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ON THE UNIQUENESS OF THE POSITIVE SOLUTION OF A SINGULARLY PERTURBED PROBLEM

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0. Introduction. A number of authors have considered the existence of multiple positive solutions of

$$(1)\qquad \begin{aligned} -\varepsilon\Delta u &= u^p - u \quad \text{in } D,\\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

if ε is small, D is suitably complicated in R^n and

$$1 < p < (n-2)^{-1}(n+2).$$

See [**1, 20, 24** and **25**]. (Some of these consider the Neumann problem.) Here we consider the opposite situation and show that, if D has n distinct symmetries and some other properties (for example some form of generalized ellipsoid) and if $1 < p < (n-2)^{-1}(n+2)$, then the positive solution is unique for small positive ε. This provides an interesting contrast with the results above. Note that the results in [**3**] suggest that some strong geometric conditions on D are necessary for this result to be true. We actually discuss rather more general nonlinearities. Note that the behavior of the positive solutions for small ε is quite different from the cases in [**5**].

We also make some remarks on the case of large ε and the very different behavior of the Neumann problem. The very different behavior of the problem under different boundary conditions is another source of interest in the problem.

1. The main result. In this section we prove the main result. We consider a domain $D \subseteq R^n$ such that $0 \in D$, D has C^3 boundary,

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D is invariant under the n reflections in the coordinate planes and such that, in addition, if $1 \le i \le n$ and if $0 < t < s < \tilde{t}_i$, then $(I - P_i)D_{i,t} \supseteq (I - P_i)D_{i,s}$. Here P_i is the orthogonal projection onto span e_i, $D_{i,s} = \{x \in D : x_i = s\}$, $\tilde{t}_i = \sup\{x_i : x \in D\}$, and $\{e_i\}$ denotes the usual basis for R^n. We say that such a domain is of type R_n.

Theorem 1. *Assume that* $1 < p < (n-2)^{-1}(n+2)$ $(p > 1$ *if* $n = 1, 2)$ *and* D *is of type* R_n. *Then* (1) *has a unique nontrivial positive solution for small positive* ε.

The key to the above proof is to establish the possible asymptotic behavior of positive solutions as $\varepsilon \to 0$. Here we consider a more general problem. We assume that $g : R \to R$ is C^1, $g(0) = 0$, $g'(0) < 0$ and there exists $a > 0$ such that $g(y) < 0$ on $(0, a)$, $g(y) > 0$ on (a, ∞), $g(y) \sim y^p$ as $y \to \infty$ where $1 < p < (n-2)^{-1}(n+2)$ $(p > 1$ if $n = 1, 2)$. In addition, if $n \ge 3$, we assume that there is a $\tau \le n/(n-2)$ such that $g(y) \ge K_1(y - a)^\tau$ if y is near a and $y \ge a$ (where $K_1 > 0$). Alternatively, one could replace this last condition to assume that g is increasing on $[a, \infty)$ and $(y - a)^{-(n+2)(n-2)^{-1}} g(y)$ is strictly decreasing on (a, ∞).

We consider the equation

$$\begin{aligned} -\varepsilon \Delta u &= g(u) \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{2}$$

Proposition 1. *Assume that the above conditions on* g *hold, that* D *is of type* R_n *and* u_i *are positive solutions of* (2) *for* $\varepsilon = \varepsilon_i$ *where* $\varepsilon_i \to 0$ *as* $i \to \infty$. *By choosing a subsequence, if necessary, there is a positive radial solution* v *of*

$$-\Delta u = g(u) \tag{3}$$

on R^n *such that* $v(x) \to 0$ *as* $||x|| \to \infty$ *and* $u_i - v(\varepsilon^{-1/2} x)$ *converges uniformly to zero on* D *as* $i \to \infty$.

This is the key result for the proof of the theorem. Note that there can be essentially different behavior in other types of domains. (For example, consider radially symmetric solutions on annuli.) We could prove related results by various other growth conditions at infinity by combining the ideas here with those in [**5**] and the weak Harnack inequality. We discuss this later. We will prove the proposition by a series of lemmas. Define $\mu \in (a, \infty)$ by $\int_0^\mu g = 0$.

Lemma 1. *Assume that u_i are as in the statement of Proposition 1. Then there is a $K > 0$ such that $\mu \leq ||u_i||_\infty \leq K$ for large i.*

Proof. The first inequality follows from Theorem 2 in [**10**]. The second inequality will be proved by blowing up. Suppose the second inequality is false. Then, by choosing a subsequence, if necessary, we can assume that $||u_i||_\infty \to \infty$ as $i \to \infty$. Note that, by the Gidas-Ni-Nirenberg theorem, u_i has its maximum at zero. We write $u_i = ||u_i||_\infty w_i$ and rescale the coordinates by a change of variable $X_j = \tau_i x_j$ for $1 \leq j \leq n$ where $\tau_i^2 = \varepsilon_i^{-1}(||u_i||_\infty)^{p-1}$. By a simple calculation, $||w_i||_\infty = 1$, w_i is defined on $\tau_i D$ and

$$-\Delta w_i = (||u_i||_\infty)^{-p} g(||u_i||_\infty w_i) \quad \text{on } \tau_i D. \tag{4}$$

Here the Laplacian is in the new variables. By our assumption on g, and the definition of τ_i, we easily see that

$$(||u_i||_\infty)^{-p} g(||u_i||_\infty w_i(x)) - c(w_i(x))^p$$

tends to zero uniformly on any set T where $w_i(x)$ has a positive lower bound (or more generally on any set T such that $||u_i||_\infty w_i(x)$ tends to infinity uniformly on T as i tends to infinity). Here we use that $y^{-p}g(y) \to c$ as $y \to \infty$. On the other hand, if $\{||u_i||_\infty w_i(x)\}$ is bounded, $\{g(||u_i||_\infty w_i(x))\}$ is bounded and hence $(||u_i||_\infty)^{-p} g(||u||_\infty w_i(x))$ is small for large i. Hence we see that $(||u_i||_\infty)^{-p} g(||u_i||_\infty w_i(x)) - cw_i(x)^p$ tends to zero uniformly as $i \to \infty$. Since u_i has its maximum at 0, so does w_i (where $w_i(0) = 1$) and since the distance of 0 from the boundary of $\tau_i D$ tends to infinity as $i \to \infty$, we can use a standard limiting argument (cf. page 441 of [**5**]) to deduce that a subsequence of the w_i converges uniformly on compact subsets of R^n to a solution w of $-\Delta w = cw^p$ in R^n such that $||w||_\infty \leq 1$, $w(0) = 1$, $w \geq 0$. By a result of Gidas and Spruck [**13**], this is impossible and hence our original assumption is false and $\{||u_i||_\infty\}$ is bounded. This completes the proof of the lemma. □

Remark. By a more careful proof (using boundary blow ups) this result is true for any smooth bounded domain D in R^n. Moreover, if $\widetilde{K} > 0$ the bound holds for all positive solutions u corresponding to ε with $0 < \varepsilon \leq \widetilde{K}$.

Lemma 2. *There exist $l \in (0, a)$ and $K_2 > 0$ such that $u_i(x) \leq l$ if $x \in D$ and $|x_j| \geq K_2 \varepsilon_i^{1/2}$ for all j.*

Proof. Suppose not. We can choose a subsequence of the u_i and $x^i \in D$ such that $u_i(x^i) \to a$ as $i \to \infty$ and $\varepsilon_i^{-1/2} x_j^i \geq \tilde{\tau}_i$, for $1 \leq j \leq n$ where $\tilde{\tau}_i \to \infty$ as $i \to \infty$. Here x_j^i are the components of x^i. (We use the evenness to ensure $x_j^i \geq 0$.) We use a blow up argument again. This time we rescale $X_j = \varepsilon_i^{-1/2} x_j$ for $1 \leq j \leq n$ and use $\tilde{u}_i$ to denote u_i after the rescaling of x. Note that $\tilde{u}_i(0) \geq \mu$ by Lemma 1. Much as in Lemma 1, we find that a subsequence of $\tilde{u}_i$ converges uniformly on compact subsets of R^n to a solution w of $-\Delta u = g(u)$ in R^n such that w is bounded and nonnegative and $w(0) \geq \mu$. We prove that $w < a$ somewhere in R^m. If not, $w \geq a$ in R^m and $z = w - a$ is a nonnegative bounded solution of $-\Delta z = g(a + z)$ in R^n. Since $g(a + t) \geq c_1 t^\tau$ for small positive t if $n \geq 3$ and $g(a + z)(x) > 0$ on R^n, this is impossible by Proposition 3 in [**5**]. (Note that z is nontrivial since $z(0) > \mu - a$ and that, since z is superharmonic, it follows that z is positive on R^n.) Hence there exists $\tilde{x} \in R^n$ such that $w(\tilde{x}) < a$. Since w is even in each variable (because each u_i is) we can assume $\tilde{x} \geq 0$. Hence, for large i, $\tilde{u}_i(\tilde{x}) < l < a$ (where $w(\tilde{x}) < l < a$). Now, if i is large, $\varepsilon_i^{-1/2} x^i \geq \tilde{x}$ (by our choice of x^i at the beginning of the proof). Thus, by the Gidas-Ni-Nirenberg theorem, $\tilde{u}_i(\varepsilon_i^{-1/2} x') \leq \tilde{u}_i(\tilde{x}_0) \leq l$. This contradicts our choice of x^i at the beginning of the proof. □

The main difficulty of the remainder of the proof is to show that u_i must be less than a on most of $S = \{x \in D : x_i = 0 \text{ for some } i\}$. Let S' be the set of points in D on the coordinate axes e_i, $i = 1, \dots, n$. We call this the spine. We can easily deduce from Lemma 2 by multiplying (1) by the first eigenfunction of the Laplacian that u must be small on most of D.

Lemma 3. *There exists $K > 0$ and $l_1 < a$ such that $u_i(x) \leq l_1$ if $x \in D$ and $|x| \geq K\varepsilon_i^{1/2}$.*

Remark. This is the key lemma.

Proof. This is proved by a series of blowing ups and by the consideration of full space and half space problems.

Suppose that the result is false. By the various decreasing properties of the u_i's, we see that there exists $x^i \in S'$ such that $u_i(x^i) \to a$ and $\varepsilon_i^{-1/2}|x^i| \to \infty$ as $i \to \infty$. Without loss of generality, we can assume that x^i lies on the e_j axis. There are two cases to consider. Either (after taking a subsequence if necessary) $\{\varepsilon_i^{-1/2} d(x^i, \partial D)\}$ is bounded or $\varepsilon_i^{-1/2} d(x^i, \partial D) \to \infty$ as $i \to \infty$.

Case (i). $\varepsilon_i^{-1/2} d(x^i, \partial D) \to \infty$ as $i \to \infty$. In this case we shift the origin to x^i and then rescale $X_k = \varepsilon_i^{-1/2}(x - x_k^i)$ for $1 \leq k \leq n$. Note that, in the new variables, zero is a large distance from the boundary. Note also that we may assume that $\varepsilon_i^{-1/2} x_j^i \to \infty$ as $i \to \infty$ (after a further choice of subsequence if necessary and using the evenness). Then, since $u_i(x)$ is decreasing in x_j for $x_j \geq 0$, we see that, in the new variables, u_i is decreasing in X_j on any compact subset of R^n for i large. Note that in the new variables $u_i(0) \to a$ as $i \to \infty$. Thus, by a now standard blowing up argument, we obtain a bounded nonnegative solution $\bar{u}$ of $-\Delta u = g(u)$ in R^n such that $\bar{u}(0) = a$ and $\bar{u}$ is decreasing in X_j on R^n (not necessarily strictly). In addition, $\bar{u}$ is even in X_k for $k \neq j$ and $\bar{u}$ is decreasing in X_k for $X_k \geq 0$ (again not necessarily strictly). By Lemma 2, $\bar{u}$ cannot be constant (because after the rescaling, there must be points at a bounded distance $\tilde{K}$ from zero where $\tilde{u}_i$ is less than or equal to l, and hence the limit $\bar{u}$ must have the same property). It follows that $\bar{u}$ cannot be independent of X_j. Otherwise, $\bar{u} \leq a$ in R^m (by our various decreasing properties) and $\bar{u}(0) = a$. Thus, since $g(a) = 0$, we easily see that $a - \bar{u}$ is a nonnegative nontrivial solution of a homogeneous linear elliptic equation on R^n and vanishes at 0. This is impossible by the maximum principle. Hence $\bar{u}$ depends on X_j. Now $\partial\bar{u}/\partial X_j$ satisfies a linear elliptic equation by differentiating (3) in X_j. Since $\partial\bar{u}/\partial X_j \leq 0$ by our construction and does not vanish identically (by above), it follows from the maximum principle that $\partial\bar{u}/\partial X_j < 0$ in R^n. Hence $\bar{u}(x) > a$ at some points in R^n and $\bar{u}(x) < a$ at some points on the part of the spine $X_k = 0$ if $k \neq j$. Moreover, $\bar{u} < a$ if $|X_k| \geq K_2$ for $1 \leq k \leq n$, $k \neq j$. This follows because, by Lemma 2, $\tilde{u}_i(X_j, \hat{X}_j) \leq l$ if all the components of $\hat{X}_j$ have absolute value greater than or equal to K_2 provided X_j is not large. Here $\hat{X}_j$ is the set of components of X other than X_j. Thus we see in the limit that $\bar{u}(X_j, \hat{X}_j) \leq l$ for all X_j if all the components of $\hat{X}_j$ have absolute value greater than or equal to K_2. This contradicts Lemma 4(i) below (provided we replace X_j by $-X_j$). This proves case (i).

Case (ii). This is similar, except when we blow up we possibly obtain a solution $\bar{u}$ on a half space T (vanishing on the boundary) rather than the whole space. The rest of the argument is similar except we use Lemma 4(ii) rather than Lemma 4(i). Note that because $\bar{u} = 0$ on ∂T it is easier to see more directly in this case that $\bar{u}$ must depend on X_j. □

Lemma 4. (i) *There is no nonnegative bounded solution u of $-\Delta u = g(u)$ on R^n such that u is increasing in x_1, even in x_j for $j \geq 2$ and decreasing in x_j for $x_j \geq 0$ for $j \geq 2$, such that $u > a$ at some point and $u \leq l < a$ whenever $|x_j| \geq \tau > 0$ for $2 \leq j \leq n$ and such that $u < a$ at some point on the part of the spine $x_i = 0$ for $1 < i \leq n$.*

(ii) *There is no nonnegative bounded solution u of $-\Delta u = g(u)$ in $T_1 = \{x \in R^n : x_1 \geq 0\}$ such that $u = 0$ on ∂T_1, u is increasing in x_1. $u > a$ at some point, u is even in x_j for $j \geq 2$, u is decreasing in x_j for $x_j \geq 0$ and $j \geq 2$, and $u \leq l < a$ if $x_1 \geq 0$ and if $|x_j| \geq \tau > 0$ for $2 \leq j \leq n$.*

Remark. Clearly a similar result holds if we interchange the role of the coordinates. In part (ii) it can be shown that the nonnegativity forces u to be increasing in x_1.

Proof. (i) It is convenient to prove a more general result on u where:

($\tilde{a}$) u is positive and $u > a$ somewhere,

($\tilde{\text{b}}$) u is increasing in x_i for $i = 1, \dots, m$,

($\tilde{\text{c}}$) even in x_j for $m < j \le n$,

($\tilde{\text{d}}$) decreasing in x_j for $x_j > 0$ and $m < j \le n$ and

($\tilde{\text{e}}$) if $m < n$, $u \le l < a$ whenever $|x_j| \ge \tau > 0$ for $m < j \le n$, and $u < a$ somewhere on the set $x_i = 0$ whenever $i > m$.

It is easy to prove the result if $n = 1$ by using the first integral.

To prove the result, we assume that there is a counterexample with minimal n and with m maximal (among those with minimal n). We reduce to the case where $u(x) \to 0$ if $|x_{m+1}| \cdots + |x_n| \to \infty$ uniformly in x_i for $i \le m$. To do this, note that, since u is decreasing in x_{m+1}, we see as in part of the proof of Theorem 1 in [**6**], that $\hat{u}(\hat{x}_{m+1}) = \lim_{x_{m+1} \to \infty} u(x)$ is a solution of $-\Delta u = g(u)$ on R^{n-1} such that $\hat{u} < a$ somewhere on the set $x_i = 0$ whenever $i > m$ (since u has this property and since u is decreasing in x_{m+1} for $x_{m+1} \ge 0$) and $\hat{u}(x_2, \dots, x_n) \le l$ if $|x_j| \ge \tau$ for $m + 2 \le j \le n$. Thus we have reduced n by 1 if $\hat{u} > a$ somewhere. If $||\hat{u}||_\infty = a$ and $\hat{u}(x_0) = a$ for some $x_0 \in R^n$, we can argue as in the proof of Lemma 3 and deduce that $\hat{u} \equiv a$ in R^{n-1}. This is impossible since we know that $\hat{u} < a$ at some points of the spine. The next possibility is that $||\hat{u}||_\infty = a$ but the maximum is never achieved. There are two cases here. Firstly, there is the possibility that $m + 1 = n$. Thus $\hat{u}$ is increasing in all its variables. Now since $||\hat{u}||_\infty = a$, $\hat{u}$ is subharmonic and thus, by Proposition 3 in [**5**], it will approach its supremum $\hat{u}_{\text{sup}} = a$ as $x \to \infty$ along most rays through the origin. However, since $\hat{u} < a$ at some points (as noted above), $\hat{u}$ is not constant. It is easy to see the conditions that $\hat{u}$ is nonconstant, $\hat{u}$ is increasing in all the variables and $\lim_{r \to \infty} \hat{u}(re) = a$ for almost all directions e is impossible. Thus, we have a contradiction if $m + 1 = n$.

If $m + 1 < n$ and $||\hat{u}||_\infty = a$, we let $\hat{u} = \lim_{x_1 \to \infty} \hat{u}(x)$. Since $\hat{u}$ is increasing in x_1, we see as before that $\hat{u}$ is a solution of $-\Delta u = g(u)$ on R^{n-2}. Moreover, $\breve{u}(x) \le l$ if $|x_i| \ge K_2$ for $M + 1 < i \le n$ and $||\breve{u}||_\infty = a$. Once again, it is easy to obtain a contradiction if the maximum is achieved. By successively taking limits to remove the increasing variables, we would eventually end up with a solution $\tilde{u}$ of $-\Delta u = g(u)$ on R^{n-m-1} which is decreasing in all the x_i for $x_i \ge 0$, even in x_i, $\tilde{u}(x) \le l$ if $|x_i| \ge K_2$ for $1 \le i \le m - n$ and $||\tilde{u}||_\infty = a$. In this case it is easy to see that the maximum is achieved and hence as before $\tilde{u} \equiv a$, which is impossible. Thus, we have shown in all cases, $||\hat{u}||_\infty = a$ does not occur. If $0 < ||\hat{u}||_\infty < a$, we can use Proposition 3 in [**5**] to obtain a contradiction. Thus, the only remaining possibility is that $\hat{u} \equiv 0$, that is,

$$\lim_{x_{m+1}\to\infty} u = 0 \tag{5}$$

for all $\hat{x}_{m+1}$.

We prove that there is a counterexample with the same n and a larger m, unless $\lim_{x_{m+1}\to\infty} u = 0$ *uniformly* in $\hat{x}_{m+1}$. By our decreasing properties, it suffices to consider the case where the limit is not uniform for $\hat{x}_{m+1}$ in $T = \{x_i : x_i = 0 \text{ if } i > m+1\}$. Hence, if the limit is not uniform, there exist x_{m+1}^n large, $\hat{x}_{m+1}^n \in T$ and c in $(0,a)$ such that $u(x_{m+1}^n, \hat{x}_{m+1}^n) > c$ for all n. Since $u \to 0$ as $x_{m+1} \to \infty$, we can, by replacing x_{m+1}^n by a larger value, assume that $u(x_{m+1}^n, \hat{x}_{m+1}^n) = c$. By shifting the origin to $(x_{m+1}^n, \hat{x}_{m+1}^n)$ and passing to the limit as $n \to \infty$ much as before, we obtain a bounded nonnegative solution of $-\Delta u = g(u)$ on R^n such that $u(0) = c$, u is increasing in x_i for $1 \le i \le m$, decreasing in x_{m+1} (since u is decreasing in x_i for $x_i \ge 0$ and $x_{m+1}^n \to \infty$ as $n \to \infty$) u is even in x_i for $i \ge m+2$, u is decreasing in x_i for $i \ge m+2$ and $x_i \ge 0$ and $u(x) \le l$ if $|x_i| \ge K_2$ for $i \ge m+2$. By replacing x_{m+1} by $-x_{m+1}$, we can assume u is also increasing in x_{m+1}. If $||u||_\infty \le a$, we can use the argument of the previous paragraph to obtain a contradiction. Thus, $||u||_\infty > a$. Hence we have an example with the same n and a larger m if the limit in (5) is nonuniform.

We can use a similar argument to prove that, if we have a counterexample with minimal n (and then maximal m), we must have $\lim_{x_i\to\infty} u(x_i, \hat{x}_i) = 0$ uniformly in $\hat{x}_i$ for every $i \ge m+1$. By the evenness and monotonicity properties of u_i, it follows that $u(x) \to 0$ as $||Px|| \to \infty$ uniformly in $(I-P)x$ where $P\underline{x} = (0, \dots 0, x_{m+1} \dots x_n)$. Now, by the monotonicity, one sees as before that $\lim_{x_i\to\infty, 1\le i\le m} u(x)$ exists and is a solution w of $-\Delta u = g(u)$ on R^{n-m} such that w is even in x_i for $m < i \le n$, decreasing in x_i for $x_i \ge 0$ and $m < i \le n$ and $w(x) \to 0$ as $||x|| \to \infty$, $x \in R^{n-m}$. By the monotonicity and since $w \to 0$ as $||x|| \to \infty$, $x \in R^{n-m}$, one easily sees that $u(\tilde{x}) \to w(\tilde{x})$ uniformly in $P\tilde{x}$ as $x_i \to \infty$ for $1 \le i \le m$. Since $g'(0) < 0$, we can apply a result of Gidas Ni and Nirenberg [**12**] to deduce that w is radial and w decays exponentially.

By an earlier argument, we find that either u is independent of x_i (where i is fixed with $1 \leq i \leq m$) or $\partial u/\partial x_i > 0$ in R^n. The former case cannot occur by the minimality of n. Thus we can assume that u is strictly increasing in x_i for $1 \leq i \leq m$. We will prove a little later that there exists an $\alpha < 0$ such that the problem $-\Delta h = g'(w)h + \alpha h$ on R^{n-m} has a nontrivial exponentially decaying positive solution ϕ (and its first and second derivatives also decay exponentially). Define

$$z(\hat{x}) = \int_{R^{n-m}} (w(\tilde{x}) - u(\hat{x} + \tilde{x}))\phi(\tilde{x})\, d\tilde{x}$$

for $\hat{x} \in R^m$ and $\tilde{x} \in R^{n-m}$. Note that $z > 0$ on R^m since $\phi > 0$ and u is strictly increasing in $\hat{x}$ (to $\tilde{w}$). By using the exponential decay of ϕ, by the equations satisfied by w, u and ϕ and integrating by parts, we eventually find that

$$\begin{aligned} -\Delta' z &= \int_{R^{n-m}} [g(w) - g(u) - g'(w)(w-u) - \alpha(w-u)]\phi\, d\tilde{x} \\ &= \int_{R^{n-m}} (g'(\theta) - g'(w) - \alpha)(w-u)\phi\, d\tilde{x} \end{aligned}$$

where $\theta(\tilde{x}, \hat{x})$ is between $u(x)$ and $w(\tilde{x})$ (by the mean-value theorem). Here Δ' is the Laplacian on R^m. (Note that in deriving the first formula, we have used the identity

$$\int_{R^{n-m}} g(w)\phi - w(g'(w) + \alpha)\phi = 0.$$

This is obtained by multiplying the equation $-\Delta w = g(w)$ by ϕ, integrating twice by parts and by then using the equation satisfied by ϕ.) Since u converges uniformly to w as $x_i \to \infty$ for $1 \leq i \leq m$, it follows that there is a $K > 0$ such that $g'(\theta) - g'(w) \geq \alpha/2$ if $x_i \geq K$ for $1 \leq i \leq m$. Hence, we eventually find that

$$-\Delta' z + \frac{1}{2}\alpha z \geq 0 \tag{6}$$

if $x_i \geq K$ for $1 \leq i \leq m$. Choose a ball B in R^m such that the first eigenvalue of $-\Delta'$ on B with Dirichlet boundary conditions is $-\alpha/2$. We can translate B such that $B \subseteq \{x : x_i \geq K \text{ for } 1 \leq i \leq m\}$. Let

ϕ_1 be the corresponding first eigenfunction. If we multiply (6) by ϕ_1 and integrate by parts, we find that $0 \leq \int_{\partial B} z\partial\phi_1/\partial n$ where n is the outward normal and we use that $\phi_1 = 0$ on ∂B. This is impossible since $z > 0$ on ∂B and $\partial\phi_1/\partial n < 0$ on ∂B. Hence, we have a contradiction and this case does not occur.

The only remaining case is where $m = n$, that is, u is increasing in all variables. If $||u||_\infty \leq a$, we covered this case earlier in the proof. If $||u||_\infty > a$, we prove the result without the additional assumption that $u < a$ somewhere. As before, we find that $\tilde{u}(x_0) = \lim_{x_1\to\infty} u(x)$ is a solution of the same equation on R^{n-1} and $||\tilde{u}||_\infty = ||u||_\infty > a$. Thus, if we have such an example, we can reduce dimensions until we have an increasing positive solution $\bar{u}$ of $-u'' = g(u)$ on R with $||\bar{u}||_\infty > a$. It is easy to use the first integral to show that this case does not occur.

It remains to construct ϕ in the second case. This is a variant of an argument in [**6**]. By differentiating the equation for w in x_1, $z = \partial w/\partial x_1$ is a solution of the eigenvalue problem

$$-\Delta z + g'(w)z = \lambda z$$

on R^{n-m} corresponding to the eigenvalue zero. Note that, since w decays rapidly at infinity, it follows easily from the equations that $\partial w/\partial x_1$ and its derivatives also decay rapidly at infinity. Thus, $z \in W^{1,2}(R^{n-m})$. Since $w \to 0$ as $x_1 \to \pm\infty$, we easily see that z changes sign in R^{n-m}. We can then argue much as in part of the proof of Theorem 1 in [**6**] to deduce that z (or more strictly a multiple of z) does not minimize the functional $E(\tilde{w}) = \int_{R^{n-m}} |\nabla\tilde{w}|^2/2 - g'(w)\tilde{w}^2/2$ on $T = \{\tilde{w} \in W^{1,2}(R^{n-m}) : ||\tilde{w}||_2 = 1\}$ while $E(z) = 0$ (by a simple calculation). Thus, E must take negative values. Hence, $\alpha = \inf_{w\in T} E(\tilde{w})$ exists, $\alpha < 0$ and α is achieved. (To see that α is achieved, it suffices to show that the corresponding self-adjoint operator $H(v) = -\Delta v - g'(w)v$ on $L^2(R^{n-m})$ with domain $W^{2,2}(R^{n-m})$ has the property that $H - \lambda I$ is Fredholm for $\lambda \leq 0$ and is invertible for λ large negative. The second property follows easily from the Lax-Milgram lemma while the first follows since our condition that $w \to 0$ as $|x_1| \to \infty$ ensures that $H - \lambda I$ is a relatively compact perturbation of $-\Delta v - (g'(0) + \lambda)I$ and since $g'(0) + \lambda \leq g'(0) < 0$ for $\lambda \leq 0$). If ϕ minimizes E on T, we can, by replacing ϕ by $|\phi|$, assume that $\phi \geq 0$. Then ϕ is a weak solution in $L^2(R^{n-m})$ of $-\Delta v - g'(w)v = \alpha v$ on R^{n-m}. Standard regularity theory ensures that ϕ is a strong and smooth solution. Since $g'(w) + \alpha \to \alpha < 0$ as $|\hat{x}| \to \infty$, a result of Kato [**17**] ensures that ϕ decays exponentially. Standard local $W^{2,p}$ estimates ensure that the derivatives of ϕ decay rapidly as $|\hat{x}| \to \infty$. This completes the proof in the full space case.

(ii) *The half space case.* This is very similar. Once again, we make our inductive step the case where u is increasing in $x_1, \dots, x_m$ (and decreasing in x_j for $x_j \geq 0$ if $j \leq m+1$) and we require that $u \leq l$ if $x_1 \geq 0$ and $|x_j| \geq \tau$ for $m < j \leq n$. As before, if $m > 2$ we can decrease n or increase m by letting x_{m+1} tend to infinity (and keep a half space problem). If u is increasing in all variables, we can reduce as before to a half space problem with $n = 1$. This can be easily studied by using the first integral. In the case where we have nonuniform decay as some of the x_j (for $j \geq m+1$) tend to infinity when we use a blowing up argument as in part (i), we may end up with a problem on R^n. However, this can be handled by the result of part (i). This completes the proof of Lemma 4. □

Proof of Proposition 1. We first improve Lemma 3 by showing that, if $l_2 > 0$ there exist $K_3 > 0$ such that $|u_i(x)| \leq l_2$ if $x \in D$ and $||x|| \geq \varepsilon_i^{1/2} K_3$ and i is large. Suppose not. Then there exists x^i with $\varepsilon_i^{-1/2} ||x^i|| \to \infty$ as $i \to \infty$ and $u_i(x^i) \geq l_2$. If we use a change of variable $X = \varepsilon_i^{-1/2}(x - x^i)$ and note that in the new variables $T_i = \{x : ||x|| \leq \varepsilon_i^{1/2} K_2\}$ is at a distance from the origin which tends to infinity as $i \to \infty$, we see by a now familiar blowing up argument that we have a nonnegative solution U of $-\Delta u = g(u)$ on R^n such that $U(0) \geq l_2$ and $U(x) \leq l_1$ on R^n. (Here we have used Lemma 3 to ensure that $u_i \leq l_1$ of $R^n \backslash T_i$.) As before, this contradicts Proposition 3 in [**5**]. Thus, our original claim holds.

Next, if we use the change of variable $X = \varepsilon_i^{-1/2} x$, we see much as before that a subsequence of $u_i(X)$ converges uniformly on compact sets to a solution v of $-\Delta v = g(v)$ on R^n. However, by what we have proved above, $u_i(X)$ is uniformly small off compact sets. Thus, v must have the same property. Hence, $u_i(X) - v(X)$ must be uniformly small on $\varepsilon_i^{-1/2} D$. Thus, we need only prove that v is radial. By our comments above, $v \to 0$ as $||X|| \to \infty$. Since $g'(0) < 0$, as result of Gidas, Ni and Nirenberg [**12**] implies that v is radial (possibly after a change of origin). However, v has the same symmetries as u_i and hence v is even in x_j for $1 \le j \le n$ and strictly decreasing in X_j for $x_j > 0$. Hence v must be radial about the original origin. This completes the proof of Proposition 1. □

Remark. A variant of Proposition 1 holds for many other types of behavior at infinity. For example, it still holds if $g(y) \sim y$ as $y \to \infty$. (The proof uses that $-\Delta u = u$ has no bounded positive solution on R^n.) As another example, assume that there is a $b > a$ such that $g(b) = 0$, $g'(y) \le 0$ on $(b - \tau, b)$ where $\tau > 0$, $g(y) > 0$ on (a, b) and g satisfies similar assumptions as before on $[0, a)$ and at a and $\int_0^b g > 0$. (If the last condition fails there are no positive solutions.) In this case, results of Clement and Sweers [**4**] and [**22**] show that there is a $\tau_1 > 0$ such that, if λ is large, (2) has a unique positive solution u_λ with $b - \tau_1 < ||u_\lambda||_\infty < b$. Moreover, $||u_\lambda|| \to b$ as $\lambda \to \infty$ and u_λ is close to its maximum on compact subsets of Ω. This is proved by constructing families of subsolutions. On the other hand, if $\tau_2 > 0$, our proof shows that the conclusion of Proposition 1 holds for solutions u_i with $||u_i||_\infty \le b - \tau_2$ for all i (with the same proof). Since it is easy to prove that there are no positive radial solutions of $-\Delta u = g(u)$ in R^n with $||u||_\infty$ close to but less than b (by sweeping families of subsolutions or by continuous dependence), it follows that we have a rather complete answer in this case as well. Note that the results in [**1**] and the remark after Lemma 1 imply that for some domains we may have more than two positive solutions for all large λ. Thus, this case is not as simple as some of the cases in [**5**]. Analogous results hold if $g(y) > 0$ for $y > a$ and if either (i) $g(y) \to M$ as $y \to \infty$ (where $M > 0$) and $yg'(y) \to 0$ as $y \to \infty$ or if (ii) there is a $p \in (0, 1)$ and $\tilde{\mu} > 0$ such that $y^{1-p}g'(y) \to \tilde{\mu}$ as $y \to \infty$. In each of these two cases, one proves that, if λ is large there is a unique positive solution u_λ with $||u_\lambda||_\infty$ large, and other positive solutions have $||u_\lambda||_\infty$ uniformly bounded; then the conclusion of Proposition 1 holds for these other positive solutions. The idea here is to apply a Harnack inequality to $(||u||_\infty)^{-1}u$ and barriers (where u is a solution of (2)) to deduce that if $||u||_\infty$ is large, then $u(x)$ is large for a relatively large set of x and then use a sweeping family of subsolutions to deduce that u is large on nearly all of Ω (much as in [**4**] or [**5**]). Note that we can use similar arguments to bound the positive solutions of $-\Delta u = g(u)$ in R^n (with $||u||_\infty < b$ when $g(b) = 0$). It seems likely that the condition $g'(0) < 0$ can be weakened. (Most of our arguments are still valid if $g'(0) = 0$.) In particular, if $n = 2$, Proposition 1 is still valid except that it is unclear if v has to be radial. Our methods can be generalized to cover cases where $g > 0$ on $(0, a_1)$, $g < 0$ on (a_1, a_2) and $g > 0$ on (a_2, a_3) where $\int_{a_1}^{a_3} g > 0$ and provided either $g(0) > 0$ or both $g(0) = 0$ and $g'(0) > 0$. In this case positive solutions u with $u(0) \in (a_2, a_3)$ and $u(0)$ not close to a_3 have a sharp peak much as in the proposition and are close to a_1 except near 0 and $\partial\Omega$. Thus, they have two distinct layers of rapid change.

Proof of Theorem 1. Firstly, it is well known (cf. [**18**]) that $-\Delta u = u^p - u$ has a unique positive radial solution u_0 in R^n such that $u_0(r) \to 0$ as $r \to \infty$. Thus, if the u_i are positive solutions of (1) for i large, then $u(\varepsilon_i^{1/2} x) - u_0$ converges uniformly to zero as $i \to \infty$. The existence of at least one positive solution of (1) for all ε follows by a simple and standard degree argument combined with the apriori bound in Lemma 1. (Alternatively, one could use the mountain pass theorem.) Thus, it suffices to establish the uniqueness. Suppose that u_i and w_i are both positive solutions of (1) for $\varepsilon = \varepsilon_i$. Then after the rescaling $X = \varepsilon_i^{-1/2} x$, $\tilde{u}_i$ and $\tilde{w}_i$ are both positive solutions of $-\Delta u = u^p - u$ on $\varepsilon_i^{-1/2} D$ and hence $z_i = (||\tilde{u}_i - \tilde{w}_i||_\infty)^{-1}(\tilde{u}_i - \tilde{w}_i)$ is a solution of

$$-\Delta z = \frac{f(\tilde{u}_i) - f(\tilde{v}_i)}{\tilde{u}_i - \tilde{v}_i} z \tag{7}$$

on $\varepsilon_i^{-1/2} D$ such that $||z_i||_\infty = 1$ and z_i is even in x_j for $1 \le j \le n$ (where $f(y) = y^p - y$). By a now standard argument we can pass to the limit and obtain a solution z_0 of

$$-\Delta z = f'(u_0) z \tag{8}$$

on R^n such that $||z_0||_\infty \le 1$ and z is even in x_j for $1 \le j \le n$. In the following paragraph we will prove this implies that $z_0 \equiv 0$. Assuming this, it follows easily that z_i converges uniformly to zero on compact sets. Hence, if x^i is such that $z_i(x^i) = 1$ then $||x^i|| \to \infty$ as $i \to \infty$. Hence, $\tilde{u}_i(x^i)$ and $\tilde{v}_i(x^i)$ are both small (by Proposition 1) and hence we easily see that $(f(\tilde{u}, (x^i)) - f(\tilde{v}_i(x^i)))/(\tilde{u}_i(x^i) - \tilde{v}_i(x^i)) < 0$ (since $f'(0) < 0$). Hence, by the equation for z_i, $-\Delta z(x^i) < 0$. This is impossible at a local maximum and hence we have a contradiction. (Note that x^i must be in the interior of $\varepsilon_i^{-1/2} D$.)

Thus it remains to prove that (8) has no nontrivial bounded solution z_0 which is even in x_j for $1 \leq j \leq n$. Since $f'(0) < 0$, once again we can use Kato's result to deduce that z_0 decays exponentially. If $R > 0$ such that z_0 does not vanish on the sphere of radius R, choose an n-dimensional spherical harmonic $\tilde{x}$ such that $\int_S z_0(Rw)\tilde{X}(w)\,dw \neq 0$ where S is the unit sphere in R^n. (This exists by completeness.) By a simple computation, $h(r) \equiv \int_S z_0(rw)\tilde{X}(w)\,dw$ is a solution of

$$(9) \qquad \begin{aligned} -r^{1-n}\frac{d}{dr}(r^{n-1}h'(r)) + r^{-2}\tilde{\alpha}h = f'(u_0)h \quad &\text{on } (0,\infty) \\ h'(0) = 0 \quad &\text{if } \alpha = 0 \\ h(0) = 0 \quad &\text{if } \alpha > 0. \end{aligned}$$

Here $\tilde{\alpha}$ is the eigenvalue of the Laplacian on the sphere corresponding to the eigenfunction $\tilde{X}$. (Thus $\tilde{\alpha} = 0$ if $\tilde{X} \equiv 1$ and $\tilde{\alpha} \geq n-1$, otherwise as in [**22**].) Note that the boundary condition at zero is forced because z_0 must be smooth at zero (because z_0 is a solution of an elliptic equation). Since $h(R) \neq 0$ by our choice of $\tilde{X}$, h is nontrivial. Note also that h decays exponentially since z_0 does. We use different arguments for the case $\tilde{\alpha} = 0$ and $\tilde{\alpha} > 0$. If $\tilde{\alpha} = 0$, $h(r)$ is a radial solution of (8). This is impossible by Lemma 19 in Kwong [**18**]. (Note that he assumes $a > 0$ in his notation but, as he comments later, his arguments are still valid if $a = 0$. Here a is defined in [**18**].) If $\tilde{\alpha} > 0$, we see by differentiating (1) that $\partial u_0/\partial x_1$ is a solution of (8), that is, $u_0'(r)r^{-1}x_1$ is a solution of (8). By a simple calculation, this implies that $u_0'(r)$ is a solution of (9) for $\alpha = n-1$. Since u_0 is radial and C^2 at zero, we easily see that $u_0'(0) = 0$. Since u_0 decays exponentially at infinity, it follows from the equation that u_0' does as well. It follows easily that $v = -u_0'$ is the only solution of (9) for $\tilde{\alpha} = n-1$ which decays rapidly at infinity (because if w were a linearly independent solution $r^{n-1}(v'(r)w(r) - w'(r)v(r))$ would be a nonzero constant). If $\tilde{h}$ were a nontrivial solution of (9) for $\tilde{\alpha} > n-1$ with $\tilde{h}(0) = 0$ such that $\tilde{h}$ decays exponentially, then we could use the Sturm comparison theorem to prove that $\tilde{h}$ has at most one positive zero (since Gidas-Ni-Nirenberg implies that $u_0'(r) < 0$ for $r > 0$). Let c be this first zero if one exists and let $c = \infty$ otherwise. We may assume $\tilde{h} > 0$ on $(0,c)$. One easily finds that $Z(r) = r^{n-1}(v'\tilde{h} - v\tilde{h}')$ has the properties that $Z(r) \to 0$ as $r \to 0$ and Z is strictly decreasing on $(0,c)$. Thus $Z(c) < 0$ if $c < \infty$. Hence $-\tilde{h}'(c)v(c) < 0$ which is impossible since $v(c) > 0$ and

$\tilde{h}'(c) \leq 0$. Thus $c = \infty$ and $r^{n-1}(v'(r)\tilde{h}(r) - v(r)\tilde{h}'(r))$ has a negative limit (possibly $-\infty$) as $r \to \infty$. This is impossible since $\tilde{h}$ and v decay exponentially. Hence (8) has no nontrivial solution z with z decaying exponentially at infinity other than $r^{-1}u_0'(r)x_i$ (or linear combinations of these). Hence zero is the only solution of (8) which is even in the x_j, as required. This completes the proof. □

Remark 1. In the last paragraph we only use the particular structure of the nonlinearity to exclude radial solutions of (8). In fact, the main part of the argument for nonradial solutions is a variant of the proof of the Sturm comparison theorem. A similar argument shows that any bounded solution of (8) which corresponds to a negative eigenvalue must be a radial solution.

Remark 2. Note that it is unclear if the positive solution in Theorem 1 is unique for large positive ε. It is possible to use the ideas in [**5**] to show that for any domain D, the number of positive solutions for large ε is 'usually' determined by the number of positive solutions of $-\Delta u = u^p$ in D, $u = 0$ on ∂D. Unfortunately, if $n > 2$, it is not known if this problem has a unique positive solution for D as in Theorem 1. (It is unique if $n = 2$ by Theorem 5 in [**8**] and one can deduce that (1) has a unique positive solution for large ε if $n = 2$.)

Remark 3. There is an alternative method of proving the existence of the positive solution in Theorem 1. One works in $\{u \in C(D) : u \text{ is even in } x_i \text{ for } 1 \leq i \leq n\}$, one uses $v(\varepsilon^{-1/2}x)$ as an approximate solution, and one uses a contraction mapping argument rather similar to (but easier than) the proof of Theorem 3 in [**8**]. (One has to check positivity directly.) This method works in other situations. Note that it is important that $g'(0) < 0$ to ensure that 0 is not in the essential spectrum of the limits of the linearizations as ε tends to zero. On the other hand, the arguments in the proof above are often valid if $g'(0) = 0$.

Remark 4. Unfortunately, it seems that the multiplicity of positive solutions for general g (for g as in the proposition) is quite complicated if $n \geq 2$. Some sufficient conditions for uniqueness can be found in Kwong and Zhang [**19**]. We sketch briefly why there may be nonuniqueness. The methods for constructing counterexamples in Ni and Nussbaum [**20**] (which is based on an idea of Hempel [**15**] when $n = 2$) implies that we can find a positive g such that $g(0) = g'(0) = 0$ and the equation $-\Delta u = g(u)$ has a positive radial solution u_0 on R^n which decays at infinity and the radial solution h of the linearized equation satisfying $h(0) = 1$ has at least two positive zeros. We can always change g so that g behaves asymptotically like y^p for large y. Moreover, with care, we can modify g for small y (by a small amount) and u_0 for large r such that g satisfies the assumptions of Proposition 1 and u_0 still exists. Since we only modify g for small y and u_0 for large r, the radial solution of the linearized equation will still have at least two positive zeros (by continuous dependence). By a variant of a theorem of Dunford and Schwartz [**11**], it follows that the linearization of (3) at u_0 has at least two negative eigenvalues corresponding to radial eigenfunctions. On the other hand, if $\varepsilon > 0$, it is not difficult to use the strong form of the mountain pass theorem to show that (2) has a solution u_ε on D such that the linearization of (3) at u_ε on D has at most one negative eigenvalue. It follows easily from consideration of the continuity of the spectrum under perturbations that if a subsequence of $u_\varepsilon(\varepsilon^{-1/2}x)$ converges to v as $\varepsilon \to 0$ then the linearization of $-\Delta u = g(u)$ on R^n at v has at most one negative eigenvalue. (Similar arguments appear in [**6**, Section 2].) Hence, in any example where we find a solution w of $-\Delta u = g(u)$ on R^n such that the linearization at w has at least two negative eigenvalues, there will be nonuniqueness. By deforming the nonlinearity in the example of nonuniqueness to $u^p - u$ such that the assumptions of Proposition 1 are always satisfied, we obtain an example where there must be a radial solution such that the linearization at this solution of $-\Delta u = g(u)$ has a nontrivial radial solution decaying at infinity. On the other hand, if every positive solution decaying to zero of $-\Delta u = g(u)$ on R^n is nondegenerate, one can show by using the previous remark that the number of nontrivial positive solutions of (2) for small ε is equal to the number of nontrivial positive radial solutions of $-\Delta u = g(u)$ on R^n which decay to zero at infinity. It seems likely that this nondegeneracy assumption holds for 'generic' g. If g is real analytic, one can use the ideas in the proof of Theorem 5 in [**9**] to prove that (3) has only a finite (up to translations) number of positive solutions on R^n which decay to zero at infinity.

Remark 5. If $n = 2$ and g is as in Proposition 1, one can generalize the results in [**16**] slightly to prove that each component T of the set of positive solutions (u, λ) of $-\varepsilon\Delta u = g(u)$ on D with Dirichlet boundary conditions is a smooth one-manifold parameterized by $u(0)$. Moreover, there is a unique component T_1 such that $\{u(0) : u \in T_1\}$ is not bounded and if $\{u(0) : u \in T\} = (\alpha, \beta)$, then the equation $-\Delta u = g(u)$ on R^2 has bounded positive radial solutions w_1, w_2 on R^2 tending to zero at infinity such that $w_1(0) = \alpha$, $w_2(0) = \beta$. (The last result uses Proposition 1 here.) Results of this type hold for rather more general nonlinearities.

Note that the corresponding Neumann problem behaves somewhat differently. We only consider (1) but with Neumann boundary conditions. In this case $u \equiv 1$ is a solution for all ε and nonconstant positive solutions bifurcate off this whenever $\varepsilon^{-1}(p-1)$ is a positive eigenvalue of $-\Delta$ (for Neumann boundary conditions). (This follows by a rather standard bifurcation argument for equations with a variational structure.) Thus, there is bifurcation of positive solutions for arbitrarily small ε and a uniqueness theorem like Theorem 1 cannot be true. Indeed, we might expect there to be many solutions (since there are many bifurcations).

A variant of our ideas has other uses. Let v be the radial decaying positive solution of $-\Delta u = u^p - u$ on R^n. We can then use $v(\varepsilon^{-1/2}(x - x_0))$ as an approximate solution of (1) for small ε if $x_0 \in \partial\Omega$ where we now use Neumann boundary conditions. One can show that the linearization about this approximate solution has exactly $n-1$ small eigenvalues for small ε and the kernel is close to the tangent space $T_{x_0}(\partial\Omega)$. Using this and an implicit function argument, one can reduce the problem of the existence of solutions of (1) close to $\{v(\varepsilon^{-1/2}(x - x_0)) : x_0 \in \partial\Omega\}$ to the zeros of a tangent vector field on $\partial\Omega$ (where the tangent vector field is a bifurcation equation). One then uses standard multiplicity results to deduce that, for small ε, (1) with Neumann boundary conditions has at least $\text{cat}\,(\partial\Omega)$ positive solutions where cat denotes the Lysternik-Schnirelmann category. This gives an alternative proof of a result in [**23**]. (One needs a little extra work to check that these solutions are positive.)

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