ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

SMOOTH PARTITIONS OF UNITY AND APPROXIMATING NORMS IN BANACH SPACES

DAVID P. MCLAUGHLIN

1. Introduction. In the seminal paper [**1**] it is shown that C^k-smooth separable Banach spaces admit C^k-smooth partitions of unity (definitions to follow), but it is still an open question whether this result extends to nonseparable spaces; contributions to this question appear in [**15, 10, 3, 16** and **13**]. A survey of these and related results can be found in [**4**, Section 8.3] and we refer the reader to the notes and references therein.

Theorem 4 in [**15**] states that a reflexive Banach space X admits C^k-smooth partitions of unity whenever X admits an LUR norm which is C^k-smooth. Two observations motivate the main result of this paper: first, a reflexive C^k-smooth Banach space admits C^k-smooth partitions of unity; second, in general, a space with a C^k-smooth norm and an LUR norm will not have a norm which is both C^k-smooth and LUR. In fact, let us note that Asplund's averaging technique (cf. [**4**, Section 2.2.4]) for higher order smooth norms is in general not available, e.g., $c_0(\mathbf{N})$ has a C^∞-smooth norm and its dual has an LUR norm, but $c_0(\mathbf{N})$ has no LUR C^2-smooth norm [**8**]. (Note: the corresponding averaging result for WLUR is open; because the set of C^2-smooth norms is the first category, a Baire category arguments sheds no light on this question.)

We now present the main result of this paper.

Theorem 1. *Suppose a* WLUR *norm on a Banach space X can be uniformly approximated on bounded sets by equivalent C^{k+1}-smooth norms, where $k \in \mathbf{N} \cup \{\infty\}$. Then X admits C^k-smooth partitions of unity.*

Remarks a. Note that in this theorem we do not assume *a priori* that the space in question admits a mapping into some $c_0(\Gamma)$.

Received by the editors on October 1, 1993.

b. In light of the above result from [**15**], it is a natural question to ask if this result can be improved to obtain C^{k+1}-smooth partitions of unity; the method of proof presented in this paper cannot address this problem.

The proof of Theorem 1 will be given in Section 2. In Section 3 we present the following applications of this theorem, thus exhibiting a fairly general class of spaces to which this theorem applies (for example, the space JL has a C^∞-smooth norm, see [**5**] and [**13**]).

Corollary 1. *If X admits a C^{k+1}-smooth norm and if there exists a set Γ and a linear, bounded map $T : X \to c_0(\Gamma)$ with range of T^* dense in X^*, then X admits C^k-smooth partitions of unity.*

Corollary 2. *Let X have a closed subspace Y such that Y^* is separable and X/Y has a shrinking M-basis. If X has a C^{k+1}-smooth norm, then X admits C^k-smooth partitions of unity.*

Remarks a. With regards to the separability condition in Corollary 2, we note that the space constructed in [**2**] has a subspace isomorphic to $c_0(\Gamma_1)$ such that the resulting quotient is isomorphic to $c_0(\Gamma_2)$, yet the original space admits no linear bounded map into $c_0(\Gamma)$ for any Γ.

b. If X is WCG and has a C^k-smooth norm, then X has a shrinking M-basis [**12**]. It then follows from the proof of Corollary 1 that X admits a WLUR norm which is the uniform limit on bounded sets of C^k-smooth norms.

We now fix some concepts and definitions. All spaces will be real Banach spaces and differentiability will always be in the Fréchet sense. A partition of unity $\{\phi_\alpha\}_{\alpha \in A}, \phi_\alpha : X \to \mathbf{R}$, is called C^k-smooth if ϕ_α is k-times continuously Fréchet differentiable for each $\alpha \in A$; we say X admits C^k-smooth partitions of unity if for every open covering $\{V_\beta\}_{\beta \in B}$ of X, there exists a locally finite C^k-smooth partition of unity $\{\phi_\alpha\}_{\alpha \in A}$ such that, for each $\alpha \in A$, the closure of $\{x \in X : \phi_\alpha \neq 0\}$ is contained in V_β for some $\beta \in B$. We refer to the natural and rational numbers as $\mathbf{N}$ and $\mathbf{Q}$, respectively, and we say a norm $||\cdot||$ is (weakly) locally uniformly convex [(W)LUR] if $2||x||^2 + 2||x_m||^2 - ||x + x_m||^2 \to 0$

implies x_m converges (weakly) to x. We will say that a norm $||\cdot||$ is C^k-smooth if it is k-times Fréchet differentiable away from 0; if $||\cdot||$ is differentiable at $x \in X$, then we denote its derivative at x by $||\cdot||'(x)$. A biorthogonal system $(x_\alpha, f_\alpha)_{\alpha\in\Gamma} \subset X \times X^*$ satisfies $f_\alpha(x_\beta) = 1$ if $\alpha = \beta$ and equals 0 otherwise. Such a system is called a Markuševič basis (M-basis) if $X = \overline{\text{sp}}\,(x_\alpha)_{\alpha\in\Gamma}$ and $\{f_\alpha\}_{\alpha\in\Gamma}$ is total on X and an M-basis is called shrinking if $X^* = \overline{\text{sp}}\,\{f_\alpha\}_{\alpha\in\Gamma}$.

2. Proof of Theorem 1. By appealing to Theorem 1 in [**15**], it suffices to construct a set Δ and a function $h : X \to c_0(\Delta)$ such that $h_\delta(\cdot)$ is a C^k-smooth function from $X \to \mathbf{R}$, h^{-1} is continuous, and h is one-to-one, continuous and maps into $c_0(\Delta)$. Essentially, our proof entails the construction of an index set and a corresponding set of one-dimensional subspaces which are collectively dense. They will be used to show the continuity of the inverse. This, however, will introduce a large number of coordinates; to ensure that the function in fact maps into a c_0 space, additional constructs are added. The proof will be broken into two parts, the first dealing with the construction and the second showing that the construction yields the desired properties.

Proof of Theorem 1.

Part A. *Construction of* $h(x)$. In this part we will create a linear, bounded, one-to-one map into a c_0 space (Lemma 1), construct a family of projections onto one-dimensional subspaces (Lemma 2) and finally exhibit the map h.

Let $||\cdot||$ be a WLUR norm on X. The space X has an equivalent C^1-smooth norm, so by [**6**] or [**7**] there exists an ordinal set $\Gamma = [\omega, \gamma]$ where γ is the first ordinal of cardinality $\operatorname{dens} X$ such that X^* admits a long sequence of projections $\{T_\alpha\}_{\alpha\in\Gamma}$ on X^* with $T_\omega = 0$ and T_γ the identity operator. If, for each $\alpha \in [\omega, \gamma)$, we write $\tau_\alpha = T_{\alpha+1} - T_\alpha$, then

i) $||T_\alpha|| < \infty$ for all $\alpha \in \Gamma$;

ii) $T_\alpha T_\beta = T_\beta T_\alpha = T_\alpha$ for $\alpha, \beta \in \Gamma$ with $\alpha \le \beta$;

iii) $f \in \overline{\text{sp}}_{\,\alpha\in[\omega,\gamma)}\{\tau_\alpha(f)\}$ for each $f \in X^*$; and

iv) for each $\alpha < \gamma$, $\tau_\alpha(X^*)$ is separable.

Lemma 1. *There exists a linear, bounded, one-to-one operator* $T : X^* \to c_0([\omega,\gamma) \times \mathbf{N})$ *where* γ *is the first ordinal of cardinality for* dens(X). *Furthermore, if* $\alpha \in [\omega,\gamma)$ *and* $\tau_\alpha(f) \neq 0$, *then for some* $n \in \mathbf{N}$ *we have* $T_{(\alpha,n)} f \neq 0$.

Proof. Write S_{X^*} and $S_{X^{**}}$ for the dual and bidual spheres of X with respect to this norm and choose, for each $\alpha \in [\omega,\gamma)$, a sequence $\{g_n^\alpha\}_{n=1}^\infty$ dense in $S_{X^*} \cap \tau_\alpha(X^*)$ and a sequence $\{\phi_n^\alpha\}_{n=1}^\infty \subset S_{X^{**}}$ such that $\phi_n^\alpha(g_n^\alpha) = 1$. The operator T is then defined at the coordinate (α, n) by

$$T_{(\alpha,n)} f = \frac{\phi_n^\alpha(T_\alpha(f))}{n(||T_{\alpha+1}|| + ||T_\alpha||)}$$

for each $f \in X^*$. Because $\{\alpha \in [\omega,\gamma) : ||\tau_\alpha(f)|| > \varepsilon(||T_{\alpha+1}|| + ||T_\alpha||)\}$ is finite for each $\varepsilon > 0$ and each $f \in X^*$ (cf. [**5**]), T maps into $c_0([\omega,\gamma) \times \mathbf{N})$. If $\tau_\alpha(f) \neq 0$, say $||\tau_\alpha(f)|| = c$, then for some n, $||\tau_\alpha(f) - c g_n^\alpha|| < c/2$ so $\phi_n^\alpha[\tau_\alpha(f)] > 0$ and $T_{(\alpha,n)} f \neq 0$. The operator T is clearly linear, bounded and one-to-one.

Motivated by the proof of Lemma 1 in [**11**], we will define a family of projections onto spaces of dimension one or zero. To this end, enumerate $\cup_{n=1}^\infty \mathbf{Q}^n = \{\rho_i\}_{i=1}^\infty$ and $\cup_{n=1}^\infty \mathbf{N}^n = \{\eta_i\}_{i=1}^\infty$, and let $S[\omega,\gamma)$ denote the set of all finite subsets of $[\omega,\gamma)$. Denoting $S^n[\omega,\gamma)$ as the n-fold product of $S[\omega,\gamma)$, we will write $\mathcal{F}(n) \in S^n([\omega,\gamma))$ to emphasize the length of the n-tuple $\mathcal{F}(n)$.

There are C^{k+1}-smooth norms which converge uniformly on bounded sets to $||\cdot||$, so we can suppose $\{||\cdot||_j\}_{j=1}^\infty$ is a sequence of C^{k+1}-smooth norms on X such that for all $x \in X$ we have

$$(*) \qquad \Big| \, ||x||_j - ||x|| \, \Big| < ||x||/(2j).$$

Now for each $F \in S[\omega,\gamma)$ let $\{f_i^F\}_{i=1}^\infty$ be a dense sequence in $\overline{\mathrm{sp}}_{\alpha\in F} \tau_\alpha(X^*)$, and for each triple F, i, j, choose a sequence $\{x_{ijk}^F\}_{k=1}^\infty$ such that $||x_{ijk}^F||_j = 1$ and $\lim_{k\to\infty} f_i^F(x_{ijk}^F) = ||f_i^F||_j$. Define

$$S = \{(\mathcal{F}(n), a, b, c, d) \in S^n[\omega,\gamma) \times \mathbf{Q}^n \times \mathbf{N}^n \times \mathbf{N}^n \times \mathbf{N}^n,\ 1 \leq n < \infty\}.$$

If $s = (\mathcal{F}(n), \rho_m, \eta_i, \eta_j, \eta_k) \in S$ with $\mathcal{F}(n) = (F_1, \dots, F_n)$, $\rho_m = (r_{m_1}, \dots, r_{m_n})$, $\eta_i = (i_1, \dots, i_n)$, $\eta_j = (j_1, \dots, j_n)$, and $\eta_k =$

$(k_1, \dots, k_n)$, then we define π_s to be the projection of X onto sp $\sum_{l=1}^{n} r_{m_l} x_{i_l j_l k_l}^{F_l}$ where $||\pi_s|| = 1$ if $\sum_{l=1}^{n} r_{m_l} x_{i_l j_l k_l}^{F_l} \neq 0$.

Lemma 2. *If $x \in X$ and $\varepsilon > 0$ are given, then there exists $s = (\mathcal{F}(n), \rho_m, \eta_i, \eta_j, \eta_k) \in S$ with $F_l \subset \{\alpha : \tau_\alpha[|| \cdot ||'_{j_l}(x)] \neq 0\}$ for each $l \leq n$ such that $||x - \pi_s(x)|| < \varepsilon$.*

Proof. The statement is obvious if $x = 0$. Assume that $||x|| = 1$ and we fix $\varepsilon > 0$. By property iii) of the PRI $\{T_\alpha\}_{\alpha \in \Gamma}$, we choose for each j a finite set $F_j \subset \{\alpha \in [\omega, \gamma) : \tau_\alpha[|| \cdot ||'_j(x)] \neq 0\}$ so as to guarantee the existence of $f \in \text{sp} \cup_{\alpha \in F_j} \tau_\alpha(X^*)$ with $||f||_j = 1$ and $||f - || \cdot ||'_j(x)||_j < j^{-1}$. Next, find i_j such that $||f_{i_j}^{F_j} - f||_j < j^{-1}$ and k_j so that $f_{i_j}^{F_j}(x_{i_j j k_j}^{F_j}) > 1 - 2j^{-1}$. Then

$$\begin{aligned} ||x + x_{i_j j k_j}^{F_j}|| + 6j^{-1} &\geq ||x + x_{i_j j k_j}^{F_j}||_j \geq f(x + x_{i_j j k_j}^{F_j}) \\ &= f_{i_j}^{F_j}(x_{i_j j k_j}^{F_j}) - (f_{i_j}^{F_j} - f)(x_{i_j j k_j}^{F_j}) \\ &\quad + [|| \cdot ||'_j(x)](x) - [|| \cdot ||'_j(x) - f](x) \\ &\geq 1 - 2j^{-1} - ||f_{i_j}^{F_j} - f||_j \\ &\quad + ||x||_j (1 - || \, || \cdot ||'_j(x) - f||_j) \\ &\geq 1 - 2j^{-1} - j^{-1} + (1 - j^{-1})(1 - j^{-1}) \\ &\geq 2 - 5j^{-1}. \end{aligned}$$

In addition, $||x + x_{i_j j k_j}^{F_j}|| \leq ||x|| + ||x_{i_j j k_j}^{F_j}|| \leq 2 + j^{-1}$ so $\Big| ||x + x_{i_j j k_j}^{F_j}|| - 2\Big| < 11j^{-1}$. It follows by the WLUR of $|| \cdot ||$ that $x_{i_j j k_j}^{F_j} \xrightarrow{w} x$, hence there exists a sequence of convex combination of these elements which converge in norm to x. The result now follows from a simple density argument.

To construct $h(x)$, we choose for each $p \in \mathbf{N}$ a C^∞-smooth function $\psi_p : \mathbf{R} \to \mathbf{R}$ such that

$$\begin{aligned} \psi_p(t) &= \frac{1}{2p}, & |t| &\leq \frac{1}{2p}, \\ \frac{1}{2p} \leq \psi_p(t) &\leq \frac{1}{p}, & \frac{1}{2p} < |t| &< \frac{1}{p} \end{aligned}$$

$$\psi_p(t) = |t|, \qquad\qquad |t| \geq \frac{1}{p}.$$

We let Δ be the disjoint union of $S \times \mathbf{N}^3$, $S[\omega, \gamma) \times \mathbf{N}^2$, $[\omega, \gamma) \times \mathbf{N}^3$, and $\mathbf{N}^2$, and we define a function $h : X \to c_0(\Delta) = \{f : \Delta \to \mathbf{R}$ with card $\{\delta \in \Delta : |f(\delta)| \geq \varepsilon\}$ finite for each $\varepsilon > 0\}$ coordinatewise as follows: for each $x \in X$ and $\delta \in \Delta$

(1)

$$h_\delta(x) = \frac{\psi_q \circ ||x - \pi_s(x)||_l}{4mijklqp} \cdot \prod_{\substack{\alpha \in F_r \\ 1 \leq r \leq n_0}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)} \{(\psi_p \circ || \cdot ||_{j_r})'(x)\} \right)^2 \right] \right),$$

$$\text{if } \delta = (s, l, p, q) \in S \times \mathbf{N}^3, \quad s = (\mathcal{F}(n_0), \rho_m, \eta_i, \eta_j, \eta_k)$$

(2)

$$h_\delta(x) = \frac{1}{j+p} \prod_{\alpha \in F} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)} \{(\psi_p \circ || \cdot ||_j)'(x)\} \right)^2 \right] \right)$$

$$\text{if } \delta = (F, j, p) \in S[\omega, \gamma) \times \mathbf{N}^2$$

(3)

$$h_\delta(x) = \frac{1}{j+p} T_{(\alpha,n)}[(\psi_p \circ || \cdot ||_j)'(x) \quad \text{if } \delta = (\alpha, n, j, p) \in [\omega, \gamma) \times \mathbf{N}^3$$

(4)

$$h_\delta(x) = \frac{1}{j+p} (\psi_p \circ ||x||_j) \quad \text{if } \delta = (j, p) \in \mathbf{N}^2.$$

Part B. Verification of the properties of $h(x)$.

We start with a simple lemma.

Lemma 3. *Let G be a set, and let H denote the set of all finite subsets of G. If $x = (x_\alpha)_{\alpha \in G} \in c_0(G)$, then $y = (\prod_{\alpha \in F} x_a)_{F \in H} \in c_0(H)$.*

Proof. Fix $\varepsilon > 0$ and write $M = \max_{\alpha \in G} |x_\alpha| < \infty$. Since $x \in c_0(G)$ we know there exists a finite set $A \subset G$ of cardinality n such that for

$\alpha \notin A$, $|x_\alpha| < 1$, thus we can find a finite set $B \subset G$ such that $\beta \notin B$ implies $|x_\beta| < \varepsilon/M^n$. Now if $F \in H$ and there exists $\sigma \in F$ with $\sigma \notin B$, then

$$\prod_{\alpha \in F} x_\alpha = x_\sigma \left(\prod_{\substack{\alpha \in F \\ \alpha \neq \sigma}} x_\alpha \right) < \varepsilon.$$

Because $\{F \in H : F \subset B\}$ is finite, the proof is complete.

For each $j \in \mathbf{N}$, $||\cdot||_j$ is $k+1$ times differentiable away from 0 and ψ_p is constant on a neighborhood of 0 so $(\psi_p \circ ||\cdot||_j)(x)$ is $k+1$-times differentiable at all $x \in X$. In addition, all derivatives of $\tan^{-1}(x)$ are bounded and $T_{(\alpha,n)}$ is linear so

$$\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)}\{(\psi_p \circ ||\cdot||_{j_r})'(x)\} \right)^2 \right]$$

is k-times differentiable. Since $I - \pi_s$ is C^∞-smooth, we conclude that h_δ is a C^k-smooth function for each $\delta \in \Delta$.

Next, to show that h is one-to-one, we suppose that $h(x) = h(y)$ and consider two cases. If $x = 0$, then $(\psi_p ||\cdot||_1)'(x) = 0$ because $\psi_p ||\cdot||_1 = 1/(2p)$ on a neighborhood of 0. Now T is one-to-one so (3) forces $(\psi_p ||\cdot||_1)'(y) = 0$ so that $||y||_1 \le 1/p$ for each $p \in \mathbf{N}$ and thus $y = 0$.

If x and y are not 0, then for p large enough we have $\psi_p(t) = |t|$ for $\{t : |t| > \min(||x||/2, ||y||/2)\}$. By $(*)$, we have, for large j and p, $\psi_p \circ ||x||_j = ||x||_j$ so by (4), $h(x) = h(y)$ implies $||x||_j = ||y||_j$. In addition, T is one-to-one so $(\psi_p \circ ||\cdot||_j)'(x) = ||\cdot||_j'(x) = ||\cdot||_j'(y)$. Therefore,

$$\begin{aligned} ||x+y||_j &\ge [||\cdot||_j'(x)](x+y) \\ &= [||\cdot||_j'(x)](x) + [||\cdot||_j'(y)](y) \\ &= ||x||_j + ||y||_j. \end{aligned}$$

Again, by $(*)$, we have $||x+y|| = ||x|| + ||y||$. But WLUR implies strict convexity so $x = y$ and h is one-to-one.

To show that h maps X into $c_0(\Delta)$, we fix $\varepsilon > 0$ and $x \in X$. We will show that, for fixed j and p,

$$(**) \qquad \sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)}\{(\psi_p \circ ||\cdot||_j)'(x)\} \right)^2 \right] \ge \varepsilon$$

for only finitely many $\alpha \in [\omega, \gamma)$. To prove this inequality, we let $K = ||(\psi_p|| \cdot ||_j)'(x)||^*$, and choose $N \in \mathbf{N}$ such that $\sum_{n=N+1}^{\infty} 1/2^n < \varepsilon/(2K^2)$. Now T maps into $c_0([\omega, \gamma) \times \mathbf{N})$, so there exists a finite set $G \subset [\omega, \gamma)$ such that $T_{(\alpha,n)}(\psi|| \cdot ||_j)'(x) < \sqrt{\varepsilon/2}$ for all $n \leq N$ and $\alpha \notin G$. Since $||T|| \leq 1$, for all $\alpha \notin G$ we have

$$\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \Big(T_{(\alpha,n)}\{(\psi_p \circ || \cdot ||_j)'(x)\}\Big)^2 \leq \sum_{n=1}^{N} \frac{1}{2^n}\left(\frac{\varepsilon}{2}\right) + \left(\sum_{n=N+1}^{\infty} \frac{1}{2^n}\right) K^2 < \varepsilon.$$

Since $x \geq \tan^{-1}(x)$ for all positive reals, $(**)$ follows immediately.

Let us now note that $\psi_q \circ ||x||_l \leq \max\{1, ||x||_l\}$ and $||x - \pi_s(x)||_l \leq (1 + 1/l)||x - \pi_s(x)|| \leq 4||x||$, so for each δ as in (1), $h_\delta(x) \leq \max\{||x||, 1\}$. Therefore, if any one of m, i, j, k, l, p, q are larger than $\varepsilon^{-1} \max\{||x||, 1\}$, then $h_\delta(x) < \varepsilon$. Observe that when j is fixed, η_0 is also, so that if $s = (\mathcal{F}(n_0), \rho_m, \eta_i, \eta_j, \eta_k) \in S$ then $\mathcal{F}(n_0) = (F_1, \dots, F_{n_0})$. Appealing to Lemma 3, we know that for each $r \in \{1, \dots, n_0\}$ and each $p \in \mathbf{N}$ there are only finitely many choices of $F \in S[\omega, \gamma)$ satisfying

$$\prod_{\alpha \in F} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\Big(T_{(\alpha,n)}\{(\psi_p \circ || \cdot ||_{j_r})'(x)\}\Big)^2\right]\right) \geq \frac{\varepsilon}{||x||}.$$

In addition, the term on the left is bounded above by 1, so that if $\mathcal{F}(n_0)$ contains any F not satisfying the above inequality, then

$$\frac{\psi_q \circ ||x - \pi_s(x)||_l}{4mijklqp} \cdot \prod_{\substack{\alpha \in F_r \\ 1 \leq r \leq n_0}} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1}\left[\Big(T_{(\alpha,n)}\{(\psi_p \circ || \cdot ||_{j_r})'(x)\}\Big)^2\right]\right) \leq \varepsilon.$$

Thus $h_\delta(x) \geq \varepsilon$ for only a finite number of δ as in (1). For δ as in (2), (3) and (4), the fact that $h_\delta(x) \geq \varepsilon$ for only finitely many δ follows from similar arguments and thus h maps into $c_0(\Delta)$.

To see that h is continuous, let $\varepsilon > 0$ be given. If jp is large enough, then $h_\delta(x) < \varepsilon/2$ for each $x \in X$. If both j and p are small, then there are only finitely many terms of the form $(\psi_p \circ ||\cdot||_j)'(\cdot)$ and $\psi_p \circ ||\cdot||_j$, and these functions are continuous. Since T is linear and bounded, the continuity of h follows from elementary arguments.

Finally, we must show that h^{-1} is continuous. We will suppose that $h_\delta(x_m) \to h_\delta(x)$ for each $\delta \in \Delta$ and show that $x_m \to x$. If $x = 0$, then $\psi_p||x||_j = 1/(2p)$ for each $j \in \mathbf{N}$, so for m large enough $\psi_p||x_m||_j < 1/p$ and $x_m \to 0$. If $x \neq 0$, then $2 < p||x||$ implies that $\psi_p(||x||_j) = ||x||_j > 1/p$ for each $j \in \mathbf{N}$, and $\psi_p(||x_m||_j) \to ||x||_j$ forces $||x_m||_j \to ||x||_j$. By $(*)$ it then follows that $||x_m|| \to ||x||$. To finish the proof, we will show that $\{x_m\}_{m=1}^\infty$ is totally bounded.

With this aim, use Lemma 2 to find $s = (\mathcal{F}(n_0), \rho_m, \eta_i, \eta_j, \eta_k) \in S$ with $F_r \subset \{\alpha \in [\omega, \gamma) : \tau_\alpha[||\cdot||'_{j_r}(x)] \neq 0\}$ such that $||x - \pi_s(x)|| < \varepsilon/4$, and choose $2 < p||x||$. Then $(\psi_p||\cdot||_{j_r})'(x) = ||\cdot||'_{j_r}(x)$, and by $(*)$ there exists $l \in \mathbf{N}$ such that $||x - \pi_s(x)||_l < \varepsilon/2$. The choice of F_r ensures that

$$\prod_{\alpha \in F_r} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)}\{(\psi_p \circ ||\cdot||_{j_r})'(x)\} \right)^2 \right] \right) > 0$$

so $h_\delta(x_m) \to h_\delta(x)$ for each δ as in (2) implies there exists $M \in \mathbf{N}$ so that, for each $m > M$,

$$\prod_{\alpha \in F_r} \left(\sum_{n=1}^{\infty} \frac{1}{\pi 2^n} \tan^{-1} \left[\left(T_{(\alpha,n)}\{(\psi_p \circ ||\cdot||_{j_r})'(x_m)\} \right)^2 \right] \right) > 0.$$

Choosing $q > 1/(2\varepsilon)$ we see that $\psi_q||x - \pi_s(x)||_1 = ||x - \pi_s(x)||_1$. Assembling this information and noting in particular that $h_{(s,1,p,q)}(x_m) \to h_{(s,1,p,q)}(x)$ we see that

$$||x_m - \pi_s(x_m)||_1 \to ||x - \pi_s(x)||_1 < \varepsilon/2.$$

Therefore, $\{x_k\}_{k=1}^\infty$ is totally bounded and the proof is complete. □

3. Corollaries. The following are consequences of the main result.

Corollary 1. *If X admits a C^{k+1}-smooth norm and if there exists a set Γ and a linear, bounded map $T : X \to c_0(\Gamma)$ with range of T^* dense in X^*, then X admits C^k-smooth partitions of unity.*

Proof. Let T be a linear, bounded map of X into $c_0(\Gamma)$. Using a result in [**14**], we can find a sequence of C^∞-smooth norms $||\cdot||_n$ in $c_0(\Gamma)$ which converge uniformly on bounded sets to an LUR norm $|\cdot|$. If $||\cdot||$ is a C^{k+1}-smooth norm on X, then we define a sequence of norms $\{|||\cdot|||_n\}_{n=1}^\infty$ on X by $|||x|||_n^2 = ||x||^2 + ||Tx||_n^2$. Since T is bounded, $|||x|||_n$ is an equivalent C^{k+1}-smooth norm. In addition, note that

$$|||x|||_n^2 \to ||x||^2 + |Tx|^2 = |||x|||^2.$$

Because $||\cdot||_n$ converges to $|\cdot|$ uniformly on bounded sets, the same can be said of $|||\cdot|||_n$ with respect to $|||\cdot|||$. Clearly $|||\cdot|||$ is an equivalent norm which we claim to be WLUR.

To justify this claim, suppose $2|||x|||^2 + 2|||x_m|||^2 - |||x + x_m|||^2 \to 0$. Because $2||x||^2 + 2||x_m||^2 - ||x + x_m||^2 \geq (||x||^2 - ||x_m||^2)$ and $2|Tx|^2 + 2|Tx_m|^2 - |T(x+x_m)|^2 \geq (|Tx|^2 - |Tx_m|^2)$, the latter expression must converge to 0. The LUR of the norm $|\cdot|$ implies that $Tx_m \to Tx$ in $c_0(\Gamma)$. Also, $||x_m|| \to ||x||$ so that $\sup_m\{||x - x_m||\} = C < \infty$.

We will show that if $\varepsilon > 0$ and $f \in X^*$ are given, then there exists $M \in \mathbf{N}$ such that for all $m > M$, $|f(x_m - x)| < \varepsilon$. Observe that because $T^* : c_0(\Gamma)^* \to X^*$ has dense range, we can find an element $\theta \in c_0(\Gamma)^*$ such that $||T^*(\theta) - f|| < \varepsilon/2C$. Now there exists M such that $m > M$ implies $|\theta(T(x - x_m))| < \varepsilon/2$. For such an m,

$$\begin{aligned} |f(x - x_m)| &\leq |(f - T^*\theta)(x - x_m)| + |T^*\theta(x - x_m)| \\ &\leq ||f - T^*\theta||\, ||x - x_m|| + |\theta T(x - x_m)| \leq \varepsilon. \end{aligned}$$

Thus, x_m converges weakly to x so $|||\cdot|||$ is WLUR. A direct application of Theorem 1 completes the proof. □

Corollary 2. *Let X have a closed subspace Y such that Y^* is separable and X/Y has a shrinking M-basis. If X has a C^{k+1}-smooth norm, then X admits C^k-smooth partitions of unity.*

Proof. Clearly, from Corollary 1, it suffices to show that there exists a linear bounded map $T : X \to c_0(\Gamma)$ (for some Γ) such that T^* has dense range. Let $(z_\alpha, z_\alpha^*)_{\alpha\in\Gamma_1}$ be a shrinking M-basis of $Z = X/Y$, let Q be the quotient map and let $h_\alpha = z_\alpha^* \circ Q$. Because Y is separable, we can find a set $\{g_i\}_{i=1}^\infty$ dense on the unit sphere of Y^*, and we extend these to $\{f_i\}_{i=1}^{\infty}$ in X^*. Now let Γ be the disjoint union of $\mathbf{N}$ and Γ_1, and we define $T : X \to c_0(\Gamma)$ by

$$T_\gamma(x) = \begin{cases} (1/i)f_i(x) & \gamma = i \in \mathbf{N}, \\ h_\alpha(x)/||h_\alpha|| & \gamma = \alpha \in \Gamma_1. \end{cases}$$

Then T is clearly linear and bounded by one.

We now prove that T^* has dense range. Let us denote by e_γ the element in $c_0(\Gamma)^*$ which has a one in the γ coordinate and zeros elsewhere. We show first that $T^*e_\alpha = h_\alpha$. Indeed, if $x \in X$, then

$$(T^*e_\alpha - h_\alpha)(x) = e_\alpha[T(x)] - h_\alpha(x) = e_\alpha(h_\alpha(x)) - h_\alpha(x) = 0,$$

and thus T^* maps onto a dense subset D of $Y^\perp$ by linearity. In addition, note that for each $x \in Y$, $(f_i - T^*ie_i)(x) = f_i(x) - ie_i[T(x)] = 0$, so $T^*ie_i = f_i$ on Y. Thus, the element ie_i is mapped by T^* to a member of the coset $f_i + Y^\perp$, say, $f_i + h_i$. It is now easy to see that the linear span of elements of the form $f = f_i + h_i + d : i \in \mathbf{N}, d \in D$ is in the range of T^* and is dense in X^*. $\square$

Remark. The results of this paper should be compared with the recent result in [**9**] which states that if a Banach space has an equivalent LUR norm and if every Lipschitz convex function can be uniformly approximated on bounded sets by C^k-smooth functions, then it admits C^k-smooth partitions of unity.

REFERENCES

1. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. **15** (1966), 877–898.

2. K. Ciesielski and R. Pol, *A weakly Lindelöf function space $C(K)$ without any continuous injection into $c_0(T)$*, Bull. Polish Acad. Sci. Math. **32** (1984), 681–688.

3. R. Deville, G. Godefroy and V.E. Zizler, *The three space problem for smooth partitions of unity and $C(K)$ spaces*, Math. Ann. **288** (1990), 613–625.

4. ———, *Smoothness and renormings in Banach spaces*, Pitman Monographs Surveys Pure Appl. Math. **64**, 1993.

5. J. Diestel, *Geometry of Banach spaces: selected topics*, Lecture Notes in Math. **485**, Springer-Verlag, New York, 1975.

6. M. Fabian, *On projectional resolutions of the identity on the duals of certain Banach spaces*, Bull. Austral. Math. Soc. **35** (1987), 363–371.

7. M. Fabian and G. Godefroy, *The dual of an Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), 141–151.

8. M. Fabian, J.H.M. Whitfield and V. Zizler, *Norms with locally Lipschitzian derivatives*, Israel J. Math. **44** (1983), 262–276.

9. J. Frontisi, *Smooth partitions of unity in Banach spaces*, preprint, 1993.

10. G. Godefroy, S. Troyanski, J.H.M. Whitfield and V. Zizler, *Smoothness in weakly compactly generated Banach spaces*, J. Funct. Anal. **52** (1983), 344–352.

11. ———, *Locally uniformly rotund renorming and injections into* $c_0(\Gamma)$, Canad. Math. Bull. **27** (1984), 494–500.

12. K. John and V. Zizler, *Smoothness and its equivalents in the class of weakly compactly generated Banach spaces*, J. Funct. Anal. **15** (1974), 1–11.

13. D.P. McLaughlin, *Smooth partitions of unity in preduals of* WCG *spaces*, Math. Z. **211** (1992), 189–194.

14. J. Pechanec, J.H.M. Whitfield, V.E. Zizler, *Norms locally dependent on finitely many coordinates*, An. Acad. Brasil. Ciênc. **53** (1981), 415–417.

15. H. Toruńczyk, *Smooth partitions of unity on some non-separable Banach spaces*, Studia Math. **46** (1973), 43–51.

16. J. Vanderwerff, *Smooth approximations in Banach spaces*, Proc. Amer. Math. Soc. **115** (1992), 113–120.

Kwantlen University College, P.O. Box 9030, Surrey, B.C., V3T 5H8, Canada