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ON THE ABSOLUTE RIESZ SUMMABILITY FACTORS

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ABSTRACT. In this paper a theorem on $|\overline{N}, p_n|_k$ summability factors, which generalizes a theorem of Mazhar [**6**] on $|C, 1|_k$ summability factors of infinite series, has been proved. We also apply it to Fourier series.

1. Introduction. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). We denote by u_n and t_n the nth $(C, 1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C, 1|_k$, $k \geq 1$, if (see [**3**])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{1.1}$$

But since $t_n = n(u_n - u_{n-1})$ (see [**5**]) condition (1.1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{1.2}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as } n \to \infty, \qquad P_{-i} = p_{-i} = 0, \quad i \geq 1. \tag{1.3}$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence (w_n) of the Riesz means, or simply the $(\overline{N}, p_n)$ means, of the sequence (s_n) generated by the sequence of coefficients

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(p_n) (see [**4**]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [**1**])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |w_n - w_{n-1}|^k < \infty. \tag{1.5}$$

In the special case when $p_n = 1$ for all values of n (respectively, $k = 1$), then $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ (respectively, $|\overline{N}, p_n|$) summability.

If we write

$$X_n = \sum_{v=0}^{n} p_v/P_v, \tag{1.6}$$

then (X_n) is a positive increasing sequence tending to infinity with n.

Mazhar [**6**] proved the following theorem.

Theorem A. *If*

$$\lambda_n \to 0 \quad \textit{as } n \to \infty \tag{1.7}$$

$$\sum_{n=1}^{m} n \log n |\Delta^2 \lambda_n| = \mathcal{O}(1), \tag{1.8}$$

and

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = \mathcal{O}(\log m) \quad \textit{as } m \to \infty, \tag{1.9}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C, 1|_k$, $k \geq 1$.

Note. $\Delta^2 \lambda n = \Delta \lambda n - \Delta \lambda_{n+1}$ and $\Delta \lambda n = \lambda_n - \lambda_{n+1}$.

2. The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n|_k$ summability in the form of the following theorem.

Theorem 1. *Let* (p_n) *be a sequence of positive numbers such that*

$$P_n = \mathcal{O}(np_n) \quad \textit{as } n \to \infty. \tag{2.1}$$

If

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = \mathcal{O}(1) \quad \textit{as } m \to \infty \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = \mathcal{O}(X_m) \quad \textit{as } m \to \infty, \tag{2.3}$$

and (1.7) *is satisfied, then the series* $\sum a_n\lambda_n$ *is summable* $|\overline{N}, p_n|_k$, $k \geq 1$.

Note. It should be noted that if we take $p_n = 1$ for all $n \in N$ in Theorem 1 (in this case $X_n \sim \log n$), then we get Theorem A.

We need the following lemma for the proof of Theorem 1.

Lemma. *Under the conditions of Theorem* 1, *we get*

$$nX_n|\Delta\lambda_n| = \mathcal{O}(1) \quad \textit{as } n \to \infty \tag{2.4}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty \tag{2.5}$$

$$X_n|\lambda_n| = O(1) \quad \textit{as } n \to \infty. \tag{2.6}$$

Proof. Since (nX_n) is increasing, we have

$$nX_n|\Delta\lambda_n| \leq \sum_{v=n}^{\infty} vX_v|\Delta^2\lambda_v| < \infty,$$

by (2.2). Hence $nX_n|\Delta\lambda_n| = \mathcal{O}(1)$ as $n \to \infty$. Also

$$\begin{aligned}\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| &= \sum_{n=1}^{\infty} X_n|\sum_{v=n}^{\infty} \Delta^2\lambda_v| \\ &\le \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty} |\Delta^2\lambda_v| \\ &= \sum_{v=1}^{\infty} |\Delta^2\lambda_v| \sum_{n-1}^{v} X_n \\ &\le \sum_{v=1}^{\infty} vX_v|\Delta^2\lambda_v| < \infty,\end{aligned}$$

by (2.2). Finally, we have that

$$X_n|\lambda_n| \le \sum_{v=n}^{\infty} X_v|\Delta\lambda_v| \le \sum_{v=1}^{\infty} X_v|\Delta\lambda_v| < \infty,$$

by (2.5). This completes the proof of the lemma. □

3. Proof of Theorem 1. Let (T_n) be the $(\overline{N}, p_n)$ means of the series $\sum a_n\lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r\lambda_r = \frac{1}{P_n}\sum_{v=0}^{n}(P_n - P_{v-1})a_v\lambda_v. \tag{3.1}$$

Then, for $n \ge 1$, we have

$$\begin{aligned}T_n - T_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} \frac{P_{v-1}\lambda_v}{v} va_v.\end{aligned} \tag{3.2}$$

Using Abel's transformation, we get

$$\begin{aligned}
T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} \Delta\left(\frac{P_{v-1}\lambda_v}{v}\right) \sum_{r=1}^{v} r a_r \\
&\quad + \frac{p_n \lambda_n}{n P_n} \sum_{v=1}^{n} v a_v \\
&= -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v \frac{v+1}{v} t_v \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v \frac{v+1}{v} t_v \\
&\quad + \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} \frac{1}{v} t_v \\
&\quad + \frac{(n+1) p_n \lambda_n t_n}{n P_n} \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

Since

$$|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^k \le 4^k (|T_{n,1}|^k + |T_{n,2}|^k + |T_{n,3}|^k + |T_{n,4}|^k),$$

to complete the proof of Theorem 1, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3, 4. \tag{3.3}$$

Using Hölder's inequality we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} p_v |\lambda_v| \frac{v+1}{v} |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} |\lambda_v| p_v |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |\lambda_v|^k p_v |t_v|^k \right\}
\end{aligned}$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |\lambda_v|^{k-1} |\lambda_v| p_v |t_v|^k \right\}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$= O(1) \sum_{v=1}^{m} |\lambda_v| \frac{p_v}{P_v} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \frac{p_r}{P_r} |t_r|^k$$

$$+ O(1)|\lambda_m| \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v|^k$$

$$= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m$$

$$= O(1) \quad \text{as } m \to \infty,$$

by virtue of the hypotheses and lemma. Using the fact that $P_v = O(vp_v)$, by (2.1) we have

$$\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,2}|^k \le \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v |\Delta\lambda_v| \frac{v+1}{v} |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} v p_v |\Delta\lambda_v| |t_v| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^k p_v |t_v|^k \right\}$$

$$\times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}}$$

$$
\begin{aligned}
&\times \left\{ \sum_{v=1}^{n-1} (v|\Delta\lambda_v|)^{k-1} (v|\Delta\lambda_v|) p_v |t_v|^k \right\} \\
&= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| p_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} \\
&= O(1) \sum_{v=1}^{m} v|\Delta\lambda_v| \frac{p_v}{P_v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v|\Delta\lambda_v|) \sum_{r=1}^{v} \frac{p_r}{P_r} |t_r|^k \\
&\quad + O(1) m|\Delta\lambda_m| \sum_{v=1}^{m} \frac{p_v}{P_v} |t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v)\Delta\lambda_v|)| X_v + O(1) m|\Delta\lambda_m| X_m \\
&= O(1) \sum_{v=1}^{m-1} v|\Delta^2\lambda_v| X_v \\
&\quad + O(1) \sum_{v=1}^{m-1} |\Delta\lambda_{v+1}| X_v + O(1) m|\Delta\lambda_m| X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses and lemma. Again, since $P_v/v = O(p_v)$, by (2.1) we have

$$
\begin{aligned}
\sum_{n=2}^{m+1} (P_n/p_n)^{k-1} |T_{n,3}|^k &\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}| \frac{P_v}{v} |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}| p_v |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{P_n}{P_n P_{n-1}} \left\{ \sum_{v=1}^{n-1} |\lambda_{v+1}|^k p_v |t_v|^k \right\} \\
&\quad \times \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\left\{\sum_{v=1}^{n-1}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|p_v|t_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|p_v|t_v|^k\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}\\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|\frac{p_v}{P_v}|t_v|^k\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_v + O(1)|\lambda_{m+1}|X_m\\
&= O(1) \quad \text{as } m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses and lemma.

Finally, as in $T_{n,1}$, we have

$$
\begin{aligned}
\sum_{n=1}^{m}(P_n/p_n)^{k-1}|T_{n,4}|^k &= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{p_n}{P_n}|t_n|^k\\
&= O(1) \quad \text{as } m\to\infty.
\end{aligned}
$$

Therefore, we get

$$
\sum_{n=1}^{m}(P_n/p_n)^{k-1}|T_{n,r}|^k = O(1) \quad \text{as } m\to\infty, \quad \text{for } r=1,2,3,4.
$$

This completes the proof of Theorem 1. □

4. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Let

$$
f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x)
$$

$$
\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}
$$

and

$$
\phi_1=\frac{1}{t}\int_0^t\phi(u)\,du.
$$

It is well known that if $\phi_1(t) \in BV(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nA_n(x))$ (see [**2**]). Using this fact, we get the following result for Fournier series.

Theorem 2. *If $\phi_1(t) \in BV(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 1, then the series $\sum A_n(x)\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$.*

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