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GENERALIZED FEYNMAN INTEGRALS: THE $\mathcal{L}(L_2, L_2)$ THEORY

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ABSTRACT. In this paper we develop an $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$ theory for the Feynman integral of functionals of general stochastic processes.

1. Introduction. In [**1**], Cameron and Storvick introduced a very general analytic operator-valued function space *Feynman integral*, $J_q^{an}(F)$, which mapped an $L_2(\mathbf{R})$ function ψ into an $L_2(\mathbf{R})$ function $(J_q^{an}(F)\psi)(\xi)$. Further work involving the $L_2 \to L_2$ theory, the $L_1 \to L_\infty$ theory and the $L_p \to L_{p'}$ theory, $1/p + 1/p' = 1$, includes [**2, 3, 11, 12, 13**].

In [**9**], Chung and Skoug introduced the concept of a conditional Feynman integral using Yeh's definition of conditional Wiener integrals [**20**]. In [**7**], Chung, Park and Skoug expressed the Feynman integral $J_q^{an}(F) \in \mathcal{L}(L_1(\mathbf{R}), L_\infty(\mathbf{R}))$ in terms of conditional Feynman integrals.

In various Feynman integration theories, the integrand F of the Feynman integral is a functional of the standard Wiener (i.e., Brownian) process. In [**8**], Chung, Park and Skoug defined a Feynman integral for functionals of general stochastic processes. They then used the theory of the conditional Feynman integral to develop an $\mathcal{L}(K(\mathbf{R}), L_\infty(\mathbf{R}))$ theory where

$$K(\mathbf{R}) = \{\psi_1 + \psi_2 : \psi_1 \in L_1(\mathbf{R}) \text{ and } \psi_2 \in \hat{M}(\mathbf{R})\},$$

and where $\hat{M}(\mathbf{R})$ is the space of Fourier transforms of measures from $M(\mathbf{R})$, the space of $\mathbf{C}$-valued countably additive Borel measures on $\mathbf{R}$.

In this paper we develop an $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$ theory for the operator-valued Feynman integral of functionals of general stochastic processes. The $L_2 \to L_2$ theory is more relevant in quantum mechanics and other applications than the $L_1 \to L_\infty$ or the $K(\mathbf{R}) \to L_\infty(\mathbf{R})$ theory.

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Moreover, it is usually more difficult to establish the $L_2 \to L_2$ theory; partly because a summation procedure is needed since ψ need not be in $L_1(\mathbf{R})$.

2. Definitions and preliminaries. Let $C[0,T]$ denote the $\mathbf{R}$-valued continuous functions on $[0,T]$. Let $(C_0[0,T], m)$ denote Wiener space where $C_0[0,T]$ is the set of all functions $x(t)$ in $C[0,T]$ with $x(0) = 0$ and m is the Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $R(s,t) = E[x(s)x(t)] = \min(s,t)$. We denote the Wiener integral of a Wiener measurable function F by

$$E[F] = \int_{C_0[0,T]} F(x)m(dx)$$

whenever the integral exists.

Let h be an element of $L_2[0,T]$ with $||h|| > 0$ and let $Z : C_0[0,T] \times [0,T] \to \mathbf{R}$ be the Gaussian process

$$Z(x,t) = \int_0^t h(s)\,dx(s) \tag{2.1}$$

where $\int_0^t h(s)\,dx(s)$ denotes the Paley-Wiener-Zygmund stochastic integral. Also, let

$$a(t) = \int_0^t h^2(s)\,ds. \tag{2.2}$$

In defining various analytic operator-valued Feynman integrals of F, one starts [**1**, p. 517], for $\lambda > 0$, with the Wiener integral

$$\int_{C_0[0,T]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(T) + \xi)m(dx),$$

and then extends analytically in λ to the right-half complex plane. Our starting point is the Wiener integral

$$\int_{C_0[0,T]} F(\lambda^{-1/2}Z(x,\cdot) + \xi)\psi(\lambda^{-1/2}Z(x,T) + \xi)m(dx).$$

Definition. Let $\mathbf{C}, \mathbf{C}_+$ and $\mathbf{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let h be an element of $L_2[0,T]$ with $||h|| > 0$, and let $Z(x,t)$ be given by (2.1). For each $\lambda > 0$, $\psi \in L_2(\mathbf{R})$ and $\xi \in \mathbf{R}$, assume that $F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)$ is Wiener integrable with respect to x on $C_0(0,T]$, and let

(2.3)
$$\Big(h_{I_\lambda}(F)\psi\Big)(\xi) = \int_{C_0[0,T]} F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)m(dx).$$

If $h_{I_\lambda}(F)\psi$ is in $L_2(\mathbf{R})$ as a function of ξ, and if the correspondence $\psi \to h_{I_\lambda}(F)\psi$ gives an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$, we say the operator-valued space integral $h_{I_\lambda}(F)$ exists. Next suppose there exists an $\mathcal{L}$-valued function which is analytic in λ on $\mathbf{C}_+$ and agrees with $h_{I_\lambda}(F)$ on $(0,+,\infty)$; then this $\mathcal{L}$-valued function is denoted by $h_{I_\lambda^{an}}(F)$ and is called the analytic operator-valued Wiener integral of F associated with λ. For $\lambda = -iq \in \mathbf{C}_+^\sim$, suppose there exists an operator $h_{J_q^{an}}(F)$ in $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$ such that for every ψ in $L_2(\mathbf{R})$,

$$||h_{J_q^{an}}(F)\psi - h_{I_\lambda^{an}}(F)\psi||_2 \to 0 \quad \text{as } \lambda \to iq \text{ through } \mathbf{C}_+, \tag{2.4}$$

then $h_{J_q^{an}}(F)$ is called the generalized analytic operator-valued Feynman integral of F with parameter q.

Note that if $h(t) \equiv 1$ on $[0,T]$, then this definition agrees with the previous definitions of the analytic operator-valued Feynman integral [**1, 11, 13**].

In various integral representations for $h_{J_q^{an}}(F)\psi$, since ψ is not necessarily in $L_1(\mathbf{R})$, the integral is interpreted in the mean as in the theory of the L_2-Fourier transform. We use the notation

$$\int_{\mathbf{R}}^{(\xi)} f(\xi,\eta)\,d\eta = \text{l.i.m.}_{A\to+\infty}\int_{-A}^{A} f(\xi,\eta)\,d\eta$$

which means

$$\lim_{A\to+\infty}\int_{\mathbf{R}}\left|\int_{\mathbf{R}}^{(\xi)} f(\xi,\eta)\,d\eta - \int_{-A}^{A} f(\xi,\eta)\,d\eta\right|^2 d\xi = 0.$$

The following lemma [**1, 11, 13**] plays a key role in this paper.

Lemma 1. *Let s be a positive number. For all $\lambda \in \mathbf{C}_+^\sim$ and $\psi \in L_2(\mathbf{R})$ let*

$$(C_\lambda \psi)(\xi) = \left(\frac{\lambda}{2\pi s}\right)^{1/2} \int_{\mathbf{R}} \psi(\eta) \exp\left\{ -\frac{\lambda(\eta-\xi)^2}{2s}\right\} d\eta. \tag{2.5}$$

Then $C_\lambda\psi$ is in $L_2(\mathbf{R})$ and $||C_\lambda \psi||_2 \le ||\psi||_2$ (when $\operatorname{Re}\lambda = 0$, the integral is interpreted as a limit in the mean.) In addition, $||C_\lambda\psi - C_{-iq}\psi||_2 \to 0$ as $\lambda \to -iq$ through values in $\mathbf{C}_+$ and $||C_\lambda|| = 1$ for all $\lambda \in \mathbf{C}_+^\sim$ [**11**].

The following formula [**18**] for expressing conditional Wiener integrals in terms of ordinary Wiener integrals

$$\begin{aligned} &E(F(Z(x,\cdot)+\xi)|Z(x,T)+\xi = \eta) \\ &\quad = E\left[F\left(Z(x,\cdot)+\xi - \frac{a(\cdot)}{a(T)}Z(x,T) + \frac{a(\cdot)}{a(T)}(\eta-\xi)\right)\right] \end{aligned} \tag{2.6}$$

is used several times in this paper. We also use the well-known formula

$$\left(\frac{b}{2\pi}\right)^{1/2} \int_{\mathbf{R}} \exp\left\{-\frac{bu^2}{2} + iuv\right\} du = \exp\left\{-\frac{v^2}{2b}\right\}, \quad \operatorname{Re} b > 0. \tag{2.7}$$

Finally we note that the results of this paper can easily be extended to ν-dimensional Wiener space $C_0^\nu[0,T]$ for $\nu = 2, 3, \ldots$.

3. The $\mathcal{L}(L_2, L_2)$ theory for F in the Banach algebra S. In [**4**], Cameron and Storvick introduced a Banach algebra S of functionals on $C_0[0,T]$, each of which is a type of a stochastic Fourier transform of a bounded $\mathbf{C}$-valued Borel measure. Further work, including [**5, 6, 14, 15, 16, 17**], shows that S contains many classes of functionals of interest in Feynman integration theory.

The Banach algebra S consists of functions on $C_0[0,T]$ expressible in the form

$$F(x) = \int_{L_2[0,T]} \exp\left\{ i\int_0^T v(s)\,dx(s)\right\} d\sigma(v) \tag{3.1}$$

for s–almost every x in $C_0[0,T]$, that is, except on a scale invariant null set, where σ is an element of $M(L_2[0,T])$, the space of $\mathbf{C}$-valued, countably additive Borel measures on $L_2[0,T]$.

Recall that, for each $g \in L_2[0,T]$, the PWZ integral $\int_0^T g(s)\,dx(s)$ exists for s–almost every $x \in C_0[0,T]$; this result doesn't hold for all $g \in L_1[0,T]$. Thus, in our first theorem we need to require that h belongs to $L_\infty[0,T]$ as well as to $L_2[0,T]$, so that for each $v \in L_2[0,T]$,

$$\int_0^T v(s)\,dZ(x,s) = \int_0^T v(s)h(s)\,dx(s) \tag{3.2}$$

for s, almost every x in $C_0[0,T]$.

Theorem 1. *Let $F \in S$ be given by (3.1), and let $h \in L_\infty[0,T]$. Then, for all real $q\neq 0$, $h_{J_q^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$, and for each $\psi \in L_2(\mathbf{R})$, we have*

$$\begin{aligned}\Big(h_{J_q^{an}}(F)\psi\Big)(\xi) = \int_{L_2[0,T]} \exp\bigg\{ &- \frac{i\xi(v,h^2)}{a(T)} \\ &- \frac{i}{2q}\int_0^T h^2(s)\bigg[v(s) - \frac{(v,h^2)}{a(T)}\bigg]^2 ds\bigg\} \\ &\cdot \bigg[\frac{q}{2\pi i a(T)}\bigg]^{1/2} \int_{\mathbf{R}}^{(\xi)} \exp\bigg\{\frac{i\eta(v,h^2)}{a(T)}\bigg\} \\ &\qquad \cdot \psi(\eta)\exp\bigg\{\frac{iq(\eta-\xi)^2}{2a(T)}\bigg\}\,d\eta\,d\sigma(v)\end{aligned} \tag{3.3}$$

for all $\xi \in \mathbf{R}$.

Proof. Using (3.1), (2.6), (3.2), the Fubini theorem, and a fundamental Wiener integration formula, for all $(\lambda,\xi) \in (0,\infty)\times\mathbf{R}$, we obtain the formula

(3.4)
$$\begin{aligned}(h_{I_\lambda}(F)\psi)(\xi) &= E[F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)] \\ &= \int_{\mathbf{R}} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\end{aligned}$$

$$
\begin{aligned}
&\cdot \psi(\lambda^{-1/2}Z(x,T)+\xi) \mid \lambda^{-1/2}Z(x,T)+\xi=\eta) \\
&\cdot \left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta \\
&= \int_{\mathbf{R}} E\bigg[\int_{L_2[0,T]} \exp\bigg\{i\int_0^T v(s)\, d\bigg[\lambda^{-1/2} \\
&\cdot \left(Z(x,s)-\frac{a(s)}{a(T)}Z(x,T)\right)+\frac{a(s)}{a(T)}(\eta-\xi)\bigg]\bigg\} d\sigma(v)\bigg] \\
&\cdot \psi(\eta)\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta \\
&= \int_{\mathbf{R}}\int_{L_2[0,T]} E\bigg[\exp\bigg\{i\lambda^{-1/2}\int_0^T vh\, dx \\
&\quad -\frac{i\lambda^{-1/2}(v,h^2)}{a(T)}\int_0^T h\, dx+\frac{i(\eta-\xi)}{a(T)}(v,h^2)\bigg\}\bigg] d\sigma(v) \\
&\cdot \psi(\eta)\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta \\
&= \int_{\mathbf{R}}\int_{L_2[0,T]} \exp\bigg\{\frac{i(\eta-\xi)(v,h^2)}{a(T)} \\
&\quad -\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\bigg\} d\sigma(v) \\
&\cdot \psi(\eta)\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta \\
&= \int_{L_2[0,T]} \exp\bigg\{-\frac{i\xi(v,h^2)}{a(T)} \\
&\quad -\frac{1}{2\lambda}\int_0^T h^2(s)\left[v(s)-\frac{(v,h^2)}{a(T)}\right]^2 ds\bigg\}\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \\
&\cdot \int_{\mathbf{R}} \exp\left\{\frac{i\eta(v,h^2)}{a(T)}\right\}\psi(\eta)\exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta\, d\sigma(v).
\end{aligned}
$$

To show that $h_{I_\lambda}(F)$ is an $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$-valued function of λ in $\mathbf{C}_+$, it suffices to fix ψ and ϕ and show that

$$g(\lambda) = (h_{I_\lambda}(F)\psi, \phi)$$

is a scalar-valued analytic function of λ in $\mathbf{C}_+$. Using the Fubini theorem we can write

$$(3.5)\quad g(\lambda) = \int_{L_2[0,T]} \exp\bigg\{ -\frac{1}{2\lambda}\int_0^T h^2(s) \cdot \bigg[v(s) - \frac{(v,h^2)}{a(T)}\bigg]^2 ds\bigg\}\bigg(\frac{\lambda}{2\pi a(T)}\bigg)^{1/2} \cdot \int_{\mathbf{R}} \exp\bigg\{\frac{i\eta(v,h^2)}{a(T)}\bigg\} \cdot \psi(\eta)\int_{\mathbf{R}} \exp\bigg\{ -\frac{i\xi(v,h^2)}{a(T)} - \frac{\lambda(\eta-\xi)^2}{2a(T)}\bigg\}\phi(\xi)\,d\xi\,d\eta\,d\sigma(v).$$

We will use Morera's theorem to show that $g(\lambda)$ is analytic in $\mathbf{C}_+$. First an application of the dominated convergence theorem shows that $g(\lambda)$ is continuous in $\mathbf{C}_+$. Thus, we need only show that $\int_\Gamma g(\lambda)\,d\lambda = 0$ for every closed contour Γ in $\mathbf{C}_+$. But it suffices to show this for closed triangular paths. So, let Γ be a closed triangular path in $\mathbf{C}_+$. Now let $f(v,\eta,\xi,\lambda)$ denote the integrand on the righthand side of (3.5). For all $(v,\eta,\xi) \in L_2[0,T]\times\mathbf{R}\times\mathbf{R}$, $f(v,\eta,\xi,\lambda)$ is an analytic function of λ in $\mathbf{C}_+$ and so, by the Cauchy integral theorem, $\int_\Gamma f(v,\eta,\xi,\lambda)\,d\lambda = 0$ for all $(v,\eta,\xi) \in L_2[0,T]\times\mathbf{R}\times\mathbf{R}$. Let $M = \sup\{|\lambda| : \lambda \in \Gamma\}$, and let $N = \inf\{\operatorname{Re}\lambda : \lambda\in\Gamma\}$. Then N is positive and so

$$\bigg(\frac{M}{2\pi a(T)}\bigg)^{1/2} |\phi(\xi)\psi(\eta)| \exp\bigg\{-\frac{N(\eta-\xi)^2}{2a(T)}\bigg\}$$

is an integrable dominating function for $f(v,\eta,\xi,\lambda)$ on $L_2[0,T]\times\mathbf{R}\times\mathbf{R}\times\Gamma$. Hence, by the Fubini theorem,

$$\int_\Gamma g(\lambda)\,d\lambda = \int_{L_2[0,T]}\int_{\mathbf{R}}\int_{\mathbf{R}}\int_\Gamma f(v,\eta,\xi,\lambda)\,d\lambda\,d\xi\,d\eta\,d\sigma(v) = 0.$$

Hence, $h_{I_\lambda}(F)$ is analytic, and so by Lemma 1 above, $h_{I_\lambda^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$. □

Next, using the dominated convergence theorem for Bochner integrals [**10**, p. 83] and Lemma 1, we see that the generalized analytic operator-valued Feynman integral of F, $h_{J_q^{an}}(F)$ exists and is given by (3.3). In addition, we have that

$$\Big\|h_{J_q^{an}}(F)\psi\Big\|_2 \le ||\sigma||\,||\psi||_2.$$

Remark. Throughout the rest of this paper, we only need require that h be in $L_2[0,T]$ rather than requiring h to be in $L_\infty[0,T]$.

Note that, in Theorem 1, for F in S, we expressed $h_{J_q^{an}}(F)\psi$ in terms of an integral over the infinite dimensional space $L_2[0,T]$. In our next theorem we obtain a series expansion of $h_{J_q^{an}}(F)\psi$ in terms of integrals over finite dimensional spaces.

Theorem 2. *Let $h \in L_2[0,T]$, and let*

$$F(x) = \exp\left\{ \int_0^T \theta(s, x(s))\, ds \right\} \tag{3.6}$$

where $\theta : [0,T] \times \mathbf{R} \to \mathbf{C}$ is given by

$$\theta(t,u) = \int_{\mathbf{R}} \exp\{iu\eta\}\, d\sigma_t(\eta) \tag{3.7}$$

where $\{\sigma_t : 0 < t \le T\}$ is a family from $M(\mathbf{R})$ with $||\sigma_t|| \in L_1[0,T]$, and for each Borel set $B \subseteq \mathbf{R}$, $\sigma_t(B)$ is a Borel measurable function of t. Then, for all real $q \neq 0$, $h_{J_q^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$, and for $\psi \in L_2(\mathbf{R})$, we have

(3.8)

$$\begin{aligned}
(h_{J_q^{an}}(F)\psi)(\xi) = &\int_{\mathbf{R}}^{(\xi)} \left(\frac{q}{2\pi i a(T)}\right)^{1/2} \psi(\eta) \exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\} d\eta \\
&+ \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathbf{R}^n} \exp\left\{ \frac{i\xi}{a(T)} \sum_{j=1}^n w_j [a(T) - a(s_j)] \right. \\
&\left. - \frac{i[a(T)-a(s_n)]}{2qa^2(T)} \left[\sum_{m=1}^n w_m a(s_m) \right]^2 \right\} \\
\cdot \exp\Bigg\{ &- \frac{i}{2q} \sum_{k=1}^n [a(s_k) - a(s_{k-1})] \Bigg[\sum_{m=k}^n w_m \\
&- \sum_{m=1}^n \frac{w_m a(s_m)}{a(T)} \Bigg]^2 \Bigg\} \left(\frac{q}{2\pi i a(T)}\right)^{1/2} \\
\cdot &\int_{\mathbf{R}}^{(\xi)} \exp\left\{ \frac{i\eta}{a(T)} \sum_{j=1}^n w_j a(s_j) \right\} \psi(\eta)
\end{aligned}$$

$$\cdot \exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\} d\eta\, d\sigma_{s_1}(w_1)\cdots d\sigma_{s_n}(w_n)\, d\vec{s}$$

for all $\xi \in \mathbf{R}$ *where* $\Delta_n(T) = \{\vec{s} = (s_1, \dots, s_n)\colon 0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = T\}$.

Proof. Let $H_\lambda(\xi,\eta)$ denote the conditional Wiener integral

$$E(F(\lambda^{-1/2}Z(x,\cdot)+\xi) \mid \lambda^{-1/2}Z(x,T)+\xi=\eta) \quad \text{for } \lambda>0.$$

Then, using the Fubini theorem, (2.6), and a well-known Wiener integration formula for PWZ integrals, we see that for all $(\xi,\eta,\lambda) \in \mathbf{R}\times\mathbf{R}\times(0,+\infty)$,

(3.9)

$$\begin{aligned}
H_\lambda(\xi,\eta) &= E\bigg(\sum_{n=0}^{\infty}\frac{1}{n!}\bigg(\int_0^T \theta(s,\lambda^{-1/2}Z(x,s)+\xi)\,ds\bigg)^n \\
&\qquad\qquad \cdot \mid \lambda^{-1/2}Z(x,T)+\xi=\eta\bigg) \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)} E\bigg[\prod_{j=1}^{n}\theta(s_j,\lambda^{-1/2}Z(x,s_j)+\xi \\
&\qquad\qquad - \frac{a(s_j)}{a(T)}\bigg(\lambda^{-1/2}Z(x,T)+\xi\bigg)+\frac{a(s_j)}{a(T)}\eta\bigg)\bigg]\,d\vec{s} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)} E\bigg[\prod_{j=1}^{n}\theta(s_j,\xi+\lambda^{-1/2}\sum_{k=1}^{j}[a(s_k) \\
&\qquad\qquad - a(s_{k-1})]^{1/2}\int_{s_{k-1}}^{s_k}\frac{h\,dx}{[a(s_k)-a(s_{k-1})]^{1/2}} \\
&\qquad\qquad - \frac{a(s_j)\lambda^{-1/2}}{a(T)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2} \\
&\qquad\qquad \cdot\int_{s_{k-1}}^{s_k}\frac{h\,dx}{[a(s_k)-a(s_{k-1})]^{1/2}}+\frac{a(s_j)}{a(T)}(\eta-\xi)\bigg]\,d\vec{s} \\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^{n+1}}(2\pi)^{-(n+1)/2}
\end{aligned}$$

$$\cdot \exp\left\{ -\frac{1}{2}(u_1^2+\cdots+u_{n+1}^2)\right\}$$
$$\cdot \prod_{j=1}^{n} \theta(s_j, \xi+\lambda^{-1/2}\sum_{k=1}^{j}[a(s_k)-a(s_{k-1})]^{1/2}u_k+\frac{a(s_j)}{a(T)}(\eta-\xi)$$
$$-\frac{a(s_j)\lambda^{-1/2}}{a(T)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2}u_k)\,du_1\cdots du_{n+1}\,d\vec{s}.$$

Next we substitute into the last expression above using (3.7) and then we carry out the integrations with respect to $u_1,\dots,u_{n+1}$ using (2.7), and obtain

(3.10)
$$H_\lambda(\xi,\eta) = 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}(2\pi)^{-(n+1)/2}$$
$$\cdot\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}w_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{a(s_j)}{a(T)}\right\}$$
$$\cdot\int_{\mathbf{R}^{n+1}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}u_j^2+i\lambda^{-1/2}\sum_{j=1}^{n}w_j\right.$$
$$\cdot\left[\sum_{k=1}^{j}[a(s_k)-a(s_{k-1})]^{1/2}u_k\right.$$
$$\left.\left.-\frac{a(s_j)}{a(T)}\sum_{k=1}^{n+1}[a(s_k)-a(s_{k-1})]^{1/2}u_k\right]\right\}$$
$$du_1\cdots du_{n+1}\,d\sigma_{s_1}(w_1)\cdots d\sigma_{s_n}(w_n)\,d\vec{s}$$
$$=1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^n}\exp\left\{i\xi\sum_{j=1}^{n}w_j+i(\eta-\xi)\sum_{j=1}^{n}\frac{w_ja(s_j)}{a(T)}\right\}$$
$$\cdot\exp\left\{-\frac{1}{2\lambda}\left(\sum_{k=1}^{n}[a(s_k)-a(s_{k-1})]\right.\right.$$
$$\cdot\left[\sum_{m=k}^{n}w_m-\sum_{m=1}^{n}\frac{w_ma(s_m)}{a(T)}\right]^2+\frac{[a(T)-a(s_n)]}{a^2(T)}$$

$$\cdot\bigg(\sum_{m=1}^{n} w_m a(s_m)\bigg)^2\bigg)\bigg\}\, d\sigma_{s_1}(w_1)\cdots d\sigma_{s_n}(w_n)\, d\vec{s}.$$

Then, using (3.10), we see that for all $(\lambda,\xi)\in(0,+\infty)\times\mathbf{R}$,

(3.11)

$$\begin{aligned}
\Big(h_{I_\lambda}(F)\psi\Big)(\xi) &= E[F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)]\\
&= \int_{\mathbf{R}} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)\\
&\qquad \cdot\psi(\lambda^{-1/2}Z(x,T)+\xi)\mid \lambda^{-1/2}Z(x,T)+\xi=\eta)\\
&\qquad \cdot\bigg(\frac{\lambda}{2\pi a(T)}\bigg)^{1/2}\exp\bigg\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\bigg\}\,d\eta\\
&= \int_{\mathbf{R}} H_\lambda(\xi,\eta)\bigg(\frac{\lambda}{2\pi a(T)}\bigg)^{1/2}\\
&\qquad \cdot\exp\bigg\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\bigg\}\psi(\eta)\,d\eta\\
&= \bigg(\frac{\lambda}{2\pi a(T)}\bigg)^{1/2}\int_{\mathbf{R}}\psi(\eta)\exp\bigg\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\bigg\}\,d\eta\\
&\quad + \sum_{n=1}^{\infty}\int_{\Delta_n(T)}\int_{\mathbf{R}^n}\exp\bigg\{\frac{i\xi}{a(T)}\sum_{j=1}^{n} w_j[a(T)\\
&\quad - a(s_j)] - \frac{a(T)-a(s_n)}{2\lambda a^2(T)}\bigg[\sum_{m=1}^{n} w_m a(s_m)\bigg]^2\bigg\}\\
&\quad \cdot\exp\bigg\{-\frac{1}{2\lambda}\sum_{k=1}^{n}\bigg([a(s_k)-a(s_{k-1})]\bigg[\sum_{m=k}^{n} w_m\\
&\qquad - \sum_{m=1}^{n}\frac{w_m a(s_m)}{a(T)}\bigg]^2\bigg)\bigg\}\\
&\quad \cdot\bigg\{\frac{\lambda}{2\pi a(T)}\bigg)^{1/2}\int_{\mathbf{R}}\exp\bigg\{\frac{i\eta}{a(T)}\sum_{j=1}^{n} w_j a(s_j)\bigg\}\psi(\eta)\\
&\quad \cdot\exp\bigg\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\bigg\}\,d\eta\, d\sigma_{s_1}(w_1)\cdots d_{\sigma_{s_n}}(w_n)\, d\vec{s}.
\end{aligned}$$

As in the proof of Theorem 1 above, an application of Morera's theorem shows that $h_{I_\lambda}(F)$ is an $\mathcal{L}(L_2(\mathbf{R}),L_2(\mathbf{R}))$-valued analytic function of λ throughout $\mathbf{C}_+$. In addition, by Lemma 1 and the dominated convergence theorem for Bochner integrals [**10**, p. 83], $h_{J_q^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$. In fact, for each $\psi \in L_2(\mathbf{R})$,

$$\begin{aligned}||h_{J_q^{an}}(F)\psi||_2 &\le ||\psi||_2\left[1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}||\sigma_{s_1}||\cdots||\sigma_{s_n}||\,d\vec{s}\right]\\ &= ||\psi||_2\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^T(n)\int_0^T||\sigma_{s_1}||\cdots||\sigma_{s_n}||\,d\vec{s}\right]\\ &= ||\psi||_2\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\int_0^T||\sigma_t||\,dt\right)^n\right]\\ &= ||\psi||_2\exp\left\{\int_0^T||\sigma_t||\,dt\right\}\\ &< \infty,\end{aligned}$$

since, by assumption, $||\sigma_t|| \in L_1[0,T]$.

4. The $L_2 \to L_2$ theory for exponential functions. In this section we consider functionals of the form

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))\,ds\right\} \tag{4.1}$$

where $\theta(t,u)$ is continuous for almost all $(t,u) \in [0,T]\times\mathbf{R}$ and $||\theta(t,\cdot)||_\infty$ belongs to $L_1[0,T]$. Functionals of this type arise naturally in quantum mechanics. In our next theorem we obtain a series expansion for the generalized Feynman integral of functionals of the form (4.1).

Theorem 3. *Let F be given by* (4.1), *and let $h \in L_2[0,T]$. Then, for all real $q \neq 0$, $h_{J_q^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}), L_2(\mathbf{R}))$ and, for $\psi \in L_2(\mathbf{R})$, we have*

(4.2)

$$(h_{J_q^{an}}(F)\psi)(\xi) = \int_{\mathbf{R}}^{(\xi)}\left(\frac{q}{2\pi i a(T)}\right)^{1/2}\psi(\eta)\exp\left\{\frac{iq(\eta-\xi)^2}{2a(T)}\right\}d\eta$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\left(\frac{q}{2\pi i a(s_1)}\right)^{1/2}\int_{\mathbf{R}}^{(\xi)}\theta(s_1,w_1)\\
&\cdot\exp\left\{\frac{iq(w_1-\xi)^2}{2a(s_1)}\right\}\\
&\cdot\left(\frac{q}{2\pi i[a(s_2)-a(s_1)]}\right)^{1/2}\int_{\mathbf{R}}^{(w_1)}\theta(s_2,w_2)\\
&\cdot\exp\left\{\frac{iq(w_2-w_1)^2}{2[a(s_2)-a(s_1)]}\right\}\\
&\cdots\\
&\cdot\left(\frac{q}{2\pi i[a(s_n)-a(s_{n-1})]}\right)^{1/2}\int_{\mathbf{R}}^{(w_{n-1})}\theta(s_n,w_n)\\
&\cdot\exp\left\{\frac{iq(w_n-w_{n-1})^2}{2[a(s_n)-a(s_{n-1})]}\right\}\\
&\cdot\left(\frac{q}{2\pi i[a(T)-a(s_n)]}\right)^{1/2}\int_{\mathbf{R}}^{(w_n)}\psi(\eta)\\
&\cdot\exp\left\{\frac{iq(\eta-w_n)^2}{2[a(T)-a(s_n)]}\right\}d\eta\,dw_n\cdots dw_1\,d\vec{s}
\end{aligned}
$$

for all $\xi\in\mathbf{R}$ *where* $\Delta_n(T)=\{\vec{s}=(s_1,\dots,s_n)\colon\ 0=s_0<s_1<\cdots<s_n<s_{n+1}=T\}$.

Proof. Using the same notation and proceeding as in the proof of Theorem 2, we obtain equation (3.9) as before. Then, in the last expression in (3.9), first let

$$
\begin{aligned}
v_1&=\left(\frac{a(s_1)}{\lambda}\right)^{1/2}u_1,\qquad v_2=v_1+\left(\frac{a(s_2)-a(s_1)}{\lambda}\right)^{1/2}u_2,\cdots,\\
v_{n+1}&=v_n+\left(\frac{a(s_{n+1})-a(s_n)}{\lambda}\right)^{1/2}u_{n+1}
\end{aligned}
$$

and then let

$$
\begin{aligned}
w_j&=v_j+\xi-\frac{a(s_j)}{a(T)}(v_{n+1}+\xi-\eta)\quad\text{for } j=1,2,\dots,n,\\
w_{n+1}&=v_{n+1}+\xi,\quad\text{and}\quad w_0=\xi
\end{aligned}
$$

to obtain

$$
\begin{aligned}
H_\lambda(\xi,\eta) &= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n+1}\left(\frac{\lambda}{2\pi[a(s_j)-a(s_{j-1})]}\right)^{1/2}\\
&\quad\cdot\int_{\mathbf{R}^{n+1}}\left(\prod_{j=1}^{n}\theta(s_j,w_j)\right)\\
&\quad\cdot\exp\biggl\{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{1}{[a(s_j)-a(s_{j-1})]}\biggl[(w_j-w_{j-1})\\
&\qquad+\frac{a(s_j)-a(s_{j-1})}{a(T)}(w_{n+1}-\eta)\biggr]^2\\
&\qquad-\frac{\lambda}{2[a(T)-a(s_n)]}\biggl[(w_{n+1}-w_n)\\
&\qquad-\frac{a(s_n)}{a(T)}(w_{n+1}-\eta)\biggr]^2\biggr\}\,dw_{n+1}\cdots dw_1\,d\vec{s}\\
&= 1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n+1}\left(\frac{\lambda}{2\pi[a(s_j)-a(s_{j-1})]}\right)^{1/2}\\
&\quad\cdot\int_{\mathbf{R}^{n}}\left(\prod_{j=1}^{n}\theta(s_j,w_j)\right)\exp\biggl\{-\frac{\lambda}{2}\sum_{j=1}^{n}\frac{(w_j-w_{j-1})^2}{[a(s_j)-a(s_{j-1})]}\biggr\}\\
&\quad\cdot\int_{\mathbf{R}}\exp\biggl\{-\frac{\lambda}{a(T)}(w_{n+1}-\eta)(w_n-\xi)\\
&\qquad-\frac{\lambda a(s_n)}{2a^2(T)}(w_{n+1}-\eta)^2\\
&\qquad-\frac{\lambda}{2[a(T)-a(s_n)]}\biggl[(w_{n+1}-w_n)\\
&\qquad-\frac{a(s_n)}{a(T)}(w_{n+1}-\eta)\biggr]^2\biggr\}\,dw_{n+1}\cdots dw_1\,d\vec{s}.
\end{aligned}
$$

Next we carry out the integration with respect to w_{n+1} in the above expression, simplify and then multiply both sides of the resulting expression by

$$
\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2}\exp\left\{-\frac{\lambda}{2a(T)}(\eta-\xi)^2\right\}
$$

and obtain the equation

(4.3)

$$\left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi) \mid \lambda^{-1/2}Z(x,T) + \xi = \eta)$$
$$= \left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\}$$
$$+ \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^{n+1} \left(\frac{\lambda}{2\pi[a(s_j)-a(s_{j-1})]}\right)^{1/2} \cdot \int_{\mathbf{R}^n} \left(\prod_{j=1}^{n} \theta(s_j, w_j)\right)$$
$$\cdot \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n} \frac{(w_j - w_{j-1})^2}{[a(s_j)-a(s_{j-1})]} - \frac{\lambda(w_n-\eta)^2}{2[a(T)-a(s_n)]}\right\} dw_n \cdots dw_1 \, d\vec{s}.$$

Thus, using (4.3) for each $(\lambda, \xi) \in (0, +\infty) \times \mathbf{R}$, we obtain that

$$(h_{I_\lambda}(F)\psi)(\xi)$$
$$= E[F(\lambda^{-1/2}Z(x,\cdot)+\xi)\psi(\lambda^{-1/2}Z(x,T)+\xi)]$$
$$= \int_{\mathbf{R}} E(F(\lambda^{-1/2}Z(x,\cdot)+\xi)$$
$$\cdot \psi(\lambda^{-1/2}Z(x,T)+\xi) \mid \lambda^{-1/2}Z(x,T)+\xi = \eta)$$
$$\cdot \left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} d\eta$$
$$= \left(\frac{\lambda}{2\pi a(T)}\right)^{1/2} \exp\left\{-\frac{\lambda(\eta-\xi)^2}{2a(T)}\right\} \psi(\eta)\, d\eta$$
$$+ \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \left(\frac{\lambda}{2\pi a(s_1)}\right)^{1/2} \int_{\mathbf{R}} \theta(s_1, w_1)$$
$$\cdot \exp\left\{-\frac{\lambda(w_1-\xi)^2}{2a(s_1)}\right\} \left(\frac{\lambda}{2\pi[a(s_2)-a(s_1)]}\right)^{1/2} \int_{\mathbf{R}} \theta(s_2, w_2)$$
$$\cdot \exp\left\{-\frac{\lambda(w_2-w_1)^2}{2[a(s_2)-a(s_1)]}\right\} \cdots \left(\frac{\lambda}{2\pi[a(s_n)-a(s_{n-1})]}\right)^{1/2} \int_{\mathbf{R}} \theta(s_n, w_n)$$

$$\cdot \exp\left\{-\frac{\lambda(w_n-w_{n-1})^2}{2[a(s_n)-a(s_{n-1})]}\right\}\left(\frac{\lambda}{2\pi[a(T)-a(s_n)]}\right)^{1/2}\int_{\mathbf{R}}\psi(\eta)$$
$$\cdot \exp\left\{-\frac{\lambda(\eta-w_n)^2}{2[a(T)-a(s_n)]}\right\}d\eta\, dw_n\cdots dw_1\, d\vec{s}.$$

Again, using Morera's theorem, one can show that $h_{I_\lambda}(F)$ is an $\mathcal{L}(L_2(\mathbf{R}),L_2(\mathbf{R}))$-valued analytic function of λ throughout $\mathbf{C}_+$, and thus $h_{I_\lambda^{an}}(F)\psi$ is given by the last expression in (4.4).

Next, for $n=1,2,\dots$, let

$$\begin{aligned}G_n(\vec{s},\lambda)(\xi) = {}&\left(\frac{\lambda}{2\pi a(s_1)}\right)^{1/2}\int_{\mathbf{R}}\theta(s_1,w_1)\\ &\cdot \exp\left\{-\frac{\lambda(w_1-\xi)^2}{2a(s_1)}\right\}\left(\frac{\lambda}{2\pi[a(s_2)-a(s_1)]}\right)^{1/2}\int_{\mathbf{R}}\theta(s_2,w_2)\\ &\cdot \exp\left\{-\frac{\lambda(w_2-w_1)^2}{2[a(s_2)-a(s_1)]}\right\}\cdots\left(\frac{\lambda}{2\pi[a(s_n)-a(s_{n-1})]}\right)^{1/2}\\ &\quad\cdot\int_{\mathbf{R}}\theta(s_n,w_n)\\ &\cdot \exp\left\{-\frac{\lambda(w_n-w_{n-1})^2}{2[a(s_n)-a(s_{n-1})]}\right\}\left(\frac{\lambda}{2\pi[a(T)-a(s_n)]}\right)^{1/2}\int_{\mathbf{R}}\psi(\eta)\\ &\cdot \exp\left\{-\frac{\lambda(\eta-w_n)^2}{2[a(T)-a(s_n)]}\right\}d\eta\, dw_n\cdots dw_1.\end{aligned}$$

A careful examination of $G_n(\vec{s},\lambda)(\xi)$ shows that it is the composition of convolution operators ($\psi\to C_\lambda\psi$ as in Lemma 1 where $||C_\lambda||=1$) and multiplication operators (multiplication by θ), and so

$$||G_n(\vec{s},\lambda)-G_n(\vec{s},-iq)||_2\to 0\quad\text{as } \lambda\to -iq.$$

In addition, for all $\lambda\in\mathbf{C}_+^\sim$,

$$||G_n(\vec{s},\lambda)||_2\le ||\psi||_2\prod_{j=1}^n||\theta(s_j,\cdot)||_\infty$$

and so, by the dominated convergence theorem for Bochner integrals, $h_{J_q^{an}}(F)$ exists as an element of $\mathcal{L}(L_2(\mathbf{R}),L_2(\mathbf{R}))$. In addition, for each

$\psi \in L_2(\mathbf{R})$,

$$\begin{aligned}||h_{J_q^{an}}(F)\psi||_2 &\le ||\psi||_2\left[1+\sum_{n=1}^{\infty}\int_{\Delta_n(T)}\prod_{j=1}^{n}||\theta(s_j,\cdot)||_\infty\,d\vec{s}\right]\\ &=||\psi||_2\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^T(n)\int_0^T||\theta(s_1,\cdot)||_\infty\cdots||\theta(s_n,\cdot)||_\infty\,d\vec{s}\right]\\ &=||\psi||_2\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\int_0^T||\theta(t,\cdot)||_\infty\,dt\right)^n\right]\\ &=||\psi||_2\exp\left\{\int_0^T||\theta(T,\cdot)||_\infty\,dt\right\}\\ &<\infty,\end{aligned}$$

since, by assumption, $||\theta(t,\cdot)||_\infty \in L_1[0,T]$.

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