ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

OSCILLATIONS OF HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We obtain a necessary and sufficient condition for the oscillation of all solutions of the higher order neutral differential equation

$$\frac{d^n}{dt^n}(x(t) - P(t)x(t-\tau)) + Q(t)x(t-\sigma(t)) = 0,$$

where $n \geq 1$ is an odd integer, $\tau > 0$, P, Q, $\sigma \in C([t_0,\infty), R^+)$ and $\lim_{t\to\infty}(t-\sigma(t)) = \infty$. Some applications of this result are also listed. Our results extend and improve some known results in the literature. In particular, our conditions do not require

$$\int_{t_0}^{\infty} Q(s)\,ds = \infty.$$

1. Introduction. Consider the n-order neutral delay differential equation

$$\frac{d^n}{dt^n}(x(t) - P(t)x(t-\tau)) + Q(t)x(t-\sigma(t)) = 0, \tag{1}$$

where $n \geq 1$ is an odd integer,

$$\begin{aligned} \tau \in (0,\infty), \quad & P, Q, \sigma \ni C([t_0,\infty), R^+), \\ R^+ = [0,\infty), \quad & Q(t) \not\equiv 0 \end{aligned} \tag{2}$$

and

$$t - \sigma(t) \to \infty \quad \text{as } t \to \infty.$$

Received by the editors on July 24, 1992, and in revised form on February 24, 1993.
Key words and phrases. Neutral differential equations, oscillation, comparison theorem.
1991 *Mathematics Subject Classification.* 34K15, 34C10.
Project supported by the National Natural Science Foundation of P.R. China and NSERC Canada.
Research of the first author completed while visiting the University of Alberta.

The literature on the oscillation theory of neutral differential equations is growing rapidly. This study is a relatively new field and is very interesting in applications. In particular, the oscillation of Equation (1) has been investigated by many authors. For some contributions to this topic, we refer to the monograph by Györi and Ladas [**7**], the papers by Chuanxi and Ladas [**1**], Georgiou and Qian [**3**], Gopalsamy, Lalli and Zhang [**4**], Grammatikopoulos, Ladas and Meimaridou [**5**], Grove, Ladas and Schinas [**6**], Ladas, Qian and Yan [**8**], Ladas and Qian [**9**], Ladas and Sficas [**10**] and Yan [**12**]. All the papers mentioned above, however, assume that the coefficient $Q(t)$ satisfies the integral condition

$$\int_{t_0}^{\infty} Q(s)\,ds = \infty \tag{3}$$

which has played a very important role in the study of (1). Considerably less is known about the oscillatory behavior of (1) when condition (3) does not hold.

Our aim in this paper is to establish several new types of oscillation results of (1), which do not require the integral condition (3). In Section 2 we establish several key lemmas which will enable us to obtain many oscillation criteria for (1) without condition (3). These lemmas are interesting in their own right. In Section 3 we obtain a necessary and sufficient condition for the oscillation of all solutions of (1). As some applications of this result, we establish two new comparison theorems of (1). These results extend and improve several known results in the literature.

As usual, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large t.

2. Several key lemmas. The main results in this section are the following Lemmas 1–3 which will play key roles in the proofs of the theorems in Section 3.

Lemma 1. *Assume that* (2) *holds and that*

$$0 \le P(t) \le 1. \tag{4}$$

Let $x(t)$ *be an eventually positive solution of the inequality*

$$\frac{d^n}{dt^n}(x(t) - P(t)x(t-\tau)) + Q(t)x(t-\sigma(t)) \le 0 \tag{5}$$

and set

$$y(t) = x(t) - P(t)x(t-\tau). \tag{6}$$

Then $y(t)$ *is eventually positive.*

The following Lemma 2 is an improvement of Lemma 10.5.2 in [**7**].

Lemma 2. *Assume that* $n \ge 1$ *is a positive integer,*

$$P, Q, \sigma \in C([T,\infty), R^+), \qquad \tau \in (0,\infty) \tag{7}$$

and

$$\lim_{t\to\infty}(t - \sigma(t)) = \infty,$$

and either

$$P(t) + Q(t)\sigma(t) > 0 \quad \textit{for } t \ge T \tag{8}$$

or

$$\sigma(t) > 0 \quad \textit{and} \quad Q(s) \not\equiv 0 \quad \textit{for } s \in [t, T^*] \tag{9}$$

where $T^* = T^*(t) = \min\{T : T - \sigma(T) = t\}$.

Let $b = \max\{\tau, \sup_{t\ge T}\sigma(t)\}$, *and assume that the integral inequality*

$$\begin{aligned} z(t) &\ge P(t)z(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)z(s-\sigma(s))\,ds, \qquad t \ge T \end{aligned} \tag{10}$$

has a continuous positive solution $z : [T-b, \infty) \to (0, \infty)$. *Then the corresponding integral equation*

$$(11)\qquad \begin{aligned} x(t) &= P(t)x(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)x(s-\sigma(s))\,ds, \qquad t \geq T \end{aligned}$$

has a continuous positive solution $x : [T-b, \infty) \to (0, \infty)$.

Proof. Define the set of functions

$$W = \{w \in C([T-b, \infty), R^+) : 0 \leq w(t) \leq 1 \quad \text{for } t \geq T-b\},$$

and define the mapping S on W as follows

$$(Sw)(t) = \begin{cases} \frac{1}{z(t)}[P(t)w(t-\tau)z(t-\tau) \\ \quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1} \\ \qquad \times Q(s)w(s-\sigma(s))z(s-\sigma(s))\,ds], & t \geq T, \\ \frac{t-T+b}{b}(Sw)(T) + 1 - \frac{t-T+b}{b}, & T-b \leq t \leq T. \end{cases}$$

It is easy to see by using (10) that S maps W into itself, and for any $w \in W$ we have $(Sw)(t) > 0$ for $T-b \leq t < T$.

We now define the following sequence $\{w_k(t)\}$ in W:

$$\begin{aligned} w_0(t) &= 1, \qquad t \geq T-b, \\ w_{k+1}(t) &= (Sw_k)(t) \quad \text{for } k = 0, 1, 2, \dots . \end{aligned}$$

Then, by using (10) and a simple induction, we can easily see that

$$0 \leq w_{k+1}(t) \leq w_k(t) \leq 1$$

for $t \geq T-b$ and $k = 0, 1, 2, \dots$. Set

$$w(t) = \lim_{k\to\infty} w_k(t) \quad \text{for } t \geq T-b.$$

Then it follows by Lebesgue's dominated convergence theorem that $w(t)$ satisfies

$$\begin{aligned} w(t) &= \frac{1}{z(t)}[P(t)w(t-\tau)z(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)w(s-\sigma(s))z(s-\sigma(s))\,ds], \end{aligned}$$

$t \geq T$, and

$$w(t) = \frac{t-T+b}{b} w(T) + 1 - \frac{t-T+b}{b} > 0$$

for $T-b \leq t < T$. Set $x(t) = w(t)z(t)$. Then $x(t)$ satisfies the integral equation (11) and

$$x(t) > 0 \quad \text{for } T-b \leq t < T.$$

Clearly, $x(t)$ is continuous on $[T-b, T]$. Then, by the method of steps we see, in view of (11), that $x(t)$ is continuous on $[T-b, \infty)$.

Finally it remains to show that $x(t) > 0$ for $t \geq T-b$. Assume that there exists $t^* \geq T-b$ such that $x(t) > 0$ for $T-b \leq t < t^*$ and $x(t^*) = 0$. Clearly, $t^* \geq T$. Thus, by (11) we have

$$\begin{aligned} 0 = x(t^*) = P(t^*)&x(t^* - \tau) \\ &+ \frac{1}{(n-1)!} \int_{t^*}^{\infty} (s-t)^{n-1} Q(s) x(s - \sigma(s))\, ds \end{aligned}$$

which implies

$$P(t^*) = 0 \quad \text{and} \quad Q(s)x(s-\sigma(s)) = 0$$

for all $s \geq t^*$, which contradicts (8) or (9). Therefore, $x(t)$ is positive on $[T-b, \infty)$, and the proof is complete. □

Similarly, we can prove the following lemma.

Lemma 3. *Assume that* (7) *holds,* $1 \leq n^* \leq n-1$, $c > 0$ *and that the integral inequality*

$$\begin{aligned} z(t) \geq c + &P(t)z(t-\tau) \\ &+ \int_T^t (t-u)^{n^*-1} \int_u^{\infty} (s-u)^{n-n^*-1} Q(s) z(s-\sigma(s))\, ds, \end{aligned} \tag{12}$$

$t \geq T$, has a continuous positive solution $z : [T-b, \infty) \to (0, \infty)$, where b is defined as in Lemma 2. *Then the corresponding integral equation*

$$\begin{aligned} x(t) = c + &P(t)x(t-\tau) \\ &+ \int_T^t (t-u)^{n^*-1} \int_u^{\infty} (s-u)^{u-n^*-1} Q(s) x(s-\sigma(s))\, ds, \end{aligned} \tag{13}$$

also has a continuous solution $x : [T - b, \infty) \to (0, \infty)$.

3. Main results. The following first theorem provides a necessary and sufficient condition for the oscillation of all solutions of Equation (1).

Theorem 1. *Assume that* (2) *and* (4) *hold and that* (8) *or* (9) *holds. Then every solution of* (1) *is oscillatory if and only if the corresponding differential inequality* (5) *has no eventually positive solutions.*

Before we prove Theorem 1, let us first compare it with some known results in the literature.

Remark 1. When $n = 1$, Theorem 1 improves Theorem 2.1 in Lalli and Zhang [**11**] by removing the hypothesis (3) and by relaxing the hypothesis $P(t) \equiv c \in (0, 1)$. We should note that Gopalsamy, Lalli and Zhang [**4**] extend Theorem 2.1 in [**11**] to (1), which is also improved by Theorem 1.

Proof of Theorem 1. The sufficiency is obvious. Therefore, we only need to prove the necessity. To this end, we assume, for contradiction, that (5) has an eventually positive solution $x(t)$ and set $y(t)$ as in (6). Then, by Lemma 1, we have

$$y(t) > 0. \tag{14}$$

In view of (2), we get

$$y^{(n)}(t) \le -Q(t)x(t - \sigma(t)) \le 0 \quad \text{and} \quad y^{(n)}(t) \not\equiv 0 \tag{15}$$

which implies that $y^{(n-1)}(t)$ is decreasing and that $y^{(i)}(t)$ for $i = 0, 1, \dots, n-2$ is strictly monotonic. Since n is odd, it follows by (14) that there exists an even integer $n^* : 0 \le n^* \le n - 1$ such that for large t,

$$(-1)^i y^{(i)}(t) > 0 \quad \text{for } i = n^*, n^* + 1, \dots, n-1 \tag{16}$$

and

$$y^{(i)}(t) > 0 \quad \text{for } i = 0, 1, \dots, n^*.$$

First consider the case $n^* = 0$. For this case, integrating (15) from t to ∞ and using (16), one can obtain

$$y(t) \geq \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} Q(s) x(s-\sigma(s))\, ds.$$

That is,

$$x(t) \geq P(t)x(t-\tau) + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} Q(s) x(s-\sigma(s))\, ds$$

which implies, in view of Lemma 2, that the corresponding integral equation

$$z(t) = P(t)z(t-\tau) + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} Q(s) z(s-\sigma(s))\, ds$$

also has a positive solution $z(t)$. It is clear to see that this $z(t)$ is an eventually positive solution of (1) and so the proof of this theorem for the case $n^* = 0$ is complete. For the case $n^* \geq 2$ we can use Lemma 3 to prove this theorem as in the case $n^* = 0$. The proof is complete. □

We now list some applications of Theorem 1. We compare (1) with the equation

$$\frac{d^n}{dt^n}(x(t) - P^*(t)x(t-\tau)) + Q^*(t)x(t-\sigma(t)) = 0 \tag{17}$$

and state the following comparison theorem.

Theorem 2. *Assume that* (2) *holds and that* (8) *or* (9) *holds. Further, assume that*

$$P^*(t) \geq P(t) \quad \text{and} \quad Q^*(t) \geq Q(t) \quad \text{for } t \geq t_0 \tag{18}$$

and that

$$P^*(t) \leq 1. \tag{19}$$

Then every solution of (1) *oscillates implies that every solution of* (17) *also oscillates.*

Remark 2. When $n = 1$, Theorem 2 is an improvement of Theorem 2.2 in [**11**] by removing the integral condition (3) and by relaxing the hypothesis that $1 > P^*(t) \equiv p^* \geq P(t) \equiv p > 0$. It also improves the corresponding result in [**4**].

Proof of Theorem 2. Suppose the contrary, and let $x(t)$ be an eventually positive solution of (17). Set

$$z(t) = x(t) - P^*(t)x(t-\tau).$$

In view of Lemma 1, we have $z(t) > 0$. From (17) we see that $z^{(n)}(t) = -Q^*(t)x(t-\sigma(t)) \leq 0$. Hence, there exists an even integer $n^* : 0 \leq n^* \leq n-1$ such that

$$(-1)^i z^{(i)}(t) > 0 \quad \text{for } i = n^*, n^*+1, \dots, n-1 \tag{20}$$

and

$$z^{(i)}(t) > 0 \quad \text{for } i = 0, 1, \dots, n^*.$$

If $n^* = 0$, then by integrating (17) from t to ∞ n times we have

$$z(t) = z(\infty) + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1} Q^*(s)x(s-\sigma(s))\,ds$$

and so

$$\begin{aligned} x(t) &= z(\infty) + P^*(t)x(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q^*(s)x(s-\sigma(s))\,ds \\ &\geq P(t)x(t-\tau) \\ &\quad + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)x(s-\sigma(s))\,ds, \end{aligned}$$

where $z(\infty) = \lim_{t\to\infty} z(t) \geq 0$. Similar to the proof of Theorem 1, one can see that (1) also has an eventually positive solution which is a contradiction. Hence, the proof for the case $n^* = 0$ is complete. If $n^* \neq 0$ the proof can be carried out in a similar way. □

The following Theorem 3 provides a comparison result which shows that, under appropriate hypotheses, (1) has certain oscillatory properties provided that the same is true for an associated nonneutral differential equation.

In the sequel, for the sake of convenience, we define

$$\prod_{j=1}^{0} P(s - \sigma - (j-1)\tau) \equiv 1.$$

Theorem 3. *Assume that* (2) *and* (4) *hold with* $\sigma(t) = \sigma$. *Suppose also that there exist two nonnegative integers* m *and* N *with* $m \leq N$ *such that for any sufficiently large* t *there exists* $i_0 \in \{m, m+1, \dots, N\}$ *such that*

$$Q(s) \prod_{j=1}^{i_0} P(s - \sigma - (j-1)\tau) \not\equiv 0$$

for $s \in [t, t + \sigma + i_0\tau]$. *Then every solution of the delay differential equation*

$$u^{(n)}(t) + \sum_{i=m}^{N} Q(t) \left(\prod_{j=1}^{i} P(t - \sigma - (j-1)\tau) \right) u(t - \sigma - i\tau) = 0 \tag{21}$$

oscillates implies that every solution of (1) *also oscillates.*

Remark 3. Theorem 3 is an improvement of Theorem 1 in [**8**] by removing the condition (3).

Proof of Theorem 3. Otherwise, (1) has an eventually positive solution $x(t)$ and set $y(t)$ as in (6). Then by Lemma 1, we have for large t

$$y(t) > 0. \tag{22}$$

From (1) we find that

$$y^{(n)}(t) = -Q(t)x(t-\sigma) \le 0 \tag{23}$$

which implies that $y^{(i)}(t)$ for $i = 0, 1, \dots, n-1$ are eventually monotonic. Since n is odd, it follows by (21) and (23) that there exists an even integer $n^* : 0 \le n^* \le n-1$ such that (16) holds. From (1) and (6) we see that

$$y^{(n)}(t) + Q(t)y(t-\sigma) + Q(t)P(t-\sigma)x(t-\sigma-\tau) = 0,$$

and by induction,

$$y^{(n)}(t) + \sum_{i=0}^{N} Q(t)\bigg(\prod_{j=1}^{i} P(t-\sigma-(j-1)\tau)\bigg)y(t-\sigma-i\tau) + Q(t)\bigg(\prod_{j=1}^{N+1} P(t-\sigma-(j-1)\tau)\bigg)x(t-\sigma-(N+1)\tau) = 0.$$

Hence, for t sufficiently large,

$$y^{(n)}(t) + \sum_{i=m}^{N} Q(t)\bigg(\prod_{j=1}^{i} P(t-\sigma-(j-1)\tau)\bigg)y(t-\sigma-i\tau) \le 0. \tag{24}$$

We consider the following two possible cases:

Case 1. $n^* = 0$. At this time, by integrating (24) from t to ∞ n times and noting (16), we have

$$y(t) \ge \frac{1}{(n-1)!}\sum_{i=m}^{N}\int_t^{\infty}(s-t)^{n-1}Q(s) \times \bigg(\prod_{j=1}^{i} P(s-\sigma-(j-1)\tau)\bigg)y(s-\sigma-i\tau)\,ds.$$

By a slight modification of Lemma 2, one can see that the corresponding integral equation

$$z(t) = \frac{1}{(n-1)!}\sum_{i=m}^{N}\int_t^{\infty}(s-t)^{n-1}Q(s) \times \bigg(\prod_{j=1}^{i} P(s-\sigma-(j-1)\tau)\bigg)z(s-\sigma-i\tau)\,ds$$

also has a continuous and positive solution $z : [T-b,\infty) \to (0,\infty)$ for some $T > t_0$, where $b = \max\{\tau,\sigma\}$. It is clear that this $z(t)$ is a solution of (21).

Case 2. $n^* \neq 0$. Then first integrating (24) from t to ∞ $n-n^*$ times and using (16), we have

$$y^{(n^*)}(t) \geq \frac{1}{(n-n^*-1)!}\sum_{i=m}^{N}\int_t^\infty (s-t)^{n-n^*-1}Q(s) \times \left(\prod_{j=1}^{i} P(s-\sigma-(j-1)\tau)\right) y(s-\sigma-i\tau)\,ds$$

and go on integrating the above inequality from T to t n^* times and noting (16), we get

$$y(t) \geq y(T)+\frac{1}{(n^*-1)!(n-n^*-1)!} \times \sum_{i=m}^{N}\int_T^t (t-u)^{n^*-1}\int_u^\infty (s-u)^{n-n^*-1}Q(s) \times \left(\prod_{j=1}^{i} P(s-\sigma-(j-1)\tau)\right) y(s-\sigma-i\tau)\,ds\,du.$$

According to this and a slight modification of Lemma 3, we find that the corresponding integral equation

$$z(t) = y(t)+\frac{1}{(n^*-1)!(n-n^*-1)!} \times \sum_{i=m}^{N}\int_T^t (t-u)^{n^*-1}\int_u^\infty (s-u)^{n-n^*-1}Q(s) \times \left(\prod_{j=1}^{i} P(s-\sigma-(j-1)\tau)\right) z(s-\sigma-i\tau)\,ds\,ds$$

has a positive and continuous solution $z(t)$ on $[T-b,\infty)$. Clearly, $z(t)$ is a solution of (21). Therefore, the proof of Theorem 3 is complete. □

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