ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

HIGHER ORDER UNIFORMLY GÂTEAUX DIFFERENTIABLE NORMS ON ORLICZ SPACES

R.P. MALEEV

ABSTRACT. Equivalent α_M-times uniformly Gâteaux differentiable norms are constructed for large classes of Orlicz spaces $L_M(S,\Sigma,\mu)$. Especially, for the spaces $L_{2p-1}(0,1)$, $p \in \mathbf{N}$, equivalent $(2p-1)$-uniformly Gâteaux smooth norms are found.

1. Introduction. The existence of smooth bump functions on a Banach space is of some importance in many problems of the nonlinear analysis. At the end of the 1980s, several deep results of Deville [**2, 3**] showed that the existence of higher order differentiable bumps also has geometrical implications.

The problem of the best order of Fréchet differentiability of bump functions was solved for L_p-spaces in [**1, 12**] and for Orlicz sequence spaces in [**9, 10**]. Especially, it is shown [**1**] that in l_p, p odd, there is no p-times Fréchet differentiable bump and [**9**] that in $l_M, \alpha_M^0 \in \mathbf{N}$, there is no α_M^0-times Fréchet differentiable bump, excepting the case where α_M^0 is even and M is equivalent to $t_M^{\alpha^0}$ at 0.

On the other hand, in a Banach space, a norm of some order of smoothness generates a bump with the same order of smoothness and therefore every positive result on the existence of a smooth equivalent norm is transferred directly for bumps. In [**11**] equivalent p-times Gâteaux differentiable norms are found in L_p over σ-finite measure space, p odd. Our aim is to generalize and sharpen this result for Orlicz sequence spaces l_M (function spaces $L_M(0,1)$) with $\alpha_M^0(\alpha_M^\infty)$ a positive integer and M not equivalent to $t^{\alpha_M^0}(t^{\alpha_M^\infty})$ at $0(\infty)$.

2. Preliminaries. We begin with some notations and definitions. In what follows X and Y are Banach spaces, S_X and B_X the unit sphere

Received by the editors on April 20, 1993.
Research partially supported by SRF of the Bulgarian Ministry of science and education, contract no. MM-3/1992.
Mathematics Subject Classification. Primary 46B20, 46B25.
Key words. Orlicz spaces, Gâteaux differentiability, smoothness.

and the unit ball of X, respectively. $\mathbf{N}$ denotes the set of all naturals, $\mathbf{R}$ the reals, $\mathbf{R}^+ = [0,\infty)$. The space of all continuous symmetric j-linear forms

$$T : \underbrace{X \times X \times \cdots \times X}_{j \text{ times}} \to Y$$

equipped with the norm

$$||T||_1 = \sup\{||T(x_1, \dots , x_j)||; x_i \in X, ||x_i|| \le 1, 1 \le i \le j\}$$

is denoted $B^j(X,Y)$. We write $T(\underbrace{x, x, \dots , x}_{j \text{ times}}) = T(x^j)$.

An equivalent norm in $B^j(X,Y)$ (see, e.g., [**13**, p. 10]) is given by

$$||T|| = \sup\{||T(x^j)||; x \in S_X\}.$$

Definition 1 [**4**]. The map $f : X \to Y$ is said to be *Gâteaux* (*directionally*) *differentiable* at $x \in X$, if for each $h \in X$,

$$f'(x;h) = \lim_{t\to 0} t^{-1}(f(x+th) - f(x))$$

exists and is a linear continuous function in h, i.e., $f'(x) \in B(X,Y)$. The higher order Gâteaux derivatives $f^{(k)}$ are defined inductively. Suppose the $(k-1)$th derivative $f^{(k-1)}$ of f is defined in a neighborhood $U(x)$ of x, $f^{(k-1)}(y) \in B^{k-1}(X,Y)$ for every $y \in U(x)$. Then f is called *k-times Gâteaux differentiable* at x if $f^{(k-1)} : U(x) \to B^{k-1}(X,Y)$ is Gâteaux differentiable at x, i.e., if there exists $f^{(k)}(x) \in B^k(X,Y)$ such that for each $h \in X$,

$$(1) \qquad \lim_{t\to 0} t^{-1}(f^{(k-1)}(x+th;\cdot) - f^{(k-1)}(x,\cdot)) = f^{(k)}(x;\cdot h),$$

where the limit is understood with respect to the norm in $B^{k-1}(X,Y)$.

If the limit in (1) is uniform on $h \in S_X$, we say that f is *k-times Fréchet differentiable* at x. The k-linear symmetric continuous form $f^{(k)}(x)$ is called the *k-th Gâteaux derivative of* f at x in the first case and *k-th Fréchet derivative of* f at x in the second case and is denoted also by $D^kf(x)$. The class of all k-times Gâteaux (Fréchet) differentiable maps at any $x \in A \subset X$ is denoted by $G^k(A)(F^k(A))$. If f is k-times Gâteaux differentiable at every $x \in S_X$ and the limit in (1) is uniform over $x \in S_X$ for each fixed $h \in S_X$, we say that f is *k-times uniformly Gâteaux differentiable on S_X*. If this limit is uniform over x, $h \in S_X$, we say that f is *k-times uniformly Fréchet differentiable on S_X*. The classes of all k-times uniformly Gâteaux and uniformly Fréchet differentiable maps are denoted, respectively, $UG^k(S_X)$, $UF^k(S_X)$. We note that even for maps $f : X \to Y$ which have k-th *weak Gâteaux derivative* (see, e.g., [**6**, Chapter 17] continuous on $[x, x+h] = \{y \in X; y = x+th, t \in [0,1]\}$ the Taylor's formula holds true:

$$f(x+th) = f(x) + \sum_{j=1}^{k} \frac{t^j}{j!} f^{(j)}(x;h^j) + r_k(x,h,t),$$

where

$$r_k(x,h,t) = \frac{t^k}{(k-1)!} \int_0^1 (1-\lambda)^{k-1} (f^{(k)}(x+\lambda th;h^k) - f^{(k)}(x;h^k))\, d\lambda.$$

It is easy to show that, for $t \to 0$, we have $r_k(x,h,t) = o_{xh}(t^k)(o_x(t^k))$ if $f \in G^k(U(x))(F^k(U(x)))$ and that $r_k(x,h,t) = o_h(t^k)(o(t^k))$ if $f \in UG^k(S_X)(UF^k(S_X))$. Sometimes the behavior of the remainder term r_k in the Taylor's expansion (see, e.g., [**13**, 1.3.3] is used to define Gâteaux and Fréchet differentiability at a point x and uniform Gâteaux and Fréchet differentiability on S_X as well.

Definition 2. We shall say that X is *$G^k(F^k)$-smooth* if the norm in X is a function from $G^k(X\backslash\{0\})(F^k(X\backslash\{0\}))$ and *$UG^k(UF^k)$-smooth* if this norm belongs to $UG^k(S_X)(UF^k(S_X))$.

We recall that an even convex continuous function M nondecreasing in $[0,\infty]$, such that $M(0)=0$, $M(\infty)=\infty$, is called an *Orlicz function.* For a measure space (S,Σ,μ) the Banach space of all classes equivalent μ-measurable functions $x : S \to \mathbf{R}$ with

$$\tilde{M}(\lambda x) = \int_S M(\lambda x(s))\, d\mu(s) < \infty$$

for some $\lambda > 0$, normed by the formula

$$||x|| = \inf\left\{\lambda > 0; \tilde{M}\left(\frac{x}{\lambda}\right) \le 1\right\},$$

is called an *Orlicz space* and is denoted by $L_M(S, \Sigma, \mu)$. The most common examples of Orlicz spaces are the sequence spaces l_M and function spaces $L_M(0,1)$, $L_M(0,\infty)$, that correspond to the cases: S countable union of atoms of equal mass, and $S = [0,1]$, $S = [0,\infty)$, μ the usual Lebesgue measure. It is easy to observe that the properties of the spaces l_M, $L_M(0,1)$, $L_M(0,\infty)$ are essentially determined by the behavior of M near 0, ∞ and 0 and ∞, respectively. This is reflected in the following well-known result: If two Orlicz functions M and N are *equivalent* ($M \sim N$) at 0, (∞, 0 and ∞), i.e.,

$$c^{-1}M(c^{-1}t) \le N(t) \le cM(ct), \qquad t \in [0,1], t \in [1,\infty), t \in \mathbf{R}^+$$

for some positive constant c, then l_N, respectively $L_N(0,1)$, $L_N(0,\infty)$, is isomorphic to l_M, respectively $L_M(0,1)$, $L_M(0,\infty)$, see, e.g., [**7**]. Therefore, equivalent norms are easily constructed in l_M, $L_M(0,1)$ or $L_M(0,\infty)$ using suitable Orlicz functions equivalent to M at 0, at ∞ or at 0 and ∞, respectively.

Denote $G_M^p(u,v) = u^{-p}M(uv)/M(v)$. The following pairs of numbers are associated to every Orlicz function

$$\begin{aligned}
\alpha_M^0 &= \sup\{p; \sup\{G_M^p(u,v); u, v \in (0,1]\} < \infty\},\\
\beta_M^0 &= \inf\{p; \inf\{G_M^p(u,v); u, v \in (0,1]\} > 0\},\\
\alpha_M^\infty &= \sup\left\{p; \sup\left\{\frac{1}{G_M^p(u,v)}; u, v \in [1,\infty)\right\} < \infty\right\},\\
\beta_M^\infty &= \inf\left\{p; \inf\left\{\frac{1}{G_M^p(u,v)}; u, v \in [1,\infty)\right\} > 0\right\},\\
\alpha_M &= \min(\alpha_M^0, \alpha_M^\infty), \qquad \beta_M = \max(\beta_M^0, \beta_M^\infty).
\end{aligned}$$

It is readily seen that $1 \le \alpha_M^i \le \beta_M^i \le \infty$, $i = 0, \infty$. If $\beta_M^0 < \infty$ ($\beta_M^\infty < \infty$, $\beta_M < \infty$) we say that M satisfies the Δ_2-*condition* at 0 (at ∞, at 0 and at ∞). Obviously, in this case the following inequality holds

$$(2)\quad M(\lambda t) \le c_\beta \lambda^\beta M(t), \qquad \lambda \ge 1, \quad t \in [0,1], (t \in [1,\infty), t \in [0,\infty))$$

for any $\beta > \beta_M^0$ ($\beta > \beta_M^\infty, \beta > \beta_M$), where c_β is a positive constant that does not depend on t and λ.

3. Auxiliary results. Let $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f \not\equiv 0$, be a nondecreasing continuous function such that $f(0) = 0$ and for any $0 \le a \le b$ the following inequality holds

$$f(b) - f(a) \le c(b-a)\frac{f(b)}{b}, \qquad b \ne 0 \tag{3}$$

for some positive constant $c > 0$. Obviously $c \ge 1$. In what follows we refer to such functions by writing $f \in F(c)$. The following two lemmas give some useful properties of the functions from the class $F(c)$.

Lemma 1. *Let $f \in F(c)$. For any $\lambda \ge 1$, the following inequality holds*

$$f(\lambda t) \le 2\lambda^\beta f(t), \qquad t \in \mathbf{R}^+, \tag{4}$$

where $\beta = (\log_2(2c/(2c-1)))^{-1}$.

Proof. It is easy to check that (3) implies for $\alpha = 2c/(2c-1) \in (1, 2]$:

$$f(\alpha t) \le 2 f(t).$$

Now for $\lambda = \alpha^\mu$ we obviously have

$$f(\lambda t) \le f(\alpha^{[\mu]+1} t) \le 2.2^\mu f(t) = 2\lambda^\beta f(t). \qquad \square$$

Denote $I(f;t) = \int_0^t f(u)\,du$ and define inductively $I^n(f;t) = I(I^{n-1}(f);t)$.

Lemma 2. *Let $f \in F(c)$. For every $k \in \mathbf{N}$ the function $M(t) = I^k(f;|t|)$ is an Orlicz function such that*

a) *for any $\lambda \ge 1$,*

$$M(\lambda t) \le 2\lambda^\gamma M(t), \qquad \gamma = \beta + k; \tag{5}$$

b) *for any* $\lambda \in [0,1]$,

$$M(\lambda t) \leq \lambda^k M(t); \tag{6}$$

c) *for* $0 \leq j \leq i$, $0 \leq i \leq k$,

$$|M^{(i-j)}(t)| \leq |t|^j |M^{(i)}(t)| \leq |M^{(i-j)}(2^j t)|. \tag{7}$$

Proof. Let $t \geq 0$. Obviously, $M^{(j)}(t) = I^{k-j}(f;t)$, $0 \leq j \leq k-1$ and $M^{(k)}(t) = f(t)$, i.e., all the derivatives of M of order not exceeding k are positive and nondecreasing in $\mathbf{R}^+$. Therefore, M is an Orlicz function. A simple change of the variable combined with (4) gives (5) and (6). To prove the second inequality in (7), it suffices to observe that

$$M^{(i-1)}(t) \geq \int_{t/2}^{t} M^{(i)}(u)\,du \geq \frac{t}{2} M^{(i)}\left(\frac{t}{2}\right)$$

and inductively

$$M^{(i-j)}(t) \geq 2^{-j(j+1)/2} t^j M^{(i)}\left(\frac{t}{2^j}\right),$$

i.e., $M^{(i-j)}(2^j t) \geq t^j M^{(i)}(t)$ for $0 \leq j \leq i$, $0 \leq i \leq k$. The left inequality in (7) is obvious. $\square$

Remark 1. For any $t \in \mathbf{R}$, taking $i = j$ in (7), we have

$$M(t) \leq |t|^i M^{(i)}(|t|) \leq M(2^i t), \qquad 0 \leq i \leq k.$$

In the following lemmas $f \in F(c)$, $M = I^k(f)$. Let (S, Σ, μ) be a measure space with positive measure μ and $X = L_M(S, \Sigma, \mu)$ the Orlicz space generated by M. Denote $\tilde{M}_i : X \to B^i(X, \mathbf{R})$, $0 \leq i \leq k$, the map defined by the formula

$$\tilde{M}_i(x; y_1, \dots, y_i) = \int_S M^{(i)}(x(s)) y_1(s) \cdots y_i(s)\, d\mu(s),$$
$$y_j \in X, 0 \leq j \leq i,$$

where $M_0 = M$. We note that $\tilde{M}_i(x)$, $1 \le i \le k$, is a bounded i-linear symmetric functional because (7) and (5) imply

$$\begin{aligned}\sup\{|\tilde{M}_i(x; y_1, \dots , y_i)|; y_j \in B_X, 1 \le j \le i\} \\ \le \sup\left\{\tilde{M}_i(|x|; |x|^i) + \sum_{j=1}^{i} \tilde{M}_i(|y_j|; |y_j|^i); y_j \in B_X, 1 \le j \le i\right\} \\ \le \tilde{M}(2^i x) + \sum_{j=1}^{i} \tilde{M}(2^i y_j) \le 2^{\gamma k+1}(\tilde{M}(x) + k).\end{aligned}$$

Later on we shall use this inequality in the equivalent form

$$|\tilde{M}_i(x; y_1, \dots , y_i)| \le c_1(\tilde{M}(x) + k)||y_1|| \cdots ||y_i||, \quad 1 \le i \le k, \tag{8}$$

where $c_1 = 2^{\gamma k+1}$.

Define $\tilde{M}_{i,j} : X \to B^{i-j}(X, \mathbf{R})$, $0 \le j \le i$, $0 \le i \le k$, $B^0 = \mathbf{R}$, by the formula

$$\tilde{M}_{i,j}(x; y_1, \dots , y_{i-j}) = \int_S x^j(s) M^{(i)}(x(s)) y_1(s) \cdots y_{i-j}(s)\, d\mu(s).$$

Obviously, $\tilde{M}_{i,0} = \tilde{M}_i$ and from (8) the inequalities follow

$$|\tilde{M}_{i,j}(x; y_1, \dots , y_{i-j})| \le c_1(\tilde{M}(x) + k)||x||^j ||y_1|| \cdots ||y_{i-j}||, \quad 0 \le j \le i. \tag{9}$$

Put for $u, v, w, t, \alpha \in \mathbf{R}$ and $0 \le i \le k$,

$$\begin{aligned}\varphi_i(u, v, t, \alpha) &= |M^{(i)}(\alpha(u + tv)) - M^{(i)}(u)|, \\ \psi_{i,j,r}(u, v, w, t, \alpha) &= \varphi_i(u, v, t, \alpha)|u|^j |v|^r |w|^{i-j-r},\end{aligned}$$

$j, r \ge 0$, $0 \le j + r \le i$. The following technical lemma holds true.

Lemma 3. *There exist positive constants c_2, c_3, c_4 such that, for any $u, v, w, t, \alpha \in \mathbf{R}$ satisfying $|t| < 1/2$, $|1 - \alpha| < 2|t|$ the following estimates hold*:

$$\psi_{i,j,r} \le c_2|t|(M(u) + M(v) + M(w)), \qquad (i, j) \ne (k, 0). \tag{10}$$

for $0 \le r \le i-j$ *and*

(11) $\psi_{k,0,r} \le c_3|t|^{1/2}(M(u)+M(v)+M(w))+c_4 f(4|t|^{1/2}|v|)|v|^r|w|^{k-r}$

for $1 \le r \le k$.

Proof. For the sake of brevity we shall omit the variables and simply write $\varphi_i, \psi_{i,j,r}$. Put $z=\alpha(u+tv)$. Obviously $0 \le \alpha \le 2$ and

$$\begin{aligned} |z| &\le 2(|u|+|tv|) \le 2|u|+|v|, \\ |z-u| &\le 2|t|(|u|+|v|) \le |u|+|v|. \end{aligned} \tag{12}$$

First let $0 \le i < k$. The mean value theorem, (12), (7) and (5) imply that, for some $\theta \in (0,1)$,

$$\begin{aligned} \psi_{i,j,r} &= |z-u|M^{(i+1)}(|u+\theta(z-u)|)|u|^j|v|^r|w|^{i-j-r} \\ &\le 2|t|(|u|+|v|)M^{(i+1)}(2|u|+|v|)|u|^j|v|^r|w|^{i-j-r} \\ &\le 2|t|(|u|+|v|+|w|)^{i+1}M^{(i+1)}(2(|u|+|v|+|w|)) \\ &\le |t|M(2^{i+2}(|u|+|v|+|w|)) \\ &\le |t|\left(M(2^{i+4}u)+M(2^{i+4}v)+\frac{M(2^{i+4}w)}{3}\right) \\ &\le 2^{\gamma(i+4)}|t|(M(u)+M(v)+M(w)), \end{aligned}$$

i.e., (10) holds in this case.

Now let $i=k$. As $M^{(k)}(t)=(\operatorname{sign} t)^k f(|t|)$ we consider separately two cases:

Case a) k even or k odd and $uz \ge 0$. Using (3) we easily get

$$\varphi_k = |f(|z|)-f(|u|)| \le c|z-u|\frac{f(\xi)}{\xi}, \qquad \xi=\max(|u|,|z|) \tag{13}$$

and, as above,

$$\begin{aligned} \psi_{k,j,r} &\le 2c|t|(|u|+|v|)f(2|u|+|v|)|u|^{j-1}|v|^r|w|^{k-j-r} \\ &\le C2^{\gamma(k+4)}|t|(M(u)+M(v)+M(w)), \end{aligned} \tag{14}$$

for $0 < j \le k$.

If $j = 0$, we obtain from (13), using (12), (7) and (5) consecutively,
(15)
$$\begin{aligned}
\psi_{k,0,r} &\le cf(|\alpha t v|)|v|^r|w|^{k-r} \le cf(|tv|)|v|^r|w|^{k-r}, \quad u = 0,\\
\psi_{k,0,r} &\le 2cf(4|t|^{1/2}|v|)|v|^r|w|^{k-r}, \quad 0 < |u| \le |t|^{1/2}|v|,\\
\psi_{k,0,r} &\le 2c|t|^{1/2}(|t|^{1/2}+1)f(2|u|+|v|)|v|^r|w|^{k-r}\\
&\le 4c2^{\gamma(k+4)}|t|^{1/2}(M(u)+M(v)+M(w)), \quad |t|^{1/2}|v| < |u|.
\end{aligned}$$

Now we consider

Case b) k odd and $uz < 0$. Obviously $|u|, |z| \le |z-u|$. This immediately implies

$$\varphi_k \le 2f(|z-u|). \tag{16}$$

Therefore, we obtain for $0 < j \le k$ from (12) and (7)

$$\begin{aligned}
\psi_{k,j,r} &\le 2f(|z-u|)|z-u|^j|v|^r|w|^{k-j-r}\\
&\le 2^{j+1}|t|^j f(|u|+|v|+|w|)(|u|+|v|+|w|)^k\\
&\le 2^j 2^{\gamma(k+2)}|t|^j(M(u)+M(v)+M(w)).
\end{aligned} \tag{17}$$

If $j = 0$ it follows from (16) and (12), (7), (5) and (6)

$$\begin{aligned}
\psi_{k,0,r} &\le 2f(|z-u|)|v|^r|w|^{k-r}\\
&\le 2f(4|t|^{1/2}|v|)|v|^r|w|^{k-r}, \quad |t|^{1/2}|u| \le |v|,\\
\psi_{k,0,r} &\le 2|t|^{r/2}f(4|tu|)|u|^r|w|^{k-r}\\
&\le 2^{-k+2}2^{\gamma(k+2)}|t|^{r/2}(M(u)+M(w)), \quad |v| < |t|^{1/2}|u|
\end{aligned} \tag{18}$$

for $1 \le r \le k$.

Obviously, (14) and (17) imply (10) for $i = k$. Analogously, (15) and (18) imply (11). Lemma (3) is proved. □

Remark 2. We note that $\psi_{i,j,r}$ does not depend on w for $r = i-j$. We shall write $\psi_{i,j,i-j}(u,v,t,\alpha)$. For obvious reasons for $\psi_{i,j,i-j}(u,v,t,\alpha)$ the estimates (10) and (11) can be used with $w = 0$.

For $x, y, h \in X$, $t, \alpha \in \mathbf{R}$, we define for $0 \le j \le i \le k$, $0 \le r \le i-j$,

$$A_{i,j,r}(x,y,h,t,\alpha) = \tilde{M}_{i,j}(\alpha(x+th); y^{i-j-r}h^r) - \tilde{M}_{i,j}(x; y^{i-j-r}h^r).$$

As $A_{i,j,r}$ does not depend on y for $r = i - j$ we shall write as above $A_{i,j,i-j}(x, h, t, \alpha)$.

Corollary 1. *There exist positive constants* c_5, c_6, c_7 *such that, for any* $x, y, h \in S_X$ *and* $t, \alpha \in \mathbf{R}$*, satisfying* $|t| \le 1/2$*,* $|1-\alpha| \le 2|t|$*, the following inequalities hold*:

$$(19)\quad |A_{i,j,r}(x, y, h, t, \alpha)| \le c_5|t|, \quad (i, j) \ne (k, 0), \quad 0 \le r \le i - j$$

$$(20)\quad |A_{k,0,r}(x, y, h, t, \alpha)| \le c_6|t|^{1/2} + c_7\varphi^{r/k}(t, h), \quad 0 < r \le k,$$

where $\varphi(t, h) = \int_S f(4|t|^{1/2}|h(s)|)|h(s)|^k \, d\mu(s)$*.*

Proof. Put $z = \alpha(x + th)$. Just as in (12), we have

$$(21)\quad \begin{aligned} |z| \le 2(|x| + |th|) \le 2|x| + |h|, \qquad ||z|| \le 3, \\ |z - x| \le 2|t|(|x| + |h|), \qquad ||z - x|| \le 4|t|. \end{aligned}$$

Obviously,

$$|A_{i,j,r}(x, y, h, t, \alpha)| \le a^1_{i,j,r}(x, y, h, t, \alpha) + a^2_{i,j,r}(x, y, h, t, \alpha),$$

where

$$a^1_{i,j,r}(x, y, h, t, \alpha) = \int_S \psi_{i,j,r}(x(s), h(s), y(s), t, \alpha) \, d\mu(s),$$

$$a^2_{i,j,r}(x, y, h, t, \alpha) = \begin{cases} 0 & j = 0, \\ \sum_{m=1}^{j-1} \tilde{M}_{i,j-1-m}(|z|; |z - x||x^m y^{i-j-r} h^r|), \\ & j \ne 0. \end{cases}$$

For $j \ne 0$, (5), (9) and (21) imply

$$(22)\quad a^2_{i,j,r} \le 4c_1|t|(\tilde{M}(z) + k) \sum_{m=0}^{j-2} ||z||^m \le 2c_1 3^{j-1}(2.3^\gamma + k)|t|.$$

To estimate $a^1_{i,j,r}$ it is sufficient to write (10) and (11) for $u = x(s)$, $v = h(s)$, $w = y(s)$, t, α and to integrate over S the corresponding inequalities. We get for any r, satisfying $0 \le r \le i - j$:

$$(23)\quad a^1_{i,j,r} \le c_2|t|(\tilde{M}(x) + \tilde{M}(h) + \tilde{M}(y)) = 3c_2|t|, \qquad (i, j) \ne (k, 0).$$

From (22) and (23), (19) follows immediately. On the other hand, (11) implies

$$a^1_{k,0,r} \le 3c_3|t|^{1/2} + c_4 \int_S f(4|t|^{1/2}|h(s)|)|y(s)|^{k-r}|h(s)|^r \, d\mu(s), \quad 0 < r \le k. \tag{24}$$

The integral in (24) admits for $0 < r < k$ an easy estimate using the Hölder inequality and Remark 1,

$$\begin{aligned}
&\int_S f(4|t|^{1/2}|h(s)|)|y(s)|^{k-r}|h(s)|^r \, d\mu(s) \\
&\le \left(\int_S f(4|t|^{1/2}|h(s)|)|y(s)|^k \, d\mu(s) \right)^{(k-r)/k} \\
&\quad \cdot \left(\int_S f(4|t|^{1/2}|h(s)|)|h(s)|^k \, d\mu(s) \right)^{r/k} \\
&\le \left(\int_S f(|h(s)|)|h(s)|^k \, d\mu(s) + \int_S f(|y(s)|)|y(s)|^k \, d\mu(s) \right)^{(k-r)/k} \\
&\quad \cdot \left(\int_S f(4|t|^{1/2}|h(s)|)|h(s)|^k \, d\mu(s) \right)^{r/k} \\
&\le 2^{k\gamma+2} \left(\int_S f(4|t|^{1/2}|h(s)|)h(s)^k \, d\mu(s) \right)^{r/k}.
\end{aligned}$$

Now (20) follows for $0 < r < k$ from the last inequality and (24) and directly from (24) for $r = k$. □

Remark 3. If we denote $\mathcal{A} = \{\alpha \in C[-1/2; 1/2]; |1 - \alpha(t)| \le 2|t|\}$ we may reformulate Corollary 1 as follows: for $t \to 0$,

$$A_{i,j,r}(x, y, h, t, \alpha) = O(t), \quad (i, j) \ne (k, 0), \qquad 0 \le r \le i - j \tag{25}$$

uniformly on $x, y, h \in S(X)$, $\alpha \in \mathcal{A}$ and

$$A_{k,0,r}(x, y, h, t, \alpha) = o_h(1), \qquad 0 < r \le k \tag{26}$$

uniformly on $x, y \in S_X$, $\alpha \in \mathcal{A}$.

Indeed, (25) is obvious. The proof of (26) is straightforward because the Lebesgue theorem implies

$$\lim_{t\to 0}\varphi(t,h)=\lim_{t\to 0}\int_S f(4|t|^{1/2}|h(s)|)|h(s)|^k\,d\mu(s)=0$$

for every fixed $h\in S_X$.

Lemma 4. *The functional* $\tilde{M}\in UG^k(S_X)$ *and* $D^i\tilde{M}=\tilde{M}_i$, $0\le i\le k$.

Proof. Let $x,h\in S_X$. For fixed $s\in S$, Taylor's formula gives

$$\begin{aligned}&\left|M(x(s)+th(s))-\sum_{j=1}^{k}\frac{t^j}{j!}M^{(j)}(x(s))h^j(s)\right|\\&\qquad\le |t|^k\int_0^1\frac{(1-\lambda)^{k-1}}{(k-1)!}\psi_{k,0,k}(x(s),h(s),\lambda t,1)\,d\lambda.\end{aligned}$$

After integrating over S, we easily get for some $\theta\in(0,1)$,

$$\begin{aligned}&\left|\tilde{M}(x+th)-\sum_{j=0}^{k}\frac{t^j}{j!}\tilde{M}_j(x;h^j)\right|\\&\qquad\le\frac{|t|^k}{(k-1)!}\int_0^1(1-\lambda)^{k-1}|A_{k,0,k}(x,h,\lambda t,1)|\,d\lambda\\&\qquad=\frac{|t|^k}{k!}|A_{k,0,k}(x,h,\theta t,1)|\\&\qquad=o_h(t^k);\end{aligned}$$

this ends the proof. □

4. Main theorem.

Theorem. *Let* $f\in F(c)$. *For every measure space* (S,Σ,μ) *with positive measure, the Orlicz space* $X=L_M(S,\Sigma,\mu)$, $M=I^k(f)$, *is* UG^k*-smooth.*

Proof. Denote by $n(x)$ the norm of $x\in X$. Obviously, $\tilde{M}(x/n(x))=1$ for any $x\in X$. For the sake of brevity, we put $\tilde{M}_{i,j}(x/n(x))=\overline{M}_{i,j}(x)$

(recall $\tilde{M}_{i,j}(x; y_1 \dots y_{i-j}) = \int_S x^j(s) M^{(i)}(x(s)) y_1(s) \dots y_{i-j}(s)\, d\mu(s))$. We first prove that n is uniformly Gâteaux differentiable on $X \backslash \{0\}$ and

$$n'(x) = \frac{\overline{M}_{1,0}(x)}{\overline{M}_{1,1}(x)}. \tag{27}$$

Indeed, without loss of generality we may consider $x \in S_X$. The Taylor's formula for $\tilde{M}$ gives

$$\begin{aligned} 0 &= \overline{M}(x+th) - \overline{M}(x) \\ &= \tilde{M}\frac{(x+th)}{n(x+th)} - \tilde{M}(x) \\ &= \tilde{M}'(x;z) + \int_0^1 (\tilde{M}'(x+\lambda z; z) - \tilde{M}'(x;z))\, d\lambda, \end{aligned}$$

where $z = (x+th)/n(x+th) - x$.

Therefore,

$$\begin{aligned} n(x+th) - 1 = t\, &\frac{\tilde{M}'(x;h)}{\tilde{M}'(x;x)} \\ &+ \frac{n(x+th)}{\tilde{M}'(x;x)} \int_0^1 (\tilde{M}'(x+\lambda z; z) - \tilde{M}'(x,z))\, d\lambda. \end{aligned} \tag{28}$$

In order to estimate the last integral we use the representation $x+\lambda z = \alpha(x+t_1 h)$ with $\alpha = 1+\lambda(1/n(x+th)-1)$, $t_1 = \lambda t((1-\lambda)n(x+th)+\lambda)^{-1}$. It is not hard to check that $|\alpha - 1| \le 2|t_1|, |t_1| \le 4|t|/3 \le 1/3$, whenever $|t| \le 1/4$. Thus, Corollary 1 implies

$$\begin{aligned} &\left| \int_0^1 (\tilde{M}'(x+\lambda z; z) - \tilde{M}'(x;z)) d\lambda \right| \\ &\qquad \le \int_0^1 \left(\left| \frac{1}{(n(x+th)} - 1 \right| |A_{1,1,0}(x, x, h, t_1, \alpha)| \right. \\ &\qquad\qquad\qquad \left. + \frac{|t|}{n(x+th)} |A_{1,0,1}(x, h, t_1, \alpha)| \right) d\lambda \\ &\qquad \le 2|t|(c_5|t| + c_6|t|^{1/2} + c_7 \int_S f(8|t|^{1/2}|h(s)|)|h(s)|^k\, d\mu(s)) \\ &\qquad = o_h(t). \end{aligned}$$

To get (27) from (28) it is enough to observe that $n(x+th)/\tilde{M}'(x;x) \leq 2$ (see Remark 1).

Thus the thoerem is proved for $k = 1$.

Now let $k > 1$. An easy induction using Lemma 4, the chain rule and the obvious equality

$$D\left(\frac{x}{n(x)};h\right) = \frac{h}{n(x)} - \frac{x}{n^2(x)}n'(x;h)$$

allows us to claim that n is k-times Gâteaux differentiable and leads to the formula

$$(29)\quad n^{(k)}(x) = \frac{\sum_{i=0}^{k} C_k^i(-1)^i\overline{M}_{k,i}(x)\overline{M}_{1,1}^{k-i}(x)\overline{M}_{1,0}^{i}(x) + P(\overline{M}_{i,j}(x))}{n^{k-1}(x)\overline{M}_{1,1}^{k+1}(x)}$$

where $P(\overline{M}_{i,j}(x))$ is a polynomial with respect to $\overline{M}_{i,j}$, $0 \leq j \leq i < k$, such that $P(\overline{M}_{i,j}(x)) \in B^k(X,\mathbf{R})$ for any fixed x. We note that, for example,

$$\begin{aligned} &C_k^i\overline{M}_{k,i}(x)\overline{M}_{1,0}^{i}(x)(y_1,\dots,y_k) \\ &\qquad = \sum_{m\in C_{k-i}} \tilde{M}_{k,i}\left(\frac{x}{n(x)};y_{m_1}\cdots y_{m_{k-i}}\right)\prod_{j\in K\setminus m}\tilde{M}_{1,0}\left(\frac{x}{n(x)};y_j\right), \end{aligned}$$

where $C_{k-i} = \{m = (m_1,m_2,\dots,m_{k-i});\ 1 \leq m_1 < m_2 < \dots < m_{k-i} \leq k\}$, $K = \{1,2,\dots,k\}$.

To finish the proof we have to estimate the norm of $n^{(k-1)}(x+th) - n^{(k-1)}(x) - tn^{(k)}(x;h)$ as an element of $B^{(k-1)}(X,\mathbf{R})$. The Taylor's formula implies

$$\begin{aligned} &|n^{(k-1)}(x+th;z^{k-1}) - n^{(k-1)}(x;z^{k-1}) - tn^{(k)}(x;z^{k-1}h)| \\ &\qquad\qquad \leq |t|\int_0^1 |n^{(k)}(x+\lambda th;z^{k-1}h) - n^{(k)}(x;z^{k-1}h)|\,d\lambda. \end{aligned}$$

The problem that faces us is to estimate the difference $n^{(k)}(x+\lambda th;z^{k-1}h) - n^{(k)}(x;z^{k-1}h)$ for $x,y,z \in S_X$, $\lambda \in [0,1]$.

We observe first that every $(i-j)$-linear form $\overline{M}_{i,j}$ that appears in this difference according to (29) is computed at the point $(\underbrace{z,\dots,z}_{i-j-1},h)$ or at the point $(\underbrace{z,\dots,z}_{i-j})$. Obviously for $0 \le j \le i \le k$,

$$\begin{aligned} &\overline{M}_{i,j}(x+\lambda th; z^{i-j-1}h) - \overline{M}_{i,j}(x; z^{i-j-1}h) \\ &\qquad\qquad = A_{i,j,1}(x,z,h,\lambda t, \|x+\lambda th\|^{-1}), \\ &\overline{M}_{i,j}(x+\lambda th; z^{i-j}) - \overline{M}_{i,j}(x; z^{i-j}) \\ &\qquad\qquad = A_{i,j,0}(x,z,h,\lambda t, \|x+\lambda th\|^{-1}). \end{aligned} \tag{30}$$

On the other hand, for any $x, h \in S_X$, $|t| \le 1/2$,

$$\left| \frac{1}{\|x+\lambda th\|} - 1 \right| \le 2|t|$$

and therefore (30) and Remark 3 imply for $t \to 0$

$$\begin{aligned} &\overline{M}_{i,j}(x+\lambda th; z^{i-j-r}h^r) = \overline{M}_{i,j}(x; z^{i-j-r}h^r) + O(t) \\ &\qquad\qquad (i,j) \ne (k,0), r = 0,1, \\ &\overline{M}_{k,0}(x+\lambda th; z^{k-1}h) = \overline{M}_{k,0}(x; z^{k-1}h) + o_h(1), \end{aligned} \tag{31}$$

uniformly on $x, z \in S_X$, $\lambda \in [0,1]$.

Taking into account (29), (31) and (7), we easily get

$$|n^{(k)}(x+\lambda th; z^{k-1}h) - n^{(k)}(x; z^{k-1}h)| = o_h(t)$$

uniformly on $x, z \in S_X$, $\lambda \in [0,1]$. This implies, of course,

$$\|n^{(k-1)}(x+th) - n^{(k-1)}(x) - tn^{(k)}(x; \cdot h)\| = o_h(t),$$

i.e., $n^{(k-1)}$ is UG-smooth. The main theorem is proved. □

Remark 4. We note that from Remark 1 and Theorem 6 in [**9**] it follows that the norm n from above is also UF^{k-1} smooth.

5. Applications. An easy consequence of the main theorem is the following

Theorem 1. *Let M be an Orlicz function that satisfies*

i) $1 \le k \le \alpha_M \le \beta_M < \infty$,

ii) $M(uv) \le c_0 u^k M(v)$, $u \in [0,1]$, $v \in \mathbf{R}^+$ *for some positive* c_0,

iii) $\lim_{u\to 0} M(u)/u^k = 0$.

For any measure space (S, Σ, μ), μ *a positive measure, in* $X = L_M(S, \Sigma, \mu)$ *there is an equivalent* UG^k*-smooth norm.*

Proof. Put $M_1(t) = \int_0^t M(u)\,du/u$. Obviously,

$$\frac{M(t/2)}{2} \le M_1(t) \le M(t), \qquad t \in \mathbf{R}, \tag{32}$$

i.e., $M_1 \sim M$ at 0 and ∞. Denote

$$\rho(t) = \begin{cases} M_1(t)/t^k & t > 0, \\ 0 & t = 0, \end{cases}$$

and $f_M^k(t) = \max\{\rho(u); u \in [0,t]\}$, $t \ge 0$.

We first prove that $f_M^k \in F(c)$ for some positive c. Indeed, according to (2) and (32) we have for $\beta = \beta_M + 1$,

$$M(t) \le c_\beta 2^\beta M\left(\frac{t}{2}\right) \le c_\beta 2^{\beta+1} M_1(t), \qquad t \in \mathbf{R}. \tag{33}$$

Let $0 \le a < b$, $d = \max\{u \in ([0,b]; \rho(u) = f_M^k(b)\}$. Obviously, $f_M^k(a) = f_M^k(b)$, whenever $d \le a$. If $a < d$ using the convexity of M_1, (32) and (33) we get for some $\theta \in (0,1)$:

$$\begin{aligned} f_M^k(b) - f_M^k(a) &\le \frac{M_1(d) - M_1(a)}{d^k} \\ &= (d-a)\frac{M(a+\theta(d-a))}{d^k(a+\theta(d-a))} \\ &\le \frac{(d-a)M(d)}{d^{k+1}} \\ &\le c_\beta 2^{\beta+1}\frac{(b-a)f_M^k(b)}{b}, \end{aligned}$$

i.e., (3) with $c = c_\beta 2^{\beta+1}$. Obviously, f_M^k is a nondecreasing, continuous function on $[0,\infty]$ and $f_M^k(0) = 0$. According to the main theorem, $X = L_N(S,\Sigma,\mu)$, $N = I^k(f_M^k)$, is UG^k-smooth and, to finish the proof, we have to show only that $N \sim M$ at 0 and ∞. Remark 1 and ii) imply

$$\begin{aligned} N(t) &\le |t|^k f_M^k(|t|) \le N(2^k t),\\ \frac{M(t/2)}{2} &\le M_1(t) \le |t|^k f_M^k(|t|)\\ &\le \max\left\{\left(\frac{|t|}{u}\right)^k M_1(u); u \in [0,|t|]\right\}\\ &\le c_0 M(t). \end{aligned}$$

These inequalities obviously imply

$$\frac{M(t/2^{k+1})}{2} \le N(t) \le c_0 M(t), \qquad t \in \mathbf{R}^+.$$

Thus, Theorem 1 is proved. □

Corollary 2. *For μ a σ-finite measure on S and $p \in \mathbf{N}$, the spaces $L_{2p-1}(S,\Sigma,\mu)$ admit equivalent UG^{2p-1}-smooth renormings.*

Proof. As $L_{2p-1}(S,\Sigma,\mu)$ for a σ-finite measure μ, is isometric to a subspace of $L_{2p-1}(S,\Sigma,\nu)$ for a suitable probability measure ν, we may consider only the case $\mu(S) < \infty$, S free of atoms. Put

$$M(t) = \int_0^t \frac{M_1(u)du}{u}, \qquad M_1(u) = \begin{cases} u^{2p}, & u \in [0,1],\\ u^{2p-1}, & u \ge 1.\end{cases}$$

Obviously $M \sim t^{2p-1}$ at ∞, $M \sim t^{2p}$ at 0 and

i) $\alpha_M^\infty = \beta_M^\infty = \alpha_M = \beta_M = 2p-1$;

ii) $M(uv) \le u^{2p-1}M(v)$, $u \in [0,1]$, $v \in \mathbf{R}^+$;

iii) $\lim_{u\to 0} \frac{M(u)}{u^{2p-1}} = 0$.

According to Theorem 1 there is an equivalent UG^{2p-1}-smooth norm on $L_M(S,\Sigma,\mu)$. To finish the proof it is enough to observe that $L_M(S,\Sigma,\mu)$ is isomorphic to $L_{2p-1}(S,\Sigma,\mu)$. □

Remark 5. This corollary sharpens the result from [T], where G^{2p-1}-smooth equivalent norms are found on $L_{2p-1}(S,\Sigma,\mu)$, where μ is σ-finite and $p \in \mathbf{N}$. Especially, the same is true for l_{2p-1}, $L_{2p-1}(0,1)$ and $L_{2p-1}(0,\infty)$, $p \in \mathbf{N}$. We note also that the existence of UG-smooth renorming in $L_1(0,1)$ is well known and follows from general considerations.

Corollary 3. *Let $k = \alpha_M^0(\alpha_M^\infty) \le \beta_M^0(\beta_M^\infty) < \infty$, $k \in \mathbf{N}$ and*

$$(34)\qquad \begin{aligned} M(uv) &\le c_0 u^k M(v), \qquad u,v \in [0,1] \\ (M(uv) &\ge c_0 u^k M(v), \qquad u,v \in [1,\infty)) \end{aligned}$$

for some positive constant c_0. Then in $l_M(L_M(0,1))$ there exists an equivalent UG^k-smooth norm.

Proof. Let us consider first the sequence case. If $\lim_{u\to 0} M(u)/u^k = 0$ we may apply Theorem 1 for the function

$$N(t) = \int_0^{|t|} M_1(u)\frac{du}{u}, \qquad M_1(u) = \begin{cases} M(u)/M(1) & u \in [0,1], \\ u^{k+1} & u \in [1,\infty), \end{cases}$$

that is equivalent to M at 0.

If $\lim_{u\to 0} M(u)/u^k = 0$ does not hold, choose a sequence $\{u_n\}_{n=1}^\infty$, $u_n > 0$, such that $M(u_n)/u_n^k \ge a > 0$, $\lim_{n\to\infty} u_n = 0$. From the inequality $M(u)/u^k \le c_0 M(v)/v^k$, $0 < u \le v \le 1$, it obviously follows that $M(u)/u^k \ge a/c_0$, $u \in (0,1]$, that combined with $M(u) \le c_0 u^k M(1)$ implies $M \sim t^k$ at 0. For k odd the result now follows from Remark 5. If k is even, in l_M isomorphic to l_k, there exists even UF^∞-smooth norm [**1, 12**].

To prove the results for $L_M(0,1)$ it is sufficient to apply Theorem 1 for the function

$$N(t) = \int_0^{|t|} M_1(u)\frac{du}{u}, \qquad M_1(u) = \begin{cases} u^{k+1} & u \in [0,1], \\ M(u)/M(1) & u \in [1,\infty), \end{cases}$$

that is equivalent to M at ∞. $\square$

Remark 6. We note that the conditions (34) with $k = \alpha_M^0(\alpha_M^\infty)$ easily imply that in $l_M(L_M(0,1))$ there exists an equivalent G^k-smooth norm, whose $(k-1)$th derivative is locally Lipschitzian. Indeed, from [**8**, Corollary 3] it follows that $l_M(L_M(0,1))$ has an F^{k-1}-smooth norm with locally Lipschitzian $(k-1)$th derivative. Thus the assertion follows directly from [**5**, Theorem 3.1].

We finish with some

Examples. Let $M(t) = t^p(1+|\ln t|)^q$, $p \in \mathbf{N}$. It is easy to check that $\alpha_M = \beta_M = p$.

a) If $q < 0$ there is an l_M equivalent UG^p-smooth norm;

b) if $q > 0$ there is an $L_M(0,1)$ equivalent UG^p-smooth norm.

We note that in both cases there are no bump functions that are F^p-smooth.

Acknowledgments. The author acknowledges his gratitude to S. Troyanski for valuable discussions.

REFERENCES

1. R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. **15** (1966), 877–898.

2. R. Deville, *A characterization of C^∞-smooth Banach spaces*, Bull. London Math. Soc. **22** (1990), 13–17.

3. ———, *Geometrical implications of the existence of very smooth bump functions in Banach spaces*, Israel J. Math. **67** (1989), 1–22.

4. R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs Surveys Pure Appl. Math. **64**, Longman Scientific & Technical, copublished by John Wiley & Sons, New York, 1992.

5. M. Fabian, J.H.M. Whitfield and V. Zizler, *Norms with locally Lipschitzian derivatives*, Israel J. Math. **44** (1983), 262–276.

6. L.V. Kantorovich and G.P. Akilov, *Functional analysis*, Nauka, Moskow, 1977 (in Russian).

7. M. Krasnoselskii and Ya. Rutickii, *Convex functions and Orlicz spaces*, Groningen, Wolters-Noordorf, 1961.

8. R.P. Maleev, *Norms of best smoothness in Orlicz spaces*, Z. Anal. Anwendungen **12** (1993), 123–135.

9. R.P. Maleev and S.L. Troyanski, *Smooth norms in Orlicz spaces*, Canad. Math. Bull. **34** (1991), 74–82.

10. ———, *Smooth functions in Orlicz spaces*, Contemp. Math. **85** (1989), 355–370.

11. S.L. Troyanski, *Gâteaux differentiable norms in L_p*, Math. Ann. **287** (1990), 221–227.

12. K. Sundaresan, *Smooth Banach spaces*, Math. Ann. **173** (1967), 191–199.

13. K. Sundaresan and S. Swaminathan, *Geometry and nonlinear analysis in Banach spaces*, Lecture Notes in Math. **1131**, Springer-Verlag, Berlin-New York, 1985.

Department of Mathematics and Informatics, University of Sofia, 5 J Bourchier Blvd., 1126 Sofia, Bulgaria