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A NOTE ON THE NUMBER OF t-CORE PARTITIONS

KEN ONO

ABSTRACT. A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. A Ferrers graph represents a partition in the natural way. Fix a positive integer t. A partition of n is called a *t-core* partition of n if none of its hook numbers are multiples of t. Let $c_t(n)$ denote the number of t-core partitions of n. It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$. In [**7**], the author proved the conjecture for $t \geq 4$ even and for those t divisible by at least one of 5, 7, 9, or 11. Moreover if $t \geq 5$ is odd, then it was shown that $c_t(n) > 0$ for n sufficiently large. In this note we show that if $k \geq 2$, then $c_{3k}(n) > 0$ for all n using elementary arguments.

A partition of a positive integer n is a nonincreasing sequence of positive integers with sum n. Here we define a special class of partitions.

Definition 1. Let $t \geq 1$ be a positive integer. Any partition of n whose Ferrers graph have no hook numbers divisible by t is known as a *t-core partition* of n.

The hooks are important in the representation theory of finite symmetric groups and the theory of cranks associated with Ramanujan's congruences for the ordinary partition function [**3, 4, 5**].

If $t \geq 1$ and $n \geq 0$, then we define $c_t(n)$ to be the number of partitions of n that are t-core partitions. The arithmetic of $c_t(n)$ is studied in [**3, 4**]. The power series generating function for $c_t(n)$ is given by the infinite product:

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-q^n)}. \tag{1}$$

One easily verifies that $c_2(n)$ and $c_3(n)$ are zero infinitely often. Here

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are the first few terms of the relevant generating functions.

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} c_2(n)q^n$$
$$= 1 + q + q^3 + q^6 + q^{10} + q^{15} + q^{21} + \cdots$$
$$\prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} = \sum_{n\geq 0} c_3(n)q^n$$
$$= 1 + q + 2q^2 + 2q^4 + q^5 + 2q^6 + q^8 + 2q^9 + 2q^{10} + 2q^{12} + \cdots$$

In fact, it is a classical fact that

$$\prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} = \sum_{n\geq 0} q^{t_n}.$$

Here $t_n = n(n+1)/2$ are the usual triangular numbers.

Exact formulae for $c_5(n)$ and $c_7(n)$ appear in [**4**]. Given a positive integer t, is $c_t(n) > 0$ for all $n \geq 0$? In other words, does every positive integer n admit at least one t-core partition? The results in [**4**] show that $c_5(n)$ and $c_7(n)$ are positive for all $n \geq 0$. For $t \geq 5$ prime, Garvan and Olsson have asked if $c_t(n) > 0$ for all n. It has been conjectured that if $t \geq 4$, then $c_t(n) > 0$ for all $n \geq 0$.

The reader should note that $c_t(n) \leq c_{tk}(n)$ for all n. If a partition has no hook numbers divisible by t, then it certainly has no hook numbers divisible by any multiple of t. Hence the conjecture essentially is reduced to a study of $c_p(n)$ where p is prime. The only obstructions to this method is an analysis of $c_t(n)$ where t is a multiple of 2 or 3; when $t = 2$ or 3 the conjecture is false.

In [**7**], the author proved the following partial solution to this conjecture using Deligne's estimates on the Fourier coefficients of modular forms, Gauss's Eureka Theorem, and quadratic form theory.

Theorem 1. *If $t \geq 4$, then $c_t(n) > 0$ for n sufficiently large. Furthermore, if $t \geq 4$ is even, or divisible by 5, 7, 9, or 11, then $c_t(n) > 0$ for all $n \geq 0$.*

The proof of the conjecture when $t \equiv 2 \bmod 4$ is an application of Gauss's Eureka Theorem. We now show that similar methods show that $c_{3k}(n) > 0$ for all $n \geq 0$ if $k \geq 2$. First we recall the proof when $t = 9$.

Theorem 2. *If $c_9(n)$ is the number of 9-core partitions of n, then $c_9(n) > 0$ for all $n \geq 0$.*

Proof. The generating function for $c_9(n)$ is

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^n)} &= \prod_{n=1}^{\infty} \frac{(1-q^{3n})^3}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3} \\ &= \sum_{n=0}^{\infty} c_3(n) q^n \prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3}. \end{aligned} \tag{2}$$

The last infinite product corresponds to a weight 3 Eisenstein series on $\Gamma_0(3)$ with Dirichlet character $\varepsilon(n) = (n/3)$ [**6**, Theorem 6]. This means that its power series expansion is given by the divisor function $\sigma_{2,\varepsilon}(n)$ in the following way:

$$\prod_{n=1}^{\infty} \frac{(1-q^{9n})^9}{(1-q^{3n})^3} = \sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n) q^{3(n-1)}. \tag{3}$$

Here the divisor function $\sigma_{2,\varepsilon}(n)$ is defined by

$$\sigma_{2,\varepsilon}(n) = \sum_{0<d\mid n} \varepsilon\left(\frac{n}{d}\right) d^2. \tag{4}$$

It is an easy exercise to verify that all of the coefficients in (3) are positive since $\sigma_{2,\varepsilon}(n) > 0$ for all $n \in \mathbf{Z}^+$.

Combining these facts we obtain from (2) and (3)

$$\sum_{n=0}^{\infty} c_9(n) q^n = \{1 + q + 2q^2 + \cdots\} \sum_{n=0}^{\infty} \sigma_{2,\varepsilon}(n) q^{3(n-1)}.$$

Since the first 3 coefficients of the power series in braces are positive and the generalized divisor function $\sigma_{2,\varepsilon}(n)$ is always positive, we find that $c_9(n)$ is always positive. This completes the proof. $\square$

It should be noted that Fine [**2**, 3.2.351, p. 79] has an elementary proof of this fact.

Now we prove the main theorem of this note using Theorem 2.

Theorem 3. *If $k \geq 2$, then $c_{3k}(n) > 0$ for all $n \geq 0$.*

Proof. If $k \equiv 0 \mod 3$, then $9 \mid 3k$. By Theorem 2 we find that $0 < c_9(n) \leq c_{3k}(n)$ for all $n \geq 0$. Therefore we may assume that $k \not\equiv 0 \bmod 3$.

We may assume that $k = 3t + i$ with $i = 1$ or 2. The generating function for $c_{3k}(n) = c_{9t+3i}(n)$ can be factored in the following way:

$$\begin{aligned}
\sum_{n=0}^{\infty} c_{9t+3i}(n)q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{9t+3i}}{(1-q^n)} \\
&= \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{9t}}{(1-q^{(3t+i)n})^{3t}} \\
&\quad \cdot \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{3i}(1-q^{(3t+i)n})^{3t}}{(1-q^n)} \\
&= \left[\sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n) q^{(3t+i)(n-1)}\right]^t \\
&\quad \cdot \prod_{n=1}^{\infty} \frac{(1-q^{(3t+i)n})^{3t+i}}{(1-q^n)} \prod_{n=1}^{\infty} \frac{(1-q^{(9t+3i)n})^{3i}}{(1-q^{(3t+i)n})^{i}} \\
&= \left[\sum_{n=1}^{\infty} \sigma_{2,\varepsilon}(n) q^{(3t+i)(n-1)}\right]^t \\
&\quad \cdot \left[\sum_{n=0}^{\infty} c_{3t+i}(n)q^n\right]\left[\sum_{n=0}^{\infty} c_3(n) q^{(3t+i)n}\right]^i.
\end{aligned} \tag{5}$$

Since $\sigma_{2,\varepsilon}(n) > 0$ for all $n \geq 1$, we see that the first factor of (5), the tth power of the divisor function power series, has positive coefficients for exponents that are multiples of $3t + i$. The coefficients $c_{3t+i}(n)$ of the middle factor in (5) are positive for all $0 \leq n \leq 3t + i - 1$; one needs at least $3t + i$ nodes before a partition can have a $3t + i$ hook. Therefore the product of the first two factors in (5) has nothing but positive coefficients. Since $c_3(0) = 1$, we find that the coefficients of the entire product, namely $c_{9t+3i}(n)$ are all positive. □

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Department of Mathematics, The University of Georgia, Athens, Georgia 30602

email address: ono@sophie.math.uga.edu