\bigg\}\,d\sigma$$

in which the $G_{A_j}(t)$ denotes the group of operators generated by A_j, $j = 1, 2$.

C. $n = 1$, $P(D) = -D^2$. For this Laplace type initial value problem, the reader can verify that

$$\begin{aligned} K(x,t,\sigma;\phi) &= (2\pi)^{-1}\int_0^{2\pi} \{e^{t(1-\sigma)e^{i\theta}} e^{-te^{-i\theta}D^2/4}\}\phi(x)\,d\theta \\ &= (2\pi)^{-1}\int_0^{2\pi} e^{(it\sin\theta\sqrt{1-\sigma})D}\phi(x)\,d\theta \\ &= (2\pi)^{-1}\int_0^{2\pi} \phi(x + it\sin\theta\sqrt{1-\sigma})\,d\theta \end{aligned}$$

provided that $\phi(z)$ is analytic, say, in a disk in the complex plane that contains $x + it$. The solution of this initial value problem is then given by

$$u(x,t) = \frac{t}{4\pi}\int_0^1 \sigma^{-1/2}\bigg\{\int_0^{2\pi} \phi(x + it\sin\theta\sqrt{1-\sigma})\,d\theta\bigg\}\,d\sigma. \tag{4.8}$$

If $\phi(x) = x^n$, then one can easily show from this that the solution $l_{2,n}(x,t)$ corresponding to this condition satisfies the inequality

$$|l_{2,n}(x,t)| \le |t|(x^2 + t^2)^{n/2}. \tag{4.9}$$

The $l_{2,n}(x,t)$ are one of the classes of Laplace polynomials.

D. $n = 2$, $P(D) = D_1^2 - D_2^2$. We leave it to the reader to show that the solution of the corresponding ultrahyperbolic equation (3.1a) with associated initial conditions is given by

$$u(x,t) = \frac{1}{4\pi}\int_0^1 \sigma^{-1/2} \left\{ \int_0^{2\pi} \phi(x_1 + t\cos\theta\sqrt{1-\sigma}, x_2 + it\sin\theta\sqrt{1-\sigma}\, d\theta \right\} d\sigma \tag{4.10}$$

if $\phi \in C^1$ in x_1 and is analytic in x_2.

E. $n = 2$, $P(D) = -(D_1^2 + D_2^2)$. Again, we can deduce the solution formula

(4.11)

$$u(x,t) = \frac{1}{4\pi}\int_0^1 \sigma^{-1/2} \left\{ \int_0^{2\pi} \phi(x_1 + it\cos\theta\sqrt{1-\sigma}, x_2 + it\sin\theta\sqrt{1-\sigma})\, d\theta \right\} d\sigma$$

if $\phi(x_1, x_2)$ is analytic in both variables.

F. Same choices as in C. Select the data function $\phi(z)$ to be entire in z of growth (ρ, τ) with $\rho \le 2$. Using formula (3.2b) in (3.5), it then follows that

$$K(x,t,\sigma;\phi) = (2\pi)^{-1}\int_0^{2\pi} e^{t(1-\sigma)e^{i\theta}} h(x, te^{-i\theta}/4)\, d\theta \tag{4.12}$$

in which $h(x,t)$ is the solution of the heat problem (3.3) corresponding to the above choice for $\phi(z)$. Thus, a solution alternative to the one given in C above follows by introducing (4.12) back into the formula (3.4). Obviously, the solution obtained in part C is better because of the fewer analytic requirements on $\phi(z)$.

5. Generalized EPD problems. We can carry out the constructions of the solutions of (3.1b) corresponding to the choices of n and

$P(D)$ as in subsections A–F of the previous section by introducing the function $K(x,t,\sigma,\phi)$ for each of these examples into (3.6). This simply means that we integrate this $K(x,t,\sigma,\phi)$ against a different power of σ and then multiply the resulting integral by $a-1$. To avoid undue repetitions, we shall simply note the bounds on the polynomial solutions of the EPD versions of A and C of Section 4 and the modifications for the example B. Note, of course, that the EPD version of C defines the GASPT equation problem. The remainder of the section will be concerned with constructing solutions, using qips, of an initial value problem associated with (3.1b) when $-1 < a < 1$.

From (4.4) and (4.9) obtained above, it easily follows that the respective solutions $e_n(x,t)$ and $b_n(x,t)$ of the problems

(5.1)
$$u_{tt}(x,t) + \frac{a}{t}u_t(x,t) = \delta D_x^2 u(x,t), \qquad u(x,0) = x^n, \qquad u_t(x,0) = 0$$

satisfy the inequalities

$$|e_n(x,t)| \le (|x| + |t|)^n \quad \text{if } \delta = 1 \tag{5.2}$$

and

$$|b_n(x,t)| \le (x^2 + t^2)^{n/2} \quad \text{if } \delta = -1.$$

These are the Euler-Poisson-Darboux and Beltrami (or GASPT) polynomials of [**10**].

The solution of (3.1b) corresponding to the choices in example B above is given by the formula

$$u(x,t) = \frac{(a-1)}{4\pi} \int_0^1 \sigma^{(a-3)/2} \left\{ \int_0^{2\pi} \phi(x_1 + t\cos\theta\sqrt{1-\sigma}, x_2 + t\sin\theta\sqrt{1-\sigma})\, d\theta \right\} d\sigma.$$

Employing the same change to rectangular coordinates as in example B of section 4, it is not hard to show that

$$u(x,t) = \frac{(a-1)}{2\pi} t^{5-a} \int_{B(x_1,x_2;t)} (t^2 - (\xi - x_1)^2 - (\eta - x_2)^2)^{(a-3)/2} \phi(\xi,\eta)\, d\xi\, d\eta. \tag{5.3}$$

We will now turn to the problem of constructing a solution to the Cauchy problem associated with (3.1b) when $-1 < a < 1$ by means of qips. To do this, it will be necessary to develop a formula alternative to the one in (3.6), and this will require an integral formula for the function $g_b(x)$ for $0 < b < 1$ along with parametric differentiation. It is known that if this problem has a solution, then one can obtain a solution to the Cauchy problem associated with (3.1b) when $-2k-1 < a < -2k+1$, $k = 1, 2, 3, \dots$ by using Taylor's series [**4, 21** and **22**]. The choices $a = -2k-1$, $k = 1, 2, 3, \dots,$ are referred to as *exceptional or singular values* and problems associated with them will not be considered here (see [**4**] and [**15**]). It is a well known fact that uniqueness fails if $a < 0$.

If $0 < b < 1$, it follows from the definition of $g_b(x)$ in Section 2 that

$$g_b(x) = 1 + \frac{x}{b} g_{b+1}(x) = 1 + x \int_0^1 \sigma^{b-1} e^{x(1-\sigma)}\, d\sigma \tag{5.4}$$

where we have used the integral formula for $g_{b+1}(x)$. Introducing this into (2.6) and using (2.3b), we find that

(5.5)

$$\begin{aligned} F_{b,1}(xy) &= g_b(\underline{x}) \circ e^{\underline{y}} = 1 \circ e^{\underline{y}} + \left(\underline{x} \int_0^1 \sigma^{b-1} e^{\underline{x}(1-\sigma)}\, d\sigma \right) \circ e^{\underline{y}} \\ &= 1 + x \int_0^1 \sigma^{b-1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} e^{x(1-\sigma)e^{i\theta}} e^{ye^{-i\theta}}\, d\theta \right\} d\sigma \\ &= 1 + x \frac{\partial}{\partial \lambda} \left[\int_0^1 \sigma^{b-1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{[x(1-\sigma)+\lambda]e^{i\theta}} e^{ye^{-i\theta}} d\theta \right\} d\sigma \right]_{\lambda=0}. \end{aligned}$$

Differentiation with respect to the parameter λ and its evaluation at $\lambda = 0$ in the last member of this brings back the factor $e^{i\theta}$ required in the third member of (5.8). With this, we can now follow the procedure of Section 3 for treating EPD problems by using a formal operator series. Since $b = (a+1)/2$ and this falls between 0 and 1 when $-1 < a < 1$, we can use the formalism of (5.5) to show that a solution of the initial value problem associated with (3.1b) when $-1 < a < 1$ is given by the formula

$$u(x,t) = \phi(x) + t \frac{\partial}{\partial \lambda} \left[\int_0^1 \sigma^{(a-1)/2} K^*(x, t, \sigma, \lambda, \phi)\, d\sigma \right]_{\lambda=0} \tag{5.6}$$

where

$$K^*(x,t,\sigma,\lambda,\phi) = \frac{1}{2\pi}\int_0^{2\pi} e^{[t(1-\sigma)+\lambda]e^{i\theta}} e^{te^{-i\theta}P(D)/4}\phi(x)\,d\theta. \tag{5.7}$$

Note that the right hand member of this formula for K^* is in the form of a quasi inner product and factor from the terms $t(1-\sigma)+\lambda$ and $tP(D)/4$ can be moved from one exponential to the other in accordance with (2.5). In applying these formulas, one starts out by assuming that $\phi(x)$ is analytic in order to carry out complex translations to construct K^*. However, the final formula for the solution $u(x,t)$ need not involve such complex translations and the conditions on $\phi(x)$ in this final solution can be weakened to, say, $\phi(x) \in C^l$ for some appropriate choice of l. This will be made clear in our two examples of applications of (5.6)–(5.7) to hyperbolic equations.

A. $n=2$, $P(D)=D^2$. For these choices and using (2.5), we have

(5.8)

$$\begin{aligned}K^*(x,t,\sigma,\lambda;\phi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{[t(1-\sigma)+\lambda]e^{i\theta}} e^{te^{-i\theta}D^2/4}\phi(x)\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{[t\sqrt{1-\sigma}+\lambda(1-\sigma)^{-1/2}]e^{i\theta}/2} e^{te^{-i\theta}\sqrt{1-\sigma}D/2}\phi(x)\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{[t\cos\theta\sqrt{1-\sigma}+\lambda(1-\sigma)^{-1/2}e^{i\theta}/2]D}\phi(x)\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \phi(x+t\cos\theta\sqrt{1-\sigma}+\lambda e^{i\theta}(1-\sigma)^{-1/2}/2)\,d\theta.\end{aligned}$$

Hence,

$$\left.\frac{\partial K^*}{\partial\lambda}\right|_{\lambda=0} = \frac{1}{4\pi}\int_0^{2\pi} \phi'(x+t\cos\theta\sqrt{1-\sigma})e^{i\theta}(1-\sigma)^{-1/2}\,d\theta.$$

Upon inserting this into (5.6) and using the fact that $u(x,t)$ must be real when $\phi(x)$ is real, we find that

$$u(x,t) = \phi(x) + \frac{t}{4\pi}\int_0^1 \sigma^{(a-1)/2}(1-\sigma)^{-1/2} \left\{\int_0^{2\pi} \cos\theta\cdot\phi'(x+t\cos\theta\sqrt{1-\sigma})\,d\theta\right\}d\sigma. \tag{5.9}$$

Observe that the function $\phi'(x+t\cos\theta\sqrt{1-\sigma})$ in this involves only a real argument. This means that (5.9) defines a solution of the problem associated with (3.1b) for $-1<a<1$ provided that $\phi(x)\in C^2$.

Let us note that if $\phi(x)=x^n$, then $\phi'(x+t\cos\theta\sqrt{1-\sigma})=n(x+t\cos\theta\sqrt{1-\sigma})^{n-1}$. Hence the above solution $u(x,t)$ corresponding to the $\phi(x)$, which we denote by $e_{n,a}(x,t)$, satisfies the inequality

$$\begin{aligned}|e_{n,a}(x,t)| &\le |x|^n+\frac{n|t|}{4\pi}\int_0^1\sigma^{(a-1)/2}(1-\sigma)^{-1/2}\\ &\qquad\left\{\int_0^{2\pi}(|x|+|t|)^n\,d\theta\right\}d\sigma\\ &=|x|^n+\frac{1}{2}B((a+1)/2,1/2)|t|(|x|+|t|)^n\end{aligned}\tag{5.10}$$

in which $B((a+1)/2,1/2)$ denotes a beta function.

B. $n=2$, $P(D)=D_1^2+D_2^2$. Using the factorization $(D_1+iD_2)(D_1-iD_2)$ of the operator $P(D)$ as earlier, we leave it to the reader to show that

$$\begin{aligned}K^*(x,t,\sigma,\lambda;\phi)&=\frac{1}{2\pi}\int_0^{2\pi}e^{[t\cos\theta\sqrt{1-\sigma}+\lambda e^{i\theta}(1-\sigma)^{-1/2}/2]D_1}\\ &\qquad e^{-[t\sin\theta\sqrt{1-\sigma}-i\lambda e^{i\theta}(1-\sigma)^{-1/2}/2]D_2}\phi(x_1,x_2)\,d\theta\\ &=\frac{1}{2\pi}\int_0^{2\pi}\phi(x_1+t\cos\theta\sqrt{1-\sigma}+\lambda e^{i\theta}(1-\sigma)^{-1/2}/2,\\ &\qquad x_2-t\sin\theta\sqrt{1-\sigma}+i\lambda e^{i\theta}(1-\sigma)^{-1/2})\,d\theta\end{aligned}$$

and that

$$\begin{aligned}\left.\frac{\partial K^*}{\partial\lambda}\right|_{\lambda=0}&=\frac{1}{4\pi}(1-\sigma)^{-1/2}\int_0^{2\pi}\{\phi_1(x_1+t\cos\theta\sqrt{1-\sigma},x_2\\ &\qquad-t\sin\theta\sqrt{1-\sigma})\cos\theta\\ &\quad-\phi_2(x_1+t\cos\theta\sqrt{1-\sigma},x_2-t\sin\theta\sqrt{1-\sigma})\sin\theta\}\,d\theta.\end{aligned}$$

Upon observing that we can replace $\sin\theta$ here by $-\sin\theta$ and replacing this last parametric differentiation evaluation in (5.6), we can finally obtain the solution

$$u(x,t)=\phi(x_1,x_2)+\frac{t}{4\pi}\int_0^1\sigma^{(a-1)/2}(1-\sigma)^{-1/2}\psi(x_1,x_2,t,\sigma)\,d\sigma\tag{5.11}$$

with

$$\psi(x_1,x_2,t,\sigma)=\int_0^{2\pi}\{\phi_1(x_1+t\cos\theta\sqrt{1-\sigma},x_2+t\sin\theta\sqrt{1-\sigma})\cos\theta \\ +\phi_2(x_1+t\cos\theta\sqrt{1-\sigma},x_2+t\sin\theta\sqrt{1-\sigma})\sin\theta\}\,d\theta.$$

An analysis of this shows that we must require that $\phi(x_1,x_2)$ have continuous second derivatives in both x_1 and x_2. A rectangular integral version of the solution in (5.11) can be obtained by following the steps for example B of the previous section.

6. Yukawa and Helmholtz type problems. Initial value problems of the type (3.1a) for the standard Yukawa and Helmholtz equations are given, respectively, by

(6.1)

$$\text{(a)}\quad Y_{tt}(x,t)=-(D^2-\mu^2)Y(x,t),\quad Y(x,0)=0,\quad Y(x,0)=\phi(x)$$

$$\text{(b)}\quad H_{tt}(x,t)=-(D^2+\mu^2)H(x,t),\quad H(x,0)=0,\quad H_t(x,0)=\phi(x)$$

in which μ is a positive parameter. The construction of representations for solutions of these, using standard transmutation techniques, leads to a number of complicated technical questions relating to the convergence of improper integrals. Further, the solutions of these corresponding to polynomial data are not polynomials. Rather, they are sums of certain types of Bessel functions. Obtaining bounds on these special solutions is tedious and this, in turn, leads to difficult convergence proofs. In developing full function theories for the solutions of the equations in (6.1), some of the noted difficulties cannot be avoided [**12**]. However, certain aspects of the subject can be considerably simplified by using qips. The integral representations that we obtain for the solution (6.1) will provide relatively convenient integrals for constructing solutions corresponding to polynomial data and for obtaining bounds on them. Series representation theorems for the solutions of (6.1a), (6.1b) and the EPD problem considered in Section 5 will be considered in Section 7.

A. An integral solution for (6.1a). Writing $-(D^2-\mu^2)=[i(D+\mu)]\cdot$

$[i(D-\mu)]$ and taking $\phi(x)$ to be analytic, it follows from (3.5) that

$$\begin{aligned} K(x,t,\sigma;\phi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{t(1-\sigma)e^{i\theta}} e^{-te^{-i\theta}(D^2-\mu^2)/4}\phi(x)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{tie^{i\theta}(D+\mu)\sqrt{1-\sigma}/2} e^{tie^{-i\theta}(D-\mu)\sqrt{1-\sigma}/2}\phi(x)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-t\mu\cos\theta\sqrt{1-\sigma}} e^{(it\cos\theta\sqrt{1-\sigma})D}\phi(x)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-t\mu\sin\theta\sqrt{1-\sigma}}\phi(x+it\cos\theta\sqrt{1-\sigma})\,d\theta. \end{aligned}$$

By splitting the last integration in this into the sum of four integrations over the intervals $k\pi/2$ to $(k+1)\pi/2$, $k=0,1,2,3$, and making appropriate changes of variables in each, one can finally show that

$$(6.2)\quad K(x,t,\sigma;\phi) = \frac{2}{\pi}\int_0^{\pi/2}\cosh(t\mu\sin\theta\sqrt{1-\sigma})v(x,t\cos\theta\sqrt{1-\sigma})\,d\theta$$

where $v(x,t)=[\phi(x+it)+\phi(x-it)]/2$ which is a solution of Laplace's equation corresponding to the conditions $v(x,0)=\phi(x)$, $v_t(x,0)=0$. Introducing (6.2) into (3.4), we finally obtain

$$(6.3)\quad Y(x,t) = \frac{t}{\pi}\int_0^1 \sigma^{-1/2}\left\{\int_0^{\pi/2}\cosh(t\mu\sin\theta\sqrt{1-\sigma})v(x,t\cos\theta\sqrt{1-\sigma})\,d\theta\right\}d\sigma.$$

Note that this has the character of a transmutation formula since it expresses the solution of the Yukawa problem as an integral transform of a solution of the Laplace equation.

Now suppose that $\phi(x)=x^n$, and let $v_n(x,t)=\{(x+it)^n+(x-it)^n\}/2$. It follows that $|v_n(x,t\cos\theta\sqrt{1-\theta})|\le(x^2+t^2)^{n/2}$ over the range of integration. Further, $|\cosh(t\mu\sin\theta\sqrt{1-\sigma})|\le e^{\mu|t|}$. Taking the absolute value of both sides of (6.3), and using the estimates just obtained, we find that if $Y_n(x,t)$ is the solution of (6.1a) corresponding to $\phi(x)=x^n$, then

$$(6.4)\qquad |Y_n(x,t)|\le|t|(x^2+t^2)^{n/2}e^{\mu|t|}.$$

B. An integral solution of (6.1b). If we write $D^2 + \mu^2 = (D + i\mu)(D - i\mu)$, the we leave it to the reader to show, using the same type of arguments as in part (i), that

$$(6.5)\quad H(x,t) = \frac{t}{\pi}\int_0^1 \sigma^{-1/2}\left\{\int_0^{\pi/2} \cos(t\mu\sin\theta\sqrt{1-\sigma}) \cdot v(x, t\cos\theta\sqrt{1-\sigma})\,d\theta\right\}d\sigma$$

with v as above. Further, if $H_n(x,t)$ corresponds to the choice $\phi(x) = x^n$, then

$$(6.6)\qquad |H_n(x,t)| \le |t|(x^2+t^2)^{n/2}.$$

C. Expansion sets. We now construct the solution sets $\{Y_n(x,t)\}$ and $\{H_n(x,t)\}$ for the problem (6.1). If $\phi(x) = x^n$, then we have

$$(6.7)\quad v_n(x, t\cos\theta\sqrt{1-\sigma}) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot x^{n-2k} t^{2k} \cos^{2k}(\theta) \cdot (1-\sigma)^k.$$

(a) The $\{Y_n\}$ set. Introducing the last member of (6.7) into the integral formula (6.3), we find

$$(6.8)\qquad Y_n(x,t) = \frac{t}{\pi}\sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} \cdot x^{n-2k} t^{2k} \lambda_k$$

where

$$\begin{aligned}\lambda_k &= \int_0^1 \sigma^{-1/2}\bigg\{\int_0^{\pi/2} \cosh(t\mu\sin\theta\sqrt{1-\sigma}))(1-\sigma)^k \cos^{2k}\theta\, d\theta\bigg\}\, d\sigma \\ &= \int_0^1 \sigma^{-1/2}\bigg\{\int_0^{\pi/2}\bigg[\sum_{j=0}^{\infty}\frac{t^{2j}\mu^{2j}(1-\sigma)^j}{(2j)!}\sin^{2j}\theta\bigg] \\ &\qquad\qquad \cdot (1-\sigma)^k\cos^{2k}\theta\, d\theta\bigg\}\, d\sigma \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\frac{t^{2j}\mu^{2j}}{(2j)!}\bigg\{\int_0^1 \sigma^{-1/2}(1-\sigma)^{j+k}\, d\sigma\bigg\} \\ &\qquad \cdot \{2\int_0^{\pi/2}\sin^{2j}\theta\cdot\cos^{2k}\theta\, d\theta\bigg\} \\ &= \frac{1}{2}\sum_{j=0}^{\infty}\frac{t^{2j}\mu^{2j}}{(2j)!}B(1/2, j+k+1)B(j+1/2, k+1/2)\end{aligned}$$

and the last member of this reduces to $\pi {}_0F_1(\_\_; k+3/2; t^2\mu^2/4)/(2k+1)$. But ${}_0F_1(\_\_; k+3/2; t^2\mu^2/4) = (\Gamma(k+3/2)/(t\mu/2)^{k+1/2})\cdot I_{k+1/2}(t\mu)$ in which $I_l(z)$ denotes a modified Bessel function of index l. Inserting this value of λ_k into (6.8) and simplifying, we find

$$\begin{aligned}Y_n(x,t) &= \sqrt{\pi}\cdot t\sum_{k=0}^{[n/2]}(-1)^k\frac{n!}{2^{k+1/2}k!(n-2k)!} \\ &\qquad\cdot\frac{x^{n-2k}t^{k-1/2}}{\mu^{k+1/2}}\cdot I_{k+1/2}(t\mu).\end{aligned} \tag{6.9}$$

If we use the formulas that express the above modified Bessel functions of half odd integers in terms of the hyperbolic functions [**18**], we can show, for example, that

$$\begin{aligned}Y_0(x,t) &= \frac{\sinh(\mu t)}{\mu}, \qquad Y_1(x,t) = \frac{x\sinh(\mu t)}{\mu}, \\ Y_2(x,t) &= \frac{x^2\sinh(\mu t)}{\mu} - \frac{t\cosh(t\mu)}{\mu^2} + \frac{\sinh(t\mu)}{\mu^3}, \text{ etc.}\end{aligned}$$

(b) The $\{H_n\}$ set. We leave it to the reader to use the integral formula (6.3) and the relations (6.7) to show that

$$H_n(x,t) = \sqrt{\pi} \cdot t \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{2^{k+1/2} k!(n-2k)!} \cdot \frac{x^{n-2k} t^{k-1/2}}{\mu^{k+1/2}} \cdot J_{k+1/2}(t\mu) \tag{6.10}$$

where $J_l(z)$ denotes the usual Bessel function of index l. We also leave it to the reader to write out the first few of the $H_n(x,t)$ in terms of $\sin(t\mu)$ and $\cos(t\mu)$.

7. Expansion theorems. Theorems on the representation of solutions of initial value problems involving the Laplace and the wave equations in terms of polynomials corresponding to data functions of the form $\phi(x) = x^n$ have been proved by D.V. Widder [**25, 26**] (see [**19**] for a corresponding treatment of the heat problem). To indicate his results, let $l_{1,n}(x,t)$ denote a solution of Laplace's equation corresponding to the data $l(x,0) = x^n$, $l_t(x,0) = 0$, and let $l_{2,n}(x,t)$ be as in Section 4. Then Widder proved that the series

$$\sum_{n=0}^{\infty} a_n l_{1,n}(x,t)$$

converges to a solution of Laplace's equation in the disk $x^2 + t^2 < R^2$ but not everywhere in any including circle. Moreover, this solution corresponds to the data

$$l(x,0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$l_t(x,0) = 0$ if $\phi(x)$ is analytic for $|x| < R$. An analogous theorem holds for sums of the type

$$\sum_{n=0}^{\infty} a_n l_{2,n}(x,t)$$

(in this case, the series converges for all x if $t = 0$). In proving these theorems, the bounds on $|l_{1,n}(x,t)|$, $|l_{2,n}(x,t)|$ were employed. Similar expansion theorems hold for solutions of the wave equation in terms of polynomial sets $\{w_{1,n}(x,t)\}$ and $\{w_{2,n}(x,t)\}$. The corresponding regions of convergence are squares $|x| + |t| < R$. In [**10**], J.W. Dettman and the author constructed the earlier mentioned solution sets $\{e_n(x,t)\}$, $\{b_n(x,t)\}$ and radial versions of the Laplace wave, EPD, and Beltrami polynomials in terms of Jacobi polynomials. Theorems on representations of the type proved by Widder were obtained for these polynomial sets by employing bounds on the Jacobi polynomials. To prove that a series of the above type diverges in some region, one requires asymptotic estimates of the polynomials. The integral formulas obtained in Sections 4-6 permitted us to determine bounds, in a relatively elementary way, on a variety of solution functions corresponding to polynomial data. We did not, however, deduce the asymptotic behavior of these polynomials for large n. In view of this, we limit the following to stating and proving theorems on convergence regions for representations of solutions of problems related to the Yukawa equation, the Helmholtz equation and the EPD equation with $-1 < a < 1$ in terms of the special solution sets developed for those equations.

Theorem 7.1 (Yukawa and Helmholtz expansions). *Let*

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n$$

be analytic in x for $|x| < R$. Then the series $\sum_{n=0}^{\infty} a_n Y_n(x,t)$ converges to a solution of the initial value problem $Y_{tt}(x,t) + Y_{xx}(x,t) - \mu^2 Y(x,t) = 0$, $Y(x,0) = 0$, $Y_t(x,0) = \phi(x)$ for all x if $t = 0$ and for $x^2 + t^2 < R^2$. Similarly, the series

$$\sum_{n=0}^{\infty} a_n H_n(x,t)$$

converges to a solution of the initial value problem $H_{tt}(x,t) + H_{xx}(x,t) + \mu^2 H(x,t) = 0$, $H(x,0) = 0$, $H_t(x,0) = \phi(x)$ for the same set of points (x,t).

Proof. We prove this for the Yukawa problem and note that the same type of argument works for the Helmholtz case. Since $\phi(x)$ is analytic for $|x| < R$, let $\widetilde{R} > 0$ be selected so that $\widetilde{R} < R$. Let $\widetilde{K}$ denote the open region $x^2 + t^2 < \widetilde{R}^2$. We shall prove that the series

$$\sum_{n=0}^{\infty} a_n Y_n(x,t)$$

and its various derivatives converge uniformly in $\widetilde{K}$. To do this, select $\hat{R}$ so that $\widetilde{R} < \hat{R} < R$. Since the series

$$\sum_{n=0}^{\infty} a_n \hat{R}^n$$

converges, there exists a positive constant M such that $|a_n| < M/\hat{R}^n$ for all n. Let $(x,t) \in \widetilde{K}$. Then it follows from (6.6) and the above bound on $|a_n|$ that

$$\begin{aligned}
\left|\sum_{n=0}^{\infty} a_n Y_n(x,t)\right| &\leq \sum_{n=0}^{\infty} |a_n| \cdot |Y_n(x,t)| \\
&\leq \sum_{n=0}^{\infty} \frac{M}{\hat{R}^n} |t| e^{\mu|t|} (x^2+t^2)^{n/2} \\
&\leq \sum_{n=0}^{\infty} \frac{M}{\hat{R}^n} \cdot \widetilde{R} \cdot e^{\mu\widetilde{R}} \widetilde{R}^n \\
&= M\widetilde{R} \cdot e^{\mu\widetilde{R}} \sum_{n=0}^{\infty} (\widetilde{R}/\hat{R})^n
\end{aligned}$$

and this last series converges. Hence,

$$\sum_{n=0}^{\infty} a_n Y_n(x,t)$$

converges uniformly in $\widetilde{K}$. To show that the various derived series converge uniformly in $\widetilde{K}$, we need to establish bounds similar to those in (6.6) for the derivatives of the $Y_n(x,t)$. From the definition of $v_n(x,t)$ in the line following (6.3), it follows that $\partial v_n(x,t)/\partial x = n v_{n-1}(x,t)$. Then, from the definition of $Y_n(x,t)$, we can show that $\partial Y_n(x,t)/\partial x =$

$nY_{n-1}(x,t)$. It then follows that $|\partial Y_n(x,t)/\partial x| = n|Y_{n-1}(x,t)| \leq n|t|e^{\mu|t|}(x^2+t^2)^{(n-1)/2}$. Using the same argument as above, we can then show that

$$\left|\sum_{n=0}^{\infty} a_n \cdot \partial Y_n(x,t)/\partial x\right| \leq M\widetilde{R}\cdot e^{\mu\widetilde{R}}\sum_{n=0}^{\infty} n\cdot(\widetilde{R}/\hat{R})^n$$

which converges and this shows that the series

$$\sum_{n=0}^{\infty} a_n \cdot \partial Y_n(x,t)/\partial x$$

converges uniformly in $\widetilde{K}$. Likewise, for the series

$$\sum_{n=0}^{\infty} a_n \cdot \partial^2 Y_n(x,t)/\partial x^2.$$

The derivatives of the $Y_n(x,t)$ with respect to t are more complicated. We have, in fact,

$$\begin{aligned}\partial Y_n(x,t)/\partial t = \pi^{-1}\int_0^1 \sigma^{-1/2}\bigg\{\int_0^{\pi/2} &\cosh(t\mu\sin\theta\cdot\sqrt{1-\sigma})\\ &\cdot v_n(x,t\cos\theta\sqrt{1-\sigma})\,d\theta\bigg\}\,d\sigma\\ +\frac{t}{\pi}\int_0^1\sigma^{-1/2}\bigg\{\int_0^{\pi/2}&\sinh(t\mu\sin\theta\cdot\sqrt{1-\sigma})(\mu\sin\theta\sqrt{1-\sigma})\\ &\cdot v_n(x,t\cos\theta\sqrt{1-\sigma})\,d\theta\bigg\}\,d\sigma\\ +\frac{t}{\pi}\int_0^1\sigma^{-1/2}\bigg\{\int_0^{\pi/2}&\cosh(t\mu\sin\theta\cdot\sqrt{1-\sigma})\\ &\cdot\partial v_n(x,t\cos\theta\sqrt{1-\sigma})/\partial t\,d\theta\bigg\}\,d\sigma.\end{aligned}$$

From the definition of $v_n(x,t)$, one can show that $|\partial v_n(x,t)/\partial t| \leq n(x^2+t^2)^{(n-1)/2}$. Finally, we leave it to the reader to establish that

$$\begin{aligned}|\partial Y_n(x,t)/\partial t &\leq (x^2+t^2)^{(n-1)/2}\cdot[\sqrt{x^2+t^2}\cdot(1+|t|e^{\mu|t|})+n]\\ &\leq \widetilde{R}^{(n-1)/2}[\widetilde{R}(1+\widetilde{R}\cdot e^{\mu\widetilde{R}})+n] \quad \text{for } (x,t)\in\widetilde{K}\end{aligned}$$

and that the series

$$\sum_{n=0}^{\infty} a_n \partial Y_n(x,t)/\partial t$$

converges uniformly in $\widetilde{K}$. By an analogous argument, so also does the series of second order time derivatives. But since the $Y_n(x,t)$ satisfy the Yukawa equation, the above argument proves that

$$\sum_{n=0}^{\infty} a_n Y_n(x,t)$$

satisfies Yukawa's equation in $\widetilde{K}$. But since $\widetilde{R}$ is arbitrary, it follows that this series satisfies the equation for $x^2+t^2<R^2$. It is a relatively easy task to check that the initial conditions are fulfilled. □

Theorem 7.2 (Euler-Poisson-Darboux expansions). *Let*

$$\phi(x)=\sum_{n=0}^{\infty} a_n x^n$$

be analytic in x *for* $|x|<R$. *Then the series*

$$\sum_{n=0}^{\infty} a_n e_{n,a}(x,t)$$

converges to a solution of the initial value problem $e_{tt}(x,t)+(a/t)e_t(x,t) = e_{xx}(x,t), e(x,0)=\phi(x), e_t(x,0)=0$ *for* $|x|+|t|<R$ *if* $a>-1$.

Proof. This is analogous to the proof of Theorem 7.1 except that we start by letting $\widetilde{K}$ denote the interior of the square $|x|+|t|=\widetilde{R}$. Then we prove that the series

$$\sum_{n=0}^{\infty} a_n e_{n,a}(x,t)$$

and its derivatives converge uniformly in $\widetilde{K}$ by using bounds of the type given in (5.10). The derivatives of the $e_{n,a}(x,t)$ satisfy these same types of bounds as can be proved from the formula (5.9). □

8. Further examples. Throughout Sections 4 to 7, we have been concerned with second order initial value problems. To gain a better perspective on the method of qips, it is useful to consider examples in which higher order t derivatives, higher order operators, and variable coefficients (aside from the factor a/t in the EPD case) appear in the equation. In the following, we provide three examples. The main purpose for the last of these is to illustrate (i) various ways of decomposing formal solution operator series into component series and, hence, component integral solution operators via qips and (ii) the rearranging of factors, including differential operators, in the exponential appearing in these integral solution operators. Most of these rearrangements are carried out to minimize the analyticity requirements on the data.

A. A higher order EPD *problem.* First we consider the problem

$$u_{tt}(x,t)+\frac{a}{t}u_t(x,t)=D^4u(x,t),\qquad u(x,0)=\phi(x),\qquad u_t(x,0)=0 \tag{8.1}$$

with $a>1$. For this example, we have

$$\begin{aligned} K(x,t,\sigma;\phi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{t(1-\sigma)e^{i\theta}}e^{te^{-i\theta}D^4/4}\phi(x)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{(t\cos\theta\sqrt{1-\sigma})D^2}\phi(x)\,d\theta. \end{aligned} \tag{8.2}$$

In view of the fact that D^2 generates a semigroup, the integrand in the last member of this fails to have a meaning whenever we choose $\phi(x)$ to be bounded and continuous. However, if we select $\phi(x)$ to be entire of growth $(2,\tau)$ and let $h(x,t)$ denote the solution of the problem $h_t(x,t)=D^2h(x,t)$, $h(x,0)=\phi(x)$, then it follows that $h(x,t)$ is defined in the time strip $|t|<1/(4\tau)$ [**19**]. If we so restrict t, it follows that

$$K(x,t,\sigma;\phi)=\frac{1}{2\pi}\int_0^{2\pi} h(x,t\cos\theta\sqrt{1-\sigma})\,d\theta.$$

By (3.6),

$$u(x,t)=\frac{(a-1)}{4\pi}\int_0^1 \sigma^{(a-3)/2}\left\{\int_0^{2\pi} h(x,t\cos\theta\sqrt{1-\sigma})\,d\theta\right\}d\sigma \tag{8.3}$$

for $-\infty < x < \infty$ and $|t| < 1/(4\tau)$.

B. A variable coefficient problem. As a preliminary to a PDE problem, let us consider the following ordinary differential equation problem:

$$y''(t) - \lambda t^2 y(t) = 0, \qquad y(0) = 0, \qquad y'(0) = 1. \tag{8.4}$$

By using series methods, one can show that

$$\begin{aligned} y(t) &= t \cdot \sum_{n=0}^{\infty} \frac{t^{4n}\lambda^n}{4^{2n} n! \cdot (5/4)_n} \\ &= t \cdot \sum_{n=0}^{\infty} \left\{ \frac{(t^3/4)^n}{(5/4)_n} \right\} \cdot \left\{ \frac{(t\lambda)^n}{n!} \right\} \\ &= \frac{1}{8\pi} \int_0^1 \sigma^{-3/4} \left\{ \int_0^{2\pi} e^{(2t^2\lambda^{1/2} \cos\theta \sqrt{1-\sigma}/4)}\, d\theta \right\} d\sigma. \end{aligned} \tag{8.5}$$

Now consider the partial differential equation problem

$$u_{tt}(x,t) = t^2 D^2 u(x,t), \qquad u(x,0) = 0, \qquad u_t(x,0) = \phi(x). \tag{8.6}$$

If we replace $\lambda^{1/2}$ in the last member of (8.5) by D and then apply the integral as an operator acting on $\phi(x)$, we obtain the solution formula

$$u(x,t) = \frac{t}{8\pi} \int_0^1 \sigma^{-3/4} \left\{ \int_0^{2\pi} \phi\left(x + \frac{1}{2} t^2 \cos\theta \sqrt{1-\sigma} \right) d\theta \right\} d\sigma \tag{8.7}$$

if $\phi(x) \in C^1$.

C. Higher order t derivatives. As the final example, we consider the initial value problem

$$\begin{aligned} D_t^4 u(x,t) = D^4 u(x,t), \qquad u(x,0) = u_t(x,0) = u_{tt}(x,0), \\ u_{ttt}(x,0) = \phi(x). \end{aligned} \tag{8.8}$$

The equation in this is of some interest since it is satisfied both by solutions of the wave equation and Laplace's equation as is clearly evident from the following factorizations of it: $(D_t^2 - D^2)(D_t^2 + D^2)u(x,t) =$

$(D_t^2+D^2)(D_t^2-D^2)u(x,t)$. The polynomial $x^8+70x^4t^4+t^8$ is an example of a solution of that fourth order equation that is not a solution of the wave or Laplace equations. One can solve the equation in (8.8) by reducing it, for example, to the pair of equations $(D_t^2-D^2)U(x,t)=0$ where $(D_t^2+D^2)u(x,t)=U(x,t)$, and this entails solving a wave problem followed by solving a Poisson problem. One must, of course, choose the solutions of these so that the initial conditions in (8.8) are satisfied, and this involves an extensive number of considerations. Were we to replace the equation in (8.8) by the equation $D_t^4u(x,t)=tD^2u(x,t)$, then this factorization approach would not even be available to us. As a consequence, we shall employ the method of qips to solve (8.8) and note that it will also apply to an analogous problem with the basic equation replaced by one of the type mentioned.

An associated ordinary differential equation problem for (8.8) is given by $y^{(iv)}(t)=\lambda y(t)$, $y^{(0)}=y'(0)=y''(0)=0$, $y'''(0)=1$. By using series methods and making appropriate rearrangements in the terms of the series, one can show that

$$(8.9)\quad y(t)=\frac{t^3}{6}\cdot\sum_{n=0}^{\infty}\frac{\tau^n\lambda^n}{n!\cdot(5/4)_n\cdot(3/2)_n\cdot(7/4)_n}\quad \text{where } \tau=(t/4)^4.$$

Upon replacing λ in this by D^4 and operating on $\phi(x)$, we obtain the following formal operator series solution of (8.8):

$$(8.10)\quad u(x,t)=\frac{t^3}{6}\cdot\sum_{n=0}^{\infty}\frac{\tau^nD^{4n}\phi(x)}{n!\cdot(5/4)_n\cdot(3/2)_n\cdot(7/4)_n},\qquad \tau=(t/4)^4.$$

If $\phi(x)=x^n$, n a positive integer, and if we let $U_n(x,t)$ denote the solution of (8.8) corresponding to this $\phi(x)$, then it is not hard to show from (8.10) that

$$U_n(x,t)=\sum_{j=0}^{[n/4]}\binom{n}{4j}\cdot\frac{t^{3+4j}x^{n-4j}}{(4j+1)(4j+2)(4j+3)}.$$

For other types of data, we need to rewrite (8.10) in integral solution form by suitable decompositions of the operator series in (8.10) into operators having integral forms.

Now the series for $y(t)$ in (8.9) can be variously decomposed, using qips, as

$$\begin{aligned} y(t) &= \{F_{3/2,7/4}(\underline{\tau}) \circ F_{5/4,1}(\underline{\lambda})\} \cdot (t^3/3) \\ &= \{F_{5/4,7/4}(\underline{\tau}) \circ F_{3/2,1}(\underline{\lambda})\} \cdot (t^3/3) \\ &= \{F_{5/4,3/2}(\underline{\tau}) \circ F_{7/4,1}(\underline{\lambda})\} \cdot (t^3/3). \end{aligned} \tag{8.11}$$

In the following we will make use of the second of these qips and note that the other two lead to similar conclusions. We then have

$$y(t) = \frac{t^3}{12\pi} \int_0^{2\pi} F_{5/4,7/4}(\tau e^{i\theta}) F_{3/2,1}(\lambda e^{-i\theta})\, d\theta.$$

Hence, the formal operator series solution in (8.10) can be replaced by the formal integral solution formula

$$u(x,t) = \frac{t^3}{12\pi} \int_0^{2\pi} F_{5/4,7/4}(t^4 e^{i\theta}/4^4) F_{3/2,1}(e^{-i\theta} D^4)\phi(x)\, d\theta. \tag{8.12}$$

Applying property (2.4) to move around the factors in the arguments of the F functions in the integrand of this, we get

$$u(x,t) = \frac{t^3}{12\pi} \int_0^{2\pi} F_{5/4,7/4}(t^2 e^{i\theta} D^2/16)\Omega(x,t,\theta;\phi)\, d\theta \tag{8.13}$$

where

(8.14)

$$\begin{aligned} \Omega(x,t,\theta;\phi) &= F_{3/2,1}(t^2 e^{-i\theta} D^2/16)\phi(x) \\ &= \frac{1}{4\pi}\int_0^1 \sigma^{-1/2}\left\{\int_0^{2\pi} e^{t(1-\sigma)e^{-i\theta}e^{i\psi}} e^{e^{-i\psi}D^2}\phi(x)\, d\theta\right\} d\sigma \\ &= \frac{1}{4\pi}\int_0^1 \sigma^{-1/2}\left\{\int_0^{2\pi} e^{(te^{-i\theta/2}\cos\psi\sqrt{(1-\sigma)}/2)D}\phi(x)\, d\psi\right\} d\sigma \\ &= \frac{1}{4\pi}\int_0^1 \sigma^{-1/2}\left\{\int_0^{2\pi} \phi(x + te^{-i\theta}\cos\psi\sqrt{(1-\sigma)}/2) d\psi\right\} d\sigma. \end{aligned}$$

Having thus computed the Ω function, we can now compute the integrand in (8.13). We leave it to the reader to make appropriate modifications of formula (2.7) to establish the result

$$(8.15)\quad F_{5/4,7/4}(te^{i\theta}D^2/16)\Omega(x,t,\theta;\phi)$$
$$= \frac{3}{32\pi}\int_0^1\int_0^1 (1-\sigma_1)^{-3/4}(1-\sigma_2)^{-1/4}$$
$$\cdot\left\{\int_0^{2\pi}\Omega(x+te^{i\theta/2}\cos\varsigma\sqrt{(1-\sigma_1)(1-\sigma_2)}/2,t,\theta;\phi)\,d\varsigma\right\}d\sigma_1\,d\sigma_2.$$

Thus, we have completed all of the preliminary calculations to obtain the integrand for the integral (8.13). If we insert (8.15) into (8.13) taking into account (8.14), we see that the final integral for $u(x,t)$ is the six-fold integral

$$\frac{3t^3}{256\pi^3}\int_0^1\int_0^1\int_0^1\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\Phi(\sigma,\sigma_1,\sigma_2,\psi,\varsigma,\theta)\,d\psi\,d\varsigma\,d\theta\,d\sigma_2\,d\sigma_1\,d\sigma$$

where

$$\Phi = \sigma^{-1/2}\sigma_1^{-3/4}\sigma_2^{-1/4}\phi$$
$$\cdot(x+\{e^{-i\theta/2}\cos\psi\sqrt{1-\sigma}+e^{i\theta/2}\cos\varsigma\sqrt{(1-\sigma_1)(1-\sigma_2)}\}t/2).$$

In view of the fact that the equation (8.8) is satisfied both by the wave and Laplace polynomials and these have different types of bounds on them, there is the question of what type of bound holds for the above defined $U_n(x,t)$. By carefully working through the integrals in (8.14) and (8.15) when $\phi(x)=x^n$, one can establish the bounding relation $|U_n(x,t)|\le C|t^3|\cdot(|x|+|t|)^n$ for C an appropriate constant. Hence, if the function

$$\phi(x)=\sum_{n=0}^{\infty}a_nx^n$$

is analytic for $|x|<R$, one can show that

$$\sum_{n=0}^{\infty}a_nU_n(x,t)$$

is a solution of the equation (8.8) in the region $|x|+|t|<R$. This is not surprising since it is the smaller of the two convergence regions for the expansions in wave and Laplace polynomials.

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