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ON THE ZETA FUNCTION VALUES $\zeta(2k+1)$, $k = 1, 2, \ldots$

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ABSTRACT. In determinantal form new series representations of the values $\zeta(2k+1) := \sum_{n=1}^{\infty} n^{-2k-1}$, $k = 1, 2, \ldots$, are presented. These follow from a certain trigonometrical identity, which seems to have some independent interest.

1. Introduction. The Riemann zeta function ζ is defined for each complex number s having real part greater than 1 as follows.

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

As intimated in the title we are here concerned about the values $\zeta(s)$ when s is restricted to odd integral values not less than 3. Apery [**1**] helped to rekindle interest in these values when he established the irrationality of $\zeta(3)$. However, for each integer $k > 1$, the arithmetical character of $\zeta(2k+1)$ is entirely unsettled. Several authors have found new series representations for some or all of the values $\zeta(2k+1)$, $k = 1, 2, \ldots$. Ramanujan [**4**] discovered (without proof) that: if α and β are positive real numbers such that $\alpha\beta = \pi^2$ and n is a positive integer, then

$$\begin{aligned}\alpha^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right\} &= (-\beta)^{-n}\left\{\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right\} \\ &\quad - 2^{2n}\sum_{k=0}^{n+1}(-1)^k \frac{B_{2k}}{(2k)!}\frac{B_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^k,\end{aligned}$$

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where the coefficients B_{2k}, $k = 0, 1, 2, \dots$, are Bernoulli numbers defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j.$$

For the particular value $\zeta(3)$ we have the following three formulas due, respectively, to Grosswald [**3**], Terras [**5**] and Apery [**1**]:

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n),$$

$$\zeta(3) = \frac{2}{45}\pi^3 - 4\sum_{n=1}^{\infty} e^{-2\pi n}\sigma_{-3}(n)\left\{2\pi^2 n^2 + \pi n + \frac{1}{2}\right\},$$

$$\zeta(3) = \frac{5}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

(Of course, $\sigma_{-3}(n) := \sum d^{-3}$, the sum extending over all positive integral divisors of n.)

In [**2**], the author showed that, for each integer $r > 2$,

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-2)\frac{\pi^{2m}}{(2m+2)!},$$

where for each pair (m, r) of integers, $m \geq 0$ and $r \geq 1$, $A_{2m}(r)$ is defined as follows:

(i) $A_{2m}(1) := B_{2m}$ and

(ii) for $r > 1$,

$$A_{2m}(r) := \sum \frac{\binom{2m}{2i, 2i_2, \dots, 2i_r} B_{2i_1}B_{2i_2}\cdots B_{2i_r}}{\{2i_1+1\}\{2(i_1+i_2)+1\}\cdots\{2(i_1+i_2+\cdots+i_{r-1})+1\}},$$

where the sum is extended over all r-tuples $(i_1, i_2, \dots, i_r)$ of nonnegative integers such that $i_1 + i_2 + \cdots + i_r = m$, and

$$\binom{2m}{2i_1, 2i_2, \dots, 2i_r}$$

is a multinomial coefficient.

In this paper we present new series representations of $\zeta(2k+1)$, $k = 1, 2, \ldots$. These are given in the third corollary of the following theorem (which also seems to have some independent interest).

Theorem 1. *For each real number* $x \in [0, \pi/2)$,

$$(1) \qquad -x\log x + x + \sum_{n=0}^{\infty} \frac{-1}{(2n+1)^2}\sin[2(2n+1)x] - 2\sum_{m=1}^{\infty} \frac{(2^{2m-1}-1)\zeta(2m)}{\pi^{2m} m(2m+1)} x^{2m+1} = i\left\{\frac{\pi}{2}x - \frac{\pi^2}{8} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos[2(2n+1)x]\right\}.$$

The details of the proof are presented in Section 2. (Note that $\lim_{\varepsilon\to 0^+}(-\varepsilon\log\varepsilon) = 0$.)

2. Proof of Theorem 1. We take as our point of departure the well-known representations of $\sin t$ and $\cos t$ by infinite products. Let $0 \le x < \pi/2$. Then, on the one hand,

$$\begin{aligned}\int_0^x \log(t\cot t)\,dt &= \int_0^x \left\{\sum_{n=0}^{\infty} \log\left\{1 - \frac{t^2}{[(2n+1)(\pi/2)]^2}\right\} - \sum_{n=1}^{\infty}\log\left\{1 - \frac{t^2}{(n\pi)^2}\right\}\right\} dt \\ &= -\int_0^x \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m}\frac{t^{2m}}{(2n+1)^{2m}(\pi/2)^{2m}}\,dt \\ &\quad + \int_0^x \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m}\frac{t^{2m}}{n^{2m}\pi^{2m}}\,dt \\ &= -\sum_{m=1}^{\infty}\frac{2^{2m}-1}{2^{2m}}\zeta(2m)\frac{2^{2m}}{m\pi^{2m}(2m+1)}x^{2m+1}\end{aligned}$$

$$+\sum_{m=1}^{\infty}\zeta(2m)\frac{1}{m\pi^{2m}(2m+1)}x^{2m+1}$$
$$=-2\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{m\pi^{2m}(2m+1)}x^{2m+1}.$$

Here we have used termwise integration and the observation

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m}}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2m}}+\sum_{n=1}^{\infty}\frac{1}{(2n)^{2m}},$$

so that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2m}}=\frac{2^{2m}-1}{2^{2m}}\zeta(2m).$$

On the other hand, since

$$\cot t=i\cdot\frac{e^{it}+e^{-it}}{e^{it}-e^{-it}}=i\cdot\frac{e^{it}(1+e^{-2it})}{e^{it}(1-e^{-2it})},$$

we have

$$\begin{aligned}\int_0^x \log(t\cot t)\,dt &= \int_0^x \log t\,dt + i\frac{\pi}{2}\int_0^x dt\\ &\quad+\int_0^x \log(1+e^{-2it})\,dt\\ &\quad-\int_0^x \log(1-e^{-2it})\,dt\\ &= x\cdot\log x - x + i\frac{\pi}{2}x\\ &\quad+\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\int_0^x e^{-2imt}\,dt\\ &\quad+\sum_{m=1}^{\infty}\frac{1}{m}\int_0^x e^{-2imt}\,dt\\ &= x\cdot\log x - x + i\frac{\pi}{2}x\\ &\quad-i\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\end{aligned}$$

$$
\begin{aligned}
&+ i\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}e^{-2(2n+1)ix}\\
&= x\cdot\log x - x\\
&\quad+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sin[2(2n+1)x]\\
&\quad+ i\left\{\frac{\pi}{2}x-\frac{\pi^2}{8}+\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos[2(2n+1)x]\right\}.
\end{aligned}
$$

Now, equating the two evaluations of $\int_0^x \log(t\cot t)\,dt$ and subsequently simplifying, we obtain (1).

Notation. For positive integers k, m, let

$$
\begin{aligned}
C(m) &:= \frac{(2^{2m-1}-1)\zeta(2m)}{(2\pi)^{2m}m},\\
\gamma(2k+1) &:= \sum_{n=0}^{\infty}(2n+1)^{-2k-1},\\
[2m+1]_{2k} &:= (2m+1)(2m+2)\cdots(2m+2k).
\end{aligned}
$$

Clearly, these are definitions to abbreviate.

Corollary 1. *For each positive integer k and each $x \in [0,\pi)$,*

$$
\begin{aligned}
\sum_{n=0}^{\infty}&\frac{1}{(2n+1)^{2k+1}}\cos[(2n+1)x]\\
&= \sum_{j=0}^{k-1}(-1)^j\gamma(2(k-j)+1)\frac{x^{2j}}{(2j)!}\\
&\quad+\frac{(-1)^{k-1}}{2(2k)!}\left\{x^{2k}\log x - x^{2k}\sum_{j=1}^{2k}\frac{1}{j}-(\log 2)x^{2k}\right\}\\
&\quad+(-1)^{k-1}\sum_{m=1}^{\infty}\frac{C(m)}{[2m+1]_{2k}}x^{2m+2k};
\end{aligned}
\tag{2}
$$

and

$$\text{(3)}\quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2k+2}}\sin[(2n+1)x]$$
$$=\sum_{j=0}^{k=1}(-1)^j\gamma(2(k-j)+1)\frac{x^{2j+1}}{(2j+1)!}$$
$$+\frac{(-1)^{k-1}}{2(2k+1)!}\left\{x^{2k+1}\log x-x^{2k+1}\sum_{j=1}^{2k+1}\frac{1}{j}-(\log 2)x^{2k+1}\right\}$$
$$+(-1)^{k-1}\sum_{m=1}^{\infty}\frac{C(m)}{[2m+1]_{2k+1}}x^{2m+2k+1}.$$

Proof. (By induction on k). Since the left side of (1) is real and the right side is purely imaginary, both sides vanish, whence

$$\sum_{n=0}^{\infty}\frac{-1}{(2n+1)^2}\sin[2(2n+1)x]$$
$$=x\log x-x+2\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{\pi^{2m}m(2m+1)}x^{2m+1}.$$

In the foregoing identity put $t:=2x$ to get

$$\sum_{n=0}^{\infty}\frac{-1}{(2n+1)^2}\sin[(2n+1)t]=\frac{1}{2}\{t\log t-(\log 2)t-t\}$$
$$+\sum_{m=1}^{\infty}\frac{C(m)}{2m+1}t^{2m+1},$$

$0\le t<\pi$. Now we operate on the foregoing identity with $\int_0^x(\cdot)\,dt$ to get

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos[(2n+1)x]$$
$$=\gamma(3)+\frac{1}{2^2}\left\{x^2\log x-x^2\left(1+\frac{1}{2}\right)-(\log 2)x^2\right\}$$
$$+\sum_{m=1}^{\infty}\frac{C(m)}{[2m+1]_2}x^{2m+2}$$

This is (2) for $k = 1$. Next, in the foregoing identity we let $x \to t$ and subsequently operate on the resulting identity with $\int_0^x (\cdot)\, dt$ to get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \sin[(2n+1)x]$$
$$= \gamma(3)x + \frac{1}{2.3!}\left\{x^3 \log x - x^3\left(1 + \frac{1}{2} + \frac{1}{3}\right) - (\log 2)x^3\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{C(m)}{[2m+1]_3} x^{2m+3}.$$

And this is (3) for $k = 1$.

Now for a fixed but arbitrary choice of k, we assume inductively that identities (2) and (3) hold. Then, in (3) we let $x \to t$, subsequently operate on the resulting identity with $\int_0^x (\cdot)\, dt$ and simplify to get an identity which is formally exactly like (2), but where k has been everywhere replaced by $k + 1$. In this last mentioned identity we then let $x \to t$ and subsequently operate on the resulting identity with $\int_0^x (\cdot)\, dt$ to get an identity which is formally exactly like (3), but where k has been everywhere replaced by $k + 1$. Inductively this establishes Corollary 1.

Corollary 2. *For each positive integer k,*

(4)
$$\sum_{j=1}^{k-1} (-1)^{k-j} \frac{(\pi/2)^{2k-2j}}{(2k-2j)!} \gamma(2j+1) + \gamma(2k+1)$$
$$= (-1)^{k+1} \frac{(\pi/2)^{2k}}{2 \cdot (2k)!} \left\{ \sum_{j=1}^{2k} \frac{1}{j} + \log \frac{4}{\pi} \right.$$
$$\left. - 2 \cdot (2k)! \sum_{m=1}^{\infty} \frac{C(m)}{[2m+1]_{2k}} \left(\frac{\pi}{2}\right)^{2m} \right\}$$
$$:= y_k, \quad \textit{say.}$$

Proof. In (2) let $x := \pi/2$.

For a given positive integer k, we adopt the further abbreviations:

$$a_{kj} := (-1)^{k-j} \frac{(\pi/2)^{2k-2j}}{(2k-2j)!}, \qquad x_j := \gamma(2j+1),$$

$j = 1, 2, \dots, k$; and realize that (4) has the form

$$\sum_{j=1}^{k-1} a_{kj} x_j + x_k = y_k.$$

Hence, for a fixed but arbitrary choice of k, we consider the first k such linear equations in $x_1, x_2, \dots, x_k$, where the matrix A_k of coefficients is given by

$$A_k := \begin{bmatrix} 1 & & & & & \\ a_{21} & 1 & & & 0 & \\ a_{31} & a_{32} & 1 & & & \\ \vdots & \vdots & \vdots & & \ddots & \\ a_{k1} & a_{k2} & a_{k3} & \cdots & a_{k,k-1} & 1 \end{bmatrix}.$$

More precisely, A_k has size $k \times k$; all entries of A_k above the diagonal are zeros; the diagonal entries of A_k are all ones; and the entries of A_k below the diagonal are as displayed. Clearly, $\det A_k = 1$. By Cramer's rule we then solve for $x_k := \gamma(2k+1)$ in terms of the y_r, $r = 1, 2, \dots, k$; and, subsequently, multiply x_k by $2^{2k+1}/(2^{2k+1}-1)$ to find $\zeta(2k+1)$. Thus, we have proved

Corollary 3. *For each positive integer k,*

$$\zeta(2k+1) = \frac{2^{2k+1}}{2^{2k+1}-1} \det \begin{bmatrix} 1 & & & & y_1 \\ a_{21} & 1 & & 0 & y_2 \\ \vdots & \vdots & \ddots & & \vdots \\ & & & 1 & y_{k-1} \\ a_{k1} & a_{k2} & \cdots & a_{k,k-1} & y_k \end{bmatrix}.$$

Now the foregoing determinantal expression for $\zeta(2k+1)$ is essentially one in "closed form." However, we should perhaps illustrate its usage for "small" values of k, say $k \in \{1,2,3\}$. First of all, we realize that for each positive integer k,

$$y_k = \frac{(-1)^{k+1}(\pi/2)^{2k}}{2\cdot(2k)!}\Bigg\{\sum_{j=1}^{2k}\frac{1}{j}+\log(4/\pi) - 2\cdot(2k)!\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_{2k}}\Bigg\}.$$

Then

$$\zeta(3) = \frac{\pi^2}{14}\Bigg\{\sum_{j=1}^{2}\frac{1}{j}+\log\frac{4}{\pi}-4\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_2}\Bigg\},$$

$$\zeta(5) = \frac{\pi^4}{124}\Bigg\{\sum_{j=1}^{2}\frac{1}{j}+\log\frac{4}{\pi}-4\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_2}\Bigg\} - \frac{\pi^4}{744}\Bigg\{\sum_{j=1}^{4}\frac{1}{j}+\log\frac{4}{\pi}-48\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_4}\Bigg\},$$

$$\zeta(7) = \frac{5\pi^6}{5,096}\Bigg\{\sum_{j=1}^{2}\frac{1}{j}+\log\frac{4}{\pi}-4\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_2}\Bigg\} - \frac{\pi^6}{5,096}\Bigg\{\sum_{j=1}^{4}\frac{1}{j}+\log\frac{4}{\pi}-48\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_4}\Bigg\} + \frac{\pi^6}{94,440}\Bigg\{\sum_{j=1}^{6}\frac{1}{j}+\log\frac{4}{\pi}-1440\sum_{m=1}^{\infty}\frac{(2^{2m-1}-1)\zeta(2m)}{2^{4m}m[2m+1]_6}\Bigg\}.$$

Concluding remarks. Earlier we noted that both sides of identity (1) vanish. The author has already observed vanishing of the right side of this identity in an investigation similar to the present one. Accordingly, consequences of this vanishing are presented elsewhere.

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