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NONLINEAR FOURTH-ORDER TWO-POINT BOUNDARY VALUE PROBLEMS

FELIX SADYRBAEV

ABSTRACT. Existence results formulated in terms of upper and lower functions and Nagumo type conditions are presented for fourth order equations and systems of two second order equations with nonlinear boundary constraints.

1. Introduction. This paper deals with the solvability of boundary value problems associated with fourth order differential equations

$$x^{(4)} = f(t, x, x', x'', x''') \tag{1}$$

as well as with systems of the form

$$\begin{aligned} x'' &= f_1(t, x, y, x', y'), \\ y'' &= f_2(t, x, y, x', y'). \end{aligned} \tag{2}$$

The results are inspired by the ones given by Bebernes and Fraker [**3**] for the boundary value problem

$$x'' = f(t, x, x'), \tag{3}$$

$$(x(0), x'(0)) \in S_1, \qquad (x(1), x'(1)) \in S_2, \tag{4}$$

where the function f is supposed to be continuous on $[0, 1] \times \mathbf{R}^2$ and the boundary sets S_1 and S_2 are subsets of $\mathbf{R}^2$. The main idea in [**3**] is to investigate the dependence of S_1 and S_2 on the a priori bounds in order that the BVP (3), (4) have a solution.

The standard set of conditions ensuring a priori bounds is the following.

(A) *Lower and upper functions.* There exist functions α and β satisfying on $[0, 1]$ the inequalities $\alpha \leq \beta$,

$$\alpha'' \geq f(t, \alpha, \alpha'), \qquad \beta'' \leq f(t, \beta, \beta'). \tag{5}$$

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(B) *The Nagumo type condition.* There exists a continuous function $\varphi : [\lambda, N] \to (0, +\infty)$ such that $|f(t,x,y)| \le \varphi(|y|)$ for all $(t,x,y) \in \omega \times \{y : \lambda \le |y| \le N\}$ and

$$\int_\lambda^N \frac{s\,ds}{\varphi(s)} \ge \max_{[0,1]} \beta(t) - \min_{[0,1]} \alpha(t),$$

where

$$\lambda = \max\{|\beta(0) - \alpha(1)|, |\beta(1) - \alpha(0)|\},$$
$$\omega = \{(t,x) : 0 \le t \le 1, \alpha(t) \le x \le \beta(t)\}.$$

One may show that the problem (3), (4) has a solution with the graph $(t, x(t), x'(t))$ lying in $G = \omega \times [-N, N]$ if some additional hypotheses are satisfied.

For example, these additional conditions in the case of the boundary constraints

$$x(a) = A, \qquad x(b) = B$$

are

$$\alpha(a) \le A \le \beta(a), \qquad \alpha(b) \le B \le \beta(b).$$

Let $G(t)$ stand for the intersection of G with the plane $t =$const. In the principal result of Bebernes and Fraker [**3**] S_1 and S_2 have to be closed connected subsets of $G(0)$ and $G(1)$, respectively and such that S_{i+1} intersects the boundary of $G(i)$ in a certain way, $i = 0, 1$.

Note that boundary conditions (4) may be written in the form

$$g_0(x(0), x'(0)) = 0, \qquad g_1(x(1), x'(1)) = 0,$$

where g_i is a function which vanishes if and only if $(x, x') \in S_i$. Then the conditions imposed above on S_i are reduced to inequalities which a continuous function g_i must satisfy on the boundary of $G(i)$. The construction of g_i whenever S_i is given easily may be carried out (see, for instance, Gudkov [**14**]).

Problems of the type (3) and (4) were investigated by Kiguradze and Lezhawa [**18**], Muldowney and Willett [**21**] (see also references in [**5**] and [**15**]).

The case of the system

$$x' = h(t,x,y), \qquad y' = f(t,x,y)$$

was treated by Bebernes and Wilhelmsen [**4**], Kiguradze and Lezhawa [**19**] and the author [**24**].

Results similar to the ones given in [**3**] for the problem (3), (4) have been obtained by Gudkov [**14**] in the case of nonseparated boundary conditions

$$g_i(x(0), x(1), x'(0), x'(1)) = 0, \qquad i = 0, 1.$$

The same type problems were considered by Muldowney and Willett [**21**] (for the intermediate value approach see [**22**]), Erbe [**11**], Baxley and Brown [**2**], Fabry and Habets [**13**], Virzhbitsky and Sadyrbaev [**27**] (see also the books [**15, 20**]).

In this paper we consider the equation (1) together with separated boundary conditions

$$g_i(x(0), x'(0), x''(0), x'''(0)) = 0, \qquad i = 0, 1, \tag{6}$$

$$h_i(x(1), x'(1), x''(1), x'''(1)) = 0, \qquad i = 0, 1, \tag{7}$$

and

$$g_i(x(0), x'(0), x''(0), x'''(0)) = 0, \qquad i = 0, 1, 2, \tag{8}$$

$$h(x(1), x'(1), x''(1), x'''(1)) = 0, \tag{9}$$

as well as the system (2) with boundary constraints

$$g_i(x(0), y(0), x'(0), y'(0)) = 0, \qquad i = 0, 1, \tag{10}$$

$$h_i(x(1), y(1), x'(1), y'(1)) = 0, \qquad i = 0, 1. \tag{11}$$

In Section 2 we present the main results for the problem (1), (6), (7) analogous to the ones given in [**3**]. Existence results for special cases of boundary conditions (6) and (7) have been obtained in [**1, 7, 10, 28**]. Our results are fairly general both in boundary constraints and in hypotheses on the right side of (1). Besides, functions α and β, which take on the roles of lower and upper functions for fourth order equations, appear in the statements of the results in Section 2. In the Appendix an example of the problem arising in the theory of semiconductors is considered.

In Section 3 the problem (2), (10), (11) is investigated. An existence result is presented in terms of vector upper and lower functions and Nagumo type conditions, which are imposed on both f_1 and f_2 separately. Usually growth restrictions are imposed on the norm of vector function f (see [**5**, Chapter 1, **16**, Chapter 12]). Vector upper and lower functions were used in [**23, 15, 6, 12**].

Our approach may be described as follows. The basic idea is to modify the given boundary value problem to a quasilinear one, to prove a priori bounds for solutions of a modified problem and then to use the theory of quasilinear problems to establish the desired existence results for the problem under consideration.

We shall assume throughout the paper that $f, f_i : I \times \mathbf{R}^4 \to \mathbf{R}$, $g_i, h_i : \mathbf{R}^4 \to \mathbf{R}$ are continuous functions, $i = 1, 2$, $I = [0, 1]$.

2. A fourth order equation. Boundary value problems of the type (1), (6), (7) are essential in describing a vast class of elastic deflections. For example, when investigating the equilibrium states of elastic beams, one must treat extremal problems for the functional of potential energy. This integral functional depends on the second derivative of unknown function and the Euler-Lagrange equation therefore is of the form (1). For a number of linear models obtained under some simplifying assumptions of smallness, see the book by Ya. Panovko and I. Gubanova [**23**].

Consider the boundary value problem (1), (6), (7) with the following assumptions.

(A1) There exist functions $\alpha, \beta \in C^2([0,1], \mathbf{R})$, $u, v \in C^3([0,1], \mathbf{R})$ such that:

(i) For all $t \in [0,1]$, $\alpha(t) \le \beta(t)$, $u'(t) \le v'(t)$, $u(t) \le \alpha'(t) \le v(t)$, $u(t) \le \beta'(t) \le v(t)$;

(ii) For all $t \in [0,1]$, $\beta''(t) \le u'(t)$, $\alpha''(t) \ge v'(t)$;

(iii) For all $(t,x,y) \in [0,1] \times [\alpha, \beta] \times [u, v]$, $v'''(t) \le f(t,x,y,v'(t),v''(t))$, $u'''(t) \ge f(t,x,y,u'(t),u''(t))$.

(A2) There exist a continuous function $\varphi : [\lambda, +\infty) \to (0, +\infty)$ and a number $N > 0$ such that:

(i) For all $(t, x, y, z) \in I \times [\alpha, \beta] \times [u, v] \times [u', v']$, for all w:

$$|w| \geq \lambda, \qquad |f(t, x, y, z, w)| \leq \varphi(|w|),$$

(ii)

$$\int_{\lambda}^{N} \frac{s ds}{\varphi(s)} \geq \max_{[0,1]} v'(t) - \min_{[0,1]} u'(t),$$

where $\lambda = \max\{|v'(0) - u'(1)|, |v'(1) - u'(0)|\}$;

(iii) For all $t \in [0, 1]$, $-N \leq u''(t) \leq N$, $-N \leq v''(t) \leq N$.

(A3) The functions g_0, h_0, g_1, h_1 in (6) and (7) satisfy the following sign conditions:

(i) For all $(z, w) \in [u'(0), v'(0)] \times [-N, N]$

$$\begin{aligned} g_0(\alpha(0), y, z, w) &\geq 0 \qquad \text{if } \alpha'(0) \leq y \leq v(0),\\ g_0(x, v(0), z, w) &\geq 0 \qquad \text{if } \alpha(0) \leq x \leq \beta(0),\\ g_0(\beta(0), y, z, w) &\leq 0 \qquad \text{if } u(0) \leq y \leq \beta'(0),\\ g_0(x, u(0), z, w) &\leq 0 \qquad \text{if } \alpha(0) \leq x \leq \beta(0); \end{aligned}$$

(ii) For all $(z, w) \in [u'(1), v'(1)] \times [-N, N]$

$$\begin{aligned} h_0(\alpha(1), y, z, w) &\geq 0 \qquad \text{if } u(1) \leq y \leq \alpha'(1),\\ h_0(\beta(1), y, z, w) &\leq 0 \qquad \text{if } \beta'(1) \leq y \leq v(1); \end{aligned}$$

(iii) For all $(x, y) \in [\alpha(0), \beta(0)] \times [u(0), v(0)]$

$$\begin{aligned} g_1(x, y, u'(0), w) &\geq 0 \qquad \text{if } u''(0) \leq w \leq N,\\ g_1(x, y, v'(0), w) &\leq 0 \qquad \text{if } -N \leq w \leq v''(0); \end{aligned}$$

(iv) For all $(x, y) \in [\alpha(1), \beta(1)] \times [u(1), v(1)]$

$$\begin{aligned} h_1(x, y, u'(1), w) &\geq 0 \qquad \text{if } -N \leq w \leq u''(1),\\ h_1(x, y, v'(1), w) &\leq 0 \qquad \text{if } v''(0) \leq w \leq N. \end{aligned}$$

Theorem 2.1. *Suppose assumptions* (A1) *to* (A3) *are satisfied. Then the boundary value problem* (1), (6) *and* (7) *has a solution.*

The proof of Theorem 2.1 will use a modified problem defined as follows. Let

$$\delta(x,y,z)=\begin{cases} z, & \text{if } y>z,\\ y, & \text{if } x\le y\le z,\\ x, & \text{if } y<x;\end{cases}$$

$$\begin{aligned} X_1&=\delta(\alpha(t),x_1,\beta(t)), & X_2&=\delta(u(t),x_2,v(t)),\\ X_3&=\delta(u'(t),x_3,v'(t)), & X_4&=\delta(-N,x_4,N),\end{aligned} \tag{12}$$

and consider the system

$$\begin{aligned} x_1'&=X_2,\\ x_2'&=X_3+\delta(0,x_1-\beta(t),1)-\delta(0,\alpha(t)-x_1,1),\\ x_3'&=X_4,\\ x_4'&=f(t,X_1,X_2,X_3,X_4)+\delta(0,x_3-v'(t),1)\\ &\quad-\delta(0,u'(t)-x_3,1),\end{aligned} \tag{13}$$

with boundary constraints

$$\begin{aligned} x_1(0)-x_2(0)&=G_0(x(0))+X_1(0)-X_2(0),\\ x_1(1)+x_2(1)&=H_0(x(1))+X_1(1)+X_2(1);\end{aligned} \tag{14}$$

$$\begin{aligned} x_3(0)-x_4(0)&=G_1(x(0))+X_3(0)-X_4(0),\\ x_3(1)+x_4(1)&=H_1(x(1))+X_3(1)+X_4(1),\end{aligned} \tag{15}$$

where

$$\begin{aligned} G_i(x)&=g_i(X_1,X_2,X_3,X_4),\\ H_i(x)&=h_i(X_1,X_2,X_3,X_4), \qquad i=0,1.\end{aligned} \tag{16}$$

Lemma 2.1. *The boundary value problem* (13)–(15) *has a solution.*

Proof. Since the right hand sides of (13), (14) and (15) are continuous and bounded, and the corresponding homogeneous boundary value problem has only the trivial solution, the solvability of (13)–(15) follows from well-known results (see, for example, [**15**, Theorem 4.1]. □

Lemma 2.2. *For any solution* $x = (x_1, x_2, x_3, x_4)$ *of the problem* (13)–(15) *the following is true for all* $t \in I$

$$\alpha(t) \le x_1(t) \le \beta(t).$$

Proof. The proof is by contradiction, considering several cases. Suppose there exists $t_0 \in (0,1)$ such that $x_1(t_0) > \beta(t_0)$.

a) Consider the case $x_2(t_0) \ge \beta'(t_0)$. Let us show that for all $t \in (t_0, 1]$,

$$x_1(t) > \beta(t) \quad \text{and} \quad x_2(t) > \beta'(t). \tag{17}$$

If it were not the case, two behaviors are possible.

First, there might exist $t_1 \in (t_0, 1]$ such that for all $t \in (t_0, t_1)$, $x_1(t) > \beta(t)$, $x_2(t) \ge \beta'(t)$ and $x_1(t_1) = \beta(t_1)$. From (13) we deduce

$$x_1'(t) - \beta(t) = \delta(u(t), x_2(t), v(t)) - \beta'(t) \ge 0.$$

Hence the difference $x_1(t) - \beta(t)$ is increasing on (t_0, t_1) and must be positive at t_1.

Second, there might exist $t_1 \in (t_0, 1]$ such that for all $t \in (t_0, t_1)$

$$x_2(t) \ge \beta'(t), \qquad x_1(t_1) > \beta(t_1) \quad \text{and} \quad x_2(t_1) = \beta'(t_1).$$

Then $x_2'(t_1) \le \beta''(t_1)$. On the other hand, from (13) and (A1) (ii) it follows that

$$\begin{aligned} x_2'(t_1) - \beta''(t_1) = \delta(u'(t_1), x_3(t_1), v'(t_1)) \\ + \delta(0, x_1(t_1) - \beta(t_1), 1) - \beta''(t_1) > 0. \end{aligned}$$

Thus we have shown that (17) is true and, specifically,

$$x_1(1) > \beta(1), \qquad x_2(1) > \beta'(1). \tag{18}$$

From (11) one gets

$$x_1(1) - \beta(1) + x_2(1) - X_2(1) = h_0(\beta(1), X_2(1), X_3(1), X_4(1)). \tag{19}$$

It follows from (A3) (ii) that if $x_2(1) > \beta'(1)$ holds then the right hand side of (19) is not positive, which contradicts (18).

b) In the case of $x_2(t_0) < \beta'(t_0)$ we can show that $x_1(0) > \beta(0)$, $x_2(0) < \beta'(0)$ and the first of boundary conditions (14) cannot be satisfied.

Thus we have shown that in fact for all $t \in (0,1)$, $x_1(t) \le \beta(t)$. The estimation from below can be carried out in a similar manner. □

Lemma 2.3. *For any solution of the problem* (13)–(15) *the following is true for all* $t \in I$:

$$u(t) \le x_2(t) \le v(t).$$

Proof. Assume that $x_2(t_0) > v(t_0)$ for some $t_0 \in (0,1)$. Then $x_2(0) > v(0)$. If this were not the case, there exists $t_1 \in [0, t_0)$ such that $x_2(t_1) > v(t_1)$, $x_2'(t_1) > v'(t_1)$. On the other hand, from (13) we obtain

$$x_2'(t_1) - v'(t_1) = \delta(u'(t_1), x_3(t_1), v'(t_1)) - v'(t_1) \le 0.$$

The contradiction shows that in fact $x_2(0) > v(0)$. From (14) one gets

$$0 > v(0) - x_2(0) = g_0(X_1(0), v(0), X_3(0), X_4(0)). \tag{20}$$

By assumption (A3) (i) the right hand side of (20) is not negative.

The contradiction proves that $x_2(t) \le v(t)$ for any $t \in I$.

A similar proof shows that for any $t \in I$, $x_2(t) \ge u(t)$. □

Lemma 2.4. *For any solution of the problem* (13)–(15) *the following is true for all* $t \in [a,b]$:

$$u'(t) \le x_3(t) \le v'(t).$$

Proof. The arguments are essentially the same as in the proof of Lemma 2.2. Suppose that $x_3(t_0) > v'(t_0)$ for some $t_0 \in (0,1)$.

There are two cases to consider, depending on whether $x_4(t_0) \ge v''(t_0)$ or $x_4(t_0) < v''(t_0)$.

a) Let us show that if the former inequality above holds then

$$x_3(1) > v'(1) \quad \text{and} \quad x_4(1) > v''(1). \tag{21}$$

If it were not the case, two behaviors are possible.

First, there might exist $t_1 \in (t_0, 1]$ such that for all $t \in (t_0, t_1)$

$$x_3(t) > v'(t), \qquad x_4(t) \geq v''(t) \quad \text{and} \quad x_3(t_1) = v'(t_1). \tag{22}$$

As in the proof of Lemma 2.2, one may show that $x_3'(t) - v''(t) \geq 0$ for any $t \in (t_0, t_1)$. Hence, the latter equality in (22) is impossible.

Secondly, it may occur that for all $t \in (t_0, t_1)$, $x_4(t) > v''(t)$, $x_3(t_1) > v'(t_1)$ and $x_4(t_1) = v''(t_1)$. Then $x_4'(t_1) \leq v'''(t_1)$. On the other hand, from (13) and (A1) (iii) we deduce $x_4'(t_1) - v'''(t_1) = f(t_1, X_1(t_1), X_2(t_1), v'(t_1), X_4(t_1)) + \delta(0, x_3(t_1) - v'(t_1), 1) - v'''(t_1) > 0$, which contradicts the previous inequality. This shows that in fact (21) holds.

It follows from (15) that

$$x_3(1) - v'(1) + x_4(1) - X_4(1) = h_1(X_1(1), X_2(1), v'(1), X_4(1)). \tag{23}$$

From (A3) (iv) and the second inequality in (21), we deduce that the right hand side in (23) is not positive, which contradicts (21) since the left hand side in (23) must be greater than zero.

b) The case of $x_4(t_0) < v''(t_0)$ may be treated analogously, and the conclusion we are led to is that $x_3(0) > v'(0)$, $x_4(0) < v''(0)$. Then it may be shown that the first boundary condition in (15) fails to be satisfied.

Hence, it is proven that for all $t \in (0, 1)$

$$x_3(t) \leq v'(t).$$

A similar proof shows the boundedness of $x_3(t)$ from below. □

Lemma 2.5. *For any solution of the problem* (13)–(15) *the following is true for all* $t \in I$:

$$|x_4(t)| \leq N.$$

Proof. By standard arguments (see [**5**, Chapter I, 1.4]) treating the equation (1) as the second order equation with respect to x'', making use of the boundedness of x'' (x_3) and noting that (A2) (i) holds uniformly in (t, x, y).

The proof of Theorem 2.1 is now a simple consequence of Lemmas 2.1–2.5. For functions $x = (x_1, x_2, x_3, x_4)$ satisfying the inequalities in assertions of Lemmas 2.2–2.5, the modified system (13)–(15) reduces to (1), (6), (7) and Theorem 2.1 follows.

Consider now equation (1) with the following boundary conditions having physical meaning:

(24a)
$$x(0) = A_0, \qquad x''(0) = A_2, \qquad x(1) = B_0, \qquad x''(1) = B_2,$$

(24b)
$$x'(0) = A_1, \qquad x'''(0) = A_3, \qquad x'(1) = B_1, \qquad x'''(1) = B_3,$$

(24c)
$$\begin{gathered} x(0) = A_0, \qquad x'''(0) = px''(0), \\ x(1) = B_0, \qquad x'''(1) = qx''(0), \\ p, q \geq 0, \ p + q > 0 \end{gathered}$$

Consider also the following set of sign conditions:

(25a)
$$\begin{aligned} \alpha(0) = A_0 = \beta(0), \qquad & \alpha(1) \leq B_0 \leq \beta(1), \\ u'(0) \leq A_2 \leq v'(0), \qquad & u'(1) \leq B_2 \leq v'(1) \end{aligned}$$

(25b)
$$\begin{aligned} \alpha'(0) \geq A_1 \geq \beta'(0), \qquad & \alpha'(1) \leq B_1 \leq \beta'(1), \\ u''(0) \geq A_3 \geq v''(0), \qquad & u''(1) \leq B_3 \leq v''(1) \end{aligned}$$

(25c)
$$\begin{aligned} \alpha(0) = A_0 = \beta(0), \qquad & \alpha(1) \leq \beta_0 \leq \beta(1), \\ pu'(0) \leq u''(0), \qquad & pv'(0) \geq v''(0), \\ pv'(0) \geq -N, \qquad & pu'(0) \leq N \\ -qu'(1) \geq u'(1), \qquad & -qv'(1) \leq v''(1), \\ -qv'(1) \geq -N, \qquad & -qu'(1) \leq N. \end{aligned}$$

Corollary 2.1. *The problems* (1) *and* (24a,b,c) *are solvable if in addition to* (A1) *and* (A2) *the condition* (25a,b,c) *holds.*

Proof. By verifying that sign conditions (A3) are fulfilled for the boundary constraints (24a,b,c) if (25a,b,c) holds. □

Corollary 2.2. *Let the following hypotheses hold.*

(B1) *There exist functions* $\alpha, \beta \in C^4(I, \mathbf{R})$ *such that*

(i) *for all* $t \in I$, $\alpha(t) \leq \beta(t)$, $\beta'(t) \leq \alpha'(t)$, $\beta''(t) \leq \alpha''(t)$;

(ii) *for all* $(t, x, y) \in I \times [\alpha, \beta] \times [\beta', \alpha']$

$$\alpha^{(4)}(t) \leq f(t, x, y, \alpha''(t), \alpha'''(t)),$$
$$\beta^{(4)}(t) \geq f(t, x, y, \beta''(t), \beta'''(t));$$

(B2) ((B3)) The conditions (A2) ((A3)) hold with u and v replaced by β', α' respectively.

Then the assertion of Theorem 2.1 is valid.

Remark 2.1. The following observations may be of value when constructing functions α, β for problems of the type (1), (6), (7). Suppose $f = \varphi(t, x, x', x'', x''') + \psi(t, x'', x''')$, where $|\varphi|$ is bounded by a positive constant M and there exist functions λ, μ such that $\lambda(t) \leq \mu(t)$ and $\lambda''(t) \geq \psi(t, \lambda(t), \lambda'(t)) + M$, $\mu''(t) \leq \psi(t, \mu(t), \mu'(t)) - M$ for any t in [0,1]. Then for α and β such that $\alpha'' = \mu$, $\beta'' = \lambda$, the inequalities (B1)(ii) hold uniformly on t, x and y.

In case f in (1) satisfies the inequalities $f \geq f_1$ and $f \leq f_2$ in some appropriate (t, x)-domain solutions α and β of the equations $x^{(4)} = f_1$ and $x^{(4)} = f_2$, respectively, also satisfies the condition (B1)(ii).

Remark 2.2. The conditions of Theorem 2.1 are also necessary in the class of functions $f(t, x, y, z, w)$ quadratic with respect to w since then the condition (A2) is always met for any choice of α and β. To prove the necessity, one might choose a solution $x(t)$ as both α and β.

Remark 2.3. The sign conditions (B3) in the case of the boundary constraints

$$(26)\qquad x(0)=A_0,\qquad x'(0)=A_1,\qquad x(1)=B_0,\qquad x'(1)=B_1$$

take the form

$$\begin{aligned}\alpha(0)=A_0=\beta(0),&\qquad \alpha(1)\le B_0\le\beta(1),\\ \alpha'(0)=A_1=\beta'(0),&\qquad \alpha'(1)=B_1=\beta'(1).\end{aligned}$$

Then it readily follows from (B1)(i) ($\beta'(t)\le\alpha'(t)$) that $\alpha(t)=\beta(t)$ on I. In the works [**7**] and [**10**] less trivial results on problems of the type (1) and (24) employing α- and β-like functions may be found.

Corollary 2.3. *The problem* (1),

$$(27)\quad x'(0)=A_1,\qquad x''(0)=A_2,\qquad x(1)=B_0,\qquad x'''(1)=B_3,$$

is solvable if in addition to (B1) *and* (B2) *the following sign conditions hold*:

$$\begin{aligned}\alpha'(0)\ge A_1\ge\beta'(0),&\qquad \alpha(1)\le B_0\le\beta(1),\\ \beta''(0)\le A_2\le\alpha''(0),&\qquad \beta'''(1)\le B_3\le\alpha'''(1).\end{aligned}$$

Remark 2.4. The problems (1) and (27) were considered among others in the work [**1**] where it was shown that they are solvable provided f is bounded. Let us construct α and β for this problem.

Suppose f is bounded in modulus by a number $K>0$. Take $\beta(t)=-\alpha(t)=(1/24)Kt^4-(1/6)(K+|B_3|)t^3-(1/2)|A_2|t^2-|A_1|t+C$, where C is defined by the equality $\beta(1)=|B_0|$. An easy computation shows that $\beta''<0$ in (0,1). Then $\beta'<0$ also in [0,1) since $\beta'(0)=-|A_1|$. Since $\beta(1)=|B_0|\ge 0$ and β is decreasing, β is positive for $t\in[0,1)$. By verifying that $\beta(0)=|B_0|$, $\beta'(1)=-|A_1|$, $\beta''(0)\le-|A_2|$, $\beta'''(1)=-|B_3|$, the construction of α and β satisfying the conditions of Corollary 2.2 is completed.

Consider (1) with the boundary conditions

$$(28)\qquad x'(0)=r_0,\qquad x(1)=r_1,\qquad x'''(0)=r_2,\qquad x'''(1)=r_3,$$

where $r_i = r_i(x(j), x'(j), x''(j), x'''(j))$ are odd continuous functions and $j(i) = 0$ for i even, $j(i) = 1$ for i odd.

Corollary 2.4. *Suppose α and β exist as in* (B1)*; the condition* (B2) *is fulfilled and the inequalities*

$$\begin{aligned} \beta'(0) &\le \min r_0, \qquad \max r_0 \le \alpha'(0),\\ \alpha(1) &\le \min r_1, \qquad \max r_1 \le \beta(1),\\ \alpha'''(0) &\le \min r_2, \qquad \max r_2 \le \beta'''(0),\\ \beta'''(1) &\le \min r_3, \qquad \max r_3 \le \alpha'''(1), \end{aligned} \tag{29}$$

hold, where maxima and minima are taken over the set

$$\begin{aligned} T = {} & [\alpha(j), \beta(j)] \times [\beta'(j), \alpha'(j)] \\ & \times [\beta''(j), \alpha''(j)] \times [-N, N], \qquad j = j(i). \end{aligned}$$

Then solutions to the problems (1) *and* (29) *exist.*

Proof. By straightforward application of Corollary 2.2. □

In order to state the next assertion, consider the following sign conditions:

(C3) The functions g_0, h_0, g_1, h_1 in (6) and (7) satisfy the following sign conditions:

(i) For all $(y, z, w) \in [\alpha'(0), \beta'(0)] \times [\beta''(0), \alpha''(0)] \times [-N, N]$

$$g_0(\alpha(0), y, z, w) \ge 0, \qquad g_0(\beta(0), y, z, w) \le 0;$$

(ii) For all $(x, z, w) \in [\alpha(1), \beta(1)] \times [\beta''(1), \alpha''(1)] \times [-N, N]$

$$h_0(x, \alpha'(1), z, w) \ge 0, \qquad h_0(x, \beta'(1), z, w) \le 0;$$

(iii) For all $(x, y) \in [\alpha(0), \beta(0)] \times [\alpha'(0), \beta'(0)]$

$$\begin{aligned} g_1(x, y, \alpha''(0), w) \ge 0 \qquad & \text{if } \alpha'''(0) \le w \le N,\\ g_1(x, y, \beta''(0), w) \le 0 \qquad & \text{if } -N \le w \le \beta'''(0); \end{aligned}$$

(iv) For all $(x,y) \in [\alpha(1),\beta(1)] \times [\alpha'(1),\beta'(1)]$

$$\begin{aligned} h_1(x,y,\alpha''(1),w) \geq 0 \qquad & \text{if } -N \leq w \leq \alpha'''(1), \\ h_1(x,y,\beta''(1),w) \leq 0 \qquad & \text{if } \beta'''(0) \leq w \leq N. \end{aligned}$$

Corollary 2.5. *Let the following assumptions be satisfied*:

(C1) *There exist functions* $\alpha,\beta \in C^4(I,\mathbf{R})$ *such that*

(i) *For all* $t \in I$, $\alpha(t) \leq \beta(t)$, $\alpha'(t) \leq \beta'(t)$, $\beta''(t) \leq \alpha''(t)$,

(ii) *for all* $(t,x,y) \in I \times [\alpha,\beta] \times [\alpha',\beta']$

$$\begin{aligned} \alpha^{(4)}(t) &\leq f(t,x,y,\alpha''(t),\alpha'''(t)), \\ \beta^{(4)}(t) &\geq f(t,x,y,\beta''(t),\beta'''(t)); \end{aligned}$$

(C2) *the conditions* (A2) *hold with* (u,v), (u',v') *replaced by* (α',β'), (β'',α''), *respectively.*

Then if, in addition, sign conditions (C3) *hold, solutions to the problems* (1), (6) *and* (7) *exist.*

Proof. By variable change t to $1-t$ and application of Corollary 2.2. □

The next result relates to that of M.Čverčko [**9**] (see also W. Kelley [**17**] for nth order version).

Consider the folllowing hypotheses:

(D1) *There exist fucntions* $\alpha,\beta \in C^4(I,\mathbf{R})$ *such that*

(i) *for all* $t \in I$, $\alpha(t) \leq \beta(t)$, $\alpha'(t) \leq \beta'(t)$, $\alpha''(t) \leq \beta''(t)$;

(ii) *for all* $(t,x,y) \in I \times [\alpha,\beta] \times [\alpha',\beta']$

$$\begin{aligned} \alpha^{(4)}(t) &\geq f(t,x,y,\alpha''(t),\alpha'''(t)), \\ \beta^{(4)}(t) &\leq f(t,x,y,\beta''(t),\beta'''(t)); \end{aligned}$$

(D2) *the condition* (A2) *holds with* u *and* v *replaced by* α' *and* β', *respectively.*

(D3) *The functions* g_0, h_0, g_1, h_1 *in (8) and (9) satisfy the following sign conditions*:

(i) *For all* $(y,z,w) \in [\alpha'(0), \beta'(0)] \times [\alpha''(0), \beta''(0)] \times [-N, N]$

$$g_0(\alpha(0), y, z, w) \geq 0, \qquad g_0(\beta(0), y, z, w) \leq 0;$$

(ii) *for all* $(x,z,w) \in [\alpha(0), \beta(0)] \times [\alpha''(0), \beta''(0)] \times [-N, N]$

$$g_1(x, \alpha'(0), z, w) \geq 0, \qquad g_1(x, \beta'(0), z, w) \leq 0;$$

(iii) *for all* $(x,y) \in [\alpha(0), \beta(0)] \times [\alpha'(0), \beta'(0)]$

$$\begin{aligned} &g_2(x, y, \alpha''(0), w) \geq 0 \qquad \text{if } \alpha'''(0) \leq w \leq N, \\ &g_2(x, y, \beta''(0), w) \leq 0 \qquad \text{if } -N \leq w \leq \beta'''(0); \end{aligned}$$

(iv) *for all* $(x,y) \in [\alpha(1), \beta(1)] \times [\alpha'(1), \beta'(1)]$

$$\begin{aligned} &h(x, y, \alpha''(1), w) \geq 0 \qquad \text{if } -N \leq w \leq \alpha'''(1), \\ &h(x, y, \beta''(1), w) \leq 0 \qquad \text{if } \beta'''(1) \leq w \leq N. \end{aligned}$$

Theorem 2.2. *Suppose assumptions* (D1)–(D3) *are satisfied. Then the boundary value problems* (1), (8) *and* (9) *have a solution* $x(t)$ *such that for any* $t \in I$,

$$\alpha \leq x \leq \beta, \qquad \alpha' \leq x' \leq \beta', \qquad \alpha'' \leq x'' \leq \beta'', \qquad -N \leq x''' \leq N. \tag{30}$$

Proof. Consider the system

(31)

$$\begin{aligned} x_1' &= X_2 - \delta(0, x_1 - \beta(t), 1) + \delta(0, \alpha(t) - x_1, 1), \\ x_2' &= X_3 - \delta(0, x_2 - \beta'(t), 1) + \delta(0, \alpha'(t) - x_2, 1), \\ x_3' &= X_4 - \delta(0, x_3 - \beta''(t), 1) + \delta(0, \alpha''(t) - x_3, 1), \\ x_4' &= f(t, X_1, X_2, X_3, X_4) + \delta(0, x_3 - \beta''(t), 1) - \delta(0, \alpha''(t) - x_3, 1) \end{aligned}$$

where X_i have the same meaning as in (13) with u and v replaced by α' and β', respectively, together with boundary conditions

$$\begin{aligned} x_1(0) &= G_0(x(0)) + \delta(\alpha(0), x_1(0), \beta(0)), \\ x_2(0) &= G_1(x(0)) + \delta(\alpha'(0), x_2(0), \beta'(0)), \\ x_3(0) &= G_2(x(0)) + \delta(\alpha''(0), x_3(0), \beta''(0)), \\ x_4(1) &= H(x(1)) + \delta(-N, x_4(1), N), \end{aligned} \tag{32}$$

where G_i and H are defined similar to (16).

Since the right hand sides of (31) and (32) are bounded and the corresponding homogeneous BVP has only the trivial solution, to prove the theorem one must show (30).

Suppose, for example, that $x_1(t_0) > \beta(t_0)$ for some $t_0 \in (0,1]$. Then there exists a $t_1 \in (a, t_0]$ such that $x_1(t_1) > \beta(t_1)$ and $x_1'(t_1) > \beta'(t_1)$. On the other hand,

$$x_1'(t_1) - \beta'(t_1) = X_2 - \delta(0, x_1(t_1) - \beta(t_1), 1) - \beta'(t_1) < 0.$$

Analogously, one can show that $x_1(t) \geq \alpha(t)$ for any $t \in (0,1]$.

In a similar manner estimates for x_2 and x_3 can be proved. The boundedness of x_4 is derived from the Nagumo type condition (D2) in a standard way. Hence, the proof. □

For Čverčko-Kelley's boundary conditions

$$(33) \qquad x(0) = A_0, \qquad x'(0) = A_1, \qquad x''(0) = A_2, \qquad x''(1) = B_2$$

one gets

Corollary 2.6. *The problems* (1) *and* (33) *are solvable if, in addition to* (D1) *and* (D2)*, the following conditions hold*:

$$\alpha(0) \leq A_0 \leq \beta(0), \qquad \alpha'(0) \leq A_1 \leq \beta'(1),$$
$$\alpha''(0) \leq A_2 \leq \beta''(0), \qquad \alpha''(1) \leq B_2 \leq \beta''(1).$$

Remark 2.5. Theorem 2.2 remains valid if the Nagumo type condition (D2) is substituted by either (D2′): the condition 2 of Theorem 1 in [**9**] (extendability of solutions to $[0,1]$) or (D″): the condition $H([0,1])$ in [**17**] (for any solution x defined on $[0,1]$ the boundedness of x'' implies the boundedness of x'''). The constants N and $-N$ in (D3)(iii) and (D3)(iv) in that case must be replaced by $+\infty$ and $-\infty$, respectively.

Corollary 2.6 with (D2) replaced by either (D2′) or (D2″) is exactly the result of Čverčko [**9**] on the boundary conditions (33) or the four-dimensional version of the result by Kelley [**17**], respectively.

Remark 2.6. In the work [**6**] the generalization of the result by Kelley [**17**] is presented. In the four dimensional version of the result in [**6**] the function h in (9) may be dependent on the values of unknown functions at both ends of the interval I, although g_0 and g_1 in (8) have the specific form $g_0 = x(0) - A_0$, $g_1 = x'(0) - A_1$.

3. A system of two second order equations. In this section we consider the boundary value problems (2), (10) and (11) with the following assumptions.

(E1) There exist functions $\alpha_i, \beta_i \in C^2(I, R)$, $i = 1, 2$, and numbers N_1 and N_2 such that

(i) for all $t \in I$, $i = 1, 2$, $\alpha_i(t) \le \beta_i(t)$, $-N_i \le \alpha_i'(t) \le N_i$, $-N_i \le \beta_i'(t) \le N_i$;

(ii) for all $(t, y, y') \in I \times [\alpha_2, \beta_2] \times [-N_2, N_2]$

$$\beta_1''(t) \le f_1(t, \beta_1(t), y, \beta_1'(t), y'),$$
$$\alpha_1''(t) \ge f_1(t, \alpha_1(t), y, \alpha_1'(t), y');$$

(iii) for all $(t, x, x') \in I \times [\alpha_1, \beta_1] \times [-N_1, N_1]$

$$\beta_2''(t) \le f_2(t, x, \beta_2(t), x', \beta_2'(t)),$$
$$\alpha_2''(t) \ge f_2(t, x, \alpha_2(t), x', \alpha_2'(t)).$$

(E2) There exist continuous functions $\varphi_i : [\lambda_i, +\infty) \to (0, +\infty)$ such that

(i) for all $(t, x, y, x', y') \in I \times [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \{x' : \lambda_1 \le |x'| \le N_1\} \times \{y' : \lambda_2 \le |y'| \le N_2\}$,

$$|f_1(t, x, y, x', y')| \le \varphi_1(|x'|),$$
$$|f_2(t, x, y, x', y')| \le \varphi_2(|y'|);$$

(ii)

$$\int_{\lambda_i}^{N_i} \frac{s\,ds}{\varphi_i(s)} > \max_I \beta_i(t) - \min_I \alpha_i(t),$$

$$\lambda_i = \max\{|\beta_i(0) - \alpha_i(1)|,\ |\beta_i(1) - \alpha_i(0)|\}, \qquad i = 1, 2;$$

(E3) The functions g_i and h_i in (10) and (11) satisfy the following sign conditions:

(i) for all $(y, y') \in [\alpha_2(0), \beta_2(0)] \times [-N_2, N_2]$

$$\begin{aligned} &g_0(\alpha_1(0), y, x', y') \geq 0 \qquad \text{if } \alpha_1'(0) \leq x' \leq N_1, \\ &g_0(\beta_1(0), y, x', y') \leq 0 \qquad \text{if } -N \leq x' \leq \beta_1'(0); \end{aligned}$$

(ii) for all $(y, y') \in [\alpha_2(1), \beta_2(1)] \times [-N_2, N_2]$,

$$\begin{aligned} &h_0(\alpha_1(1), y, x', y') \geq 0 \qquad \text{if } -N_1 \leq x' \leq \alpha_1'(1), \\ &h_0(\beta_1(0), y, x', y') \leq 0 \qquad \text{if } \beta_1'(1) \leq x' \leq N_1; \end{aligned}$$

(iii) for all $(x, x') \in [\alpha_1(0), \beta_1(0)] \times [-N_1, N_1]$

$$\begin{aligned} &g_1(x, \alpha_2(0), x', y') \geq 0 \qquad \text{if } \alpha_2'(0) \leq y' \leq N_2, \\ &g_1(x, \beta_2(0), x', y') \leq 0 \qquad \text{if } -N_2 \leq y' \leq \beta_2'(0); \end{aligned}$$

(iv) for all $(x, x') \in [\alpha_1(1), \beta_1(1)] \times [-N_1, N_1]$,

$$\begin{aligned} &h_1(x, \alpha_2(1), x', y') \geq 0 \qquad \text{if } -N_2 \leq y' \leq \alpha_2'(1), \\ &h_1(x, \beta_2(1), x', y') \leq 0 \qquad \text{if } \beta_2'(1) \leq y' \leq N_2. \end{aligned}$$

Theorem 3.1. *Suppose assumptions* (E1)–(E3) *are satisfied. Then the boundary value problems* (2), (10) *and* (11) *have a solution* (x, y) *such that, for any* $t \in I$,

(34) $$\alpha_1(t) \leq x(t) \leq \beta_1(t), \qquad \alpha_2(t) \leq y(t) \leq \beta_2(t),$$

(35) $$-N_1 \leq x'(t) \leq N_1, \qquad -N_2 \leq y'(t) \leq N_2.$$

The modified boundary value problem to consider is now

(36) $$\begin{aligned} x'' &= f_1(t, X, Y, X', Y') + \delta(0, x - \beta_1(t), 1) - \delta(0, \alpha_1(t) - x, 1), \\ y'' &= f_2(t, X, Y, X', Y') + \delta(0, y - \beta_2(t), 1) - \delta(0, \alpha_2(t) - y, 1), \end{aligned}$$

where

$$X = \delta(\alpha_1(t), x, \beta_1(t)), \qquad X' = \delta(-N_1, x', N_1)$$
$$Y = \delta(\alpha_2(t), y, \beta_2(t)), \qquad Y' = \delta(-N_2, y', N_2),$$

$$(37) \qquad \begin{aligned} x(0) - x'(0) &= G_0(x(0), y(0)) + X(0) - X'(0), \\ x(1) + x'(1) &= H_0(x(1), y(1)) + X(1) + X'(1), \end{aligned}$$

$$(38) \qquad \begin{aligned} y(0) - y'(0) &= G_1(x(0), y(0)) + Y(0) - Y'(0), \\ y(1) + y'(1) &= H_1(x(1), y(1)) + Y(1) + Y'(1), \end{aligned}$$

where $G_i(x,y) = g_i(X, Y, X', Y')$, $H_i(x,y) = h_i(X, Y, X', Y')$, $i = 1, 2$.

Lemma 3.1. *The boundary value problems* (36)–(38) *have a solution.*

Proof. The right hand sides of (36)–(38) are bounded continuous functions. Hence, the proof follows from the fact that the homogeneous boundary value problem $x'' = 0$, $y'' = 0$, $x(0) - x'(0) = 0$, $x(1) + x'(1) = 0$, $y(0) - y'(0) = 0$, $y(1) + y'(1) = 0$ has only the trivial solution. □

Lemma 3.2. *For any solution* (x, y) *of the problem* (36)–(38) *the inequalities* (34) *hold.*

Proof. Sketch. Essentially by the same arguments as in the proofs of Lemmas 2.2–2.4. First consider the case $x(t_0) > \beta_1(t_0)$ at some $t_0 \in (0,1)$. Then it might be shown that $x(1) > \beta_1(1)$, $x_1'(1) > \beta_1'(1)$ if $x'(t_0) \geq \beta_1'(t_0)$ and $x(0) > \beta_1(0)$, $x'(0) < \beta_1'(0)$ if $x'(t_0) < \beta_1'(t_0)$. Both possibilities are ruled out by assumptions (E3) (ii) or (E3) (i) since the boundary conditions (37) fail to be satisfied.

Analogously, one can treat the case of $x(t_0) < \alpha_1(t_0)$.

The estimation of $y(t)$ may be given in a similar manner. □

Lemma 3.3. *For any solution* (x, y) *of the problem* (36)–(38) *the inequalities* (35) *hold.*

Proof. By standard arguments, employing the conditions (E2) and boundedness of (x, y).

In view of (34) and (35), a solution (x,y) of a modified problem (36)–(38) is also a solution to the boundary problems (2), (10) and (11). □

Hence the proof of Theorem 3.1 is complete. □

Remark 3.1. Theorem 3.1 can be easily formulated for the case of a system of n second order equations.

In the results of [**15, 6**] and [**12**] vector α and β functions are used to get a priori bounds for a solution, but a Nagumo type condition is substituted by the assumption of existence of the so-called bounding functions. However, the application of bounding functions in the case of boundary conditions of the type (40) require $\alpha = \beta$ at one of the ends of the interval I. For discussion on the interrelation of Nagumo condition and bounding functions in the case of scalar second order equations, we refer the interested reader to [**13**].

Corollary 3.1. *Suppose positive numbers M_1, M_2, N_1, N_2 exist such that*

(F1) *for any $t \in I$, $|y| \le M_2$, $|x'| \le N_1$, $|y'| \le N_2$,*

$$xf_1(t,x,y,0,y') \ge 0 \qquad \text{if } |x| \ge M_1,$$

for any $t \in I$, $|x| \le M_1$, $|x'| \le N_1$, $|y'| \le N_2$

$$yf_2(t,x,y,x',0) \ge 0 \qquad \text{if } |y| \ge M_2;$$

(F2) ((F3)) *The condition* (E2) ((E3)) *holds with α_i and β_i replaced by $-M_i$ and M_i, respectively, $i = 1, 2$;*

Then the assertion of Theorem 3.1 *is valid.*

Example. Consider the functional

$$\int_0^1 v(t,x,y)(1 + x'^2 + y'^2)^{1/2}\, dt,$$

often arising in applications (geodesics, geometrical optics, etc., see [**8**] for examples), and the associated Euler-Lagrange system

$$\begin{aligned} x'' &= v^{-1}(v_x - v_t x')(1 + x'^2 + y'^2) \\ y'' &= v^{-1}(v_x - v_t y')(1 + x'^2 + y'^2), \end{aligned} \tag{39}$$

where v is supposed to be a positive valued continuous function.

Suppose that a positive constant M exists such that

$$\begin{aligned} xv_x(t,x,y) &\geq 0 \quad \text{if } |x| \geq M, \quad t \in I, \quad |y| \leq M \\ yv_y(t,x,y) &\geq 0 \quad \text{if } |y| \geq M, \quad t \in I, \quad |x| \leq M. \end{aligned}$$

Let

$$W = \max\{V(t,x,y) : t \in I, |x| \leq M, |y| \leq M\},$$

where $V(t,x,y) = \max\{|v_x v^{-1}|, |v_y v^{-1}|, |v_t v^{-1}|\}$. Obviously, the right sides of (39) are majorized by the functions $W(1+|x'|)(1+N^2+x'^2)$ and $W(1+|y'|)(1+N^2+y'^2)$, respectively, where N is a positive constant to define. Then the condition (F2) is fulfilled if a number N exists such that

$$W^{-1} \int_{2M}^{N} \frac{s\,ds}{(1+s)(1+N^2+s^2)} > 2M.$$

This is the case if either M or W is sufficiently small. Then, for instance, the boundary value problems (39) and (40) are solvable if, in addition, numbers A_i and B_i in the conditions

$$x(0) = A_0, \qquad y(0) = A_1, \qquad x(1) = B_0, \qquad y(1) = B_1 \tag{40}$$

are such that $|A_i| \leq M$ and $|B_i| \leq M$.

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Appendix

Consider the problem

$$y^{(4)} = a[y'' + b(y'y''' + y''^2)], \tag{A1}$$

$$\text{(A2)} \qquad \begin{aligned} y(0) &= \mu z(0), & y'(0) &= \nu z(0), \\ y(1) &= A - \gamma z(1), & y'(1) &= \delta z(1), \end{aligned}$$

where

$$\text{(A3)} \qquad z = F y''' - B y'(C + D y'')$$

and $a, b, \mu, \nu, \gamma, \delta, A, B, C, D, F$ are positive numbers.

This problem arises in the theory of transport phenomena of amorpheus semiconductors [**26**]. A solution $y(t)$ of the problem (A1), (A2), represents potential and $z(t)$ defined in (A3) means current in a semiconductor. Results on the existence of a solution to this problem were obtained in [**26**] by reducing this problem to a second order one making use of the standing assumption

$$\text{(A4)} \qquad Fab - BD = 0.$$

The case of $Fab \neq BD$ is referred to in [**26**] as complicated. In order to apply Corollary 2.4 to the problem above, rewrite the boundary conditions (A2) in the form

$$\text{(A5)} \qquad \begin{aligned} y(1) = A - \gamma z(1) = r_1, \qquad y'(0) = \nu z(0) = r_0, \\ y'''(0) = (1/F)[(1/\mu) y(0) + B y'(0)(C + D y''(0))] = r_2, \\ y'''(1) = (1/F)[(1/\mu) y(1) + B y'(1)(C + D y''(1))] = r_3. \end{aligned}$$

Choose

$$\alpha(t) = -\beta(t) = (1/24)t^4 - (1/12)t^3 + (1/2)M_2 t^2 + M_1 t - M_0,$$

where M_i are positive constants. In order for the conditions of Corollary 2.4 to be satisfied, choose M_i such that

$$\begin{aligned} \alpha''(t) &= (1/2)t^2 - (1/2)t + M_2 > 0, \\ \alpha'(t) &= (1/6)t^3 - (1/4)t^2 + M_2 t + M_1 > 0, \\ \alpha(t) &< 0 \end{aligned}$$

hold for any $t \in I$. The inequalities above imply that

$$\begin{aligned} &\min_I \alpha''(t) = M_2 - (1/8), \qquad \min_I \alpha'(t) = M_1 > 0, \\ &\max_I \alpha(t) = -m_0 = (-1/24) + (1/2)M_2 + M_1 + M_0 < 0. \end{aligned}$$

Besides, α must satisfy the inequality

$$\alpha^{(4)} \le a\alpha'' + aby'\alpha''' + ab\alpha''^2$$

for $y' \in [\beta', \alpha']$ and $t \in I$. It will hold if

$$\text{(A6)}\quad 1 \le a(M_2 - (1/8)) - (1/2)ab(M_1 + M_2 - (1/2)) + (M_2 - (1/8))^2.$$

Since $\beta = -\alpha$ and the Nagumo type condition holds with $\varphi(w)$ a linear function, all the hypotheses of Corollary 2.4 are met if additionally the sign conditions

$$\text{(A7)}\quad \begin{array}{ll} -M_1 = \beta'(0) \le \min r_0, & \max r_0 \le \alpha'(0) = M_1, \\ -m_0 = \alpha(1) \le \min r_1, & \max r_1 \le \beta(1) = m_0, \\ -1/2 = \alpha'''(0) \le \min r_2, & \max r_2 \le \beta'''(0) = 1/2, \\ -1/2 = \beta'''(1) \le \min r_3, & \max r_3 \le \alpha'''(1) = 1/2, \end{array}$$

fulfill, where minima and maxima are over the compact set T (see Corollary 2.4) depending on the choice of α and β.

Suppose $a, b, \mu, \gamma, \delta, A, B, C, D$ are given. Choose M_1 and M_2 in order for (A6) to be satisfied. Then $N = N(M_1, M_2)$ in the condition (D2) is defined which bounds $y'''(t)$. Since $r_1 = A - \gamma z(1)$, a constant M_0 can be found sufficiently large in order for the inequalities in the second line of (A7) to be satisfied. From $r_0 = \nu z(0)$ one deduces that ν can be chosen sufficiently small in order the inequalities in the first line of (A7) to be satisfied. Analogously for F large enough the rest of (A7) holds.

We summarize the results obtained in the following

Proposition. *Given $a, b, \mu, \gamma, \delta, A, B, C, D$ positive numbers ε and Δ can be found such that for $\nu < \varepsilon$ and $F > \Delta$ a solution to the problem* (A1), (A2) *exists.*

Note that since A in (A2) is positive a solution to the problem (A1) and (A2) is nontrivial.

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Institute of Mathematics and Computer Science, University of Latvia, Raynis Blvd. 29, LV-1459 Riga, Latvia