ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

FUNCTION THEORIES FOR THE YUKAWA AND HELMHOLTZ EQUATIONS

L.R. BRAGG AND J.W. DETTMAN

ABSTRACT. Transmutations operating on heat polynomials and associated heat functions are employed to develop function theories for the Yukawa and Helmholtz equations. The special functions developed by means of these transmutations are studied, including series and integral representation theorems for solutions corresponding to analytic and entire data.

1. Introduction. It is well known [**1**] that the linear second order partial differential equation in two independent variables

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y) \tag{1.1}$$

with constant coefficients can (except for one degenerate case) be transformed by changes of variables to the heat, wave, Laplace, Helmholtz or Yukawa equation. The best known function theory associated with these equations, in the case of Laplace's equation, is the analytic function theory in one complex variable. P.C. Rosenbloom and D.V. Widder [**13**] have developed a function theory for the heat equation based on the heat polynomials and associated heat functions. The authors [**5**] have shown that there is an analogous function theory for the wave equation related to these through transmutation operators. In the present paper, we show that there are analogous function theories for the Yukawa and Helmholtz equations. R.J. Duffin [**8**] has presented a function theory of the Yukawa equation from the point of view of the pseudoanalytic function theory of Bers-Vekua [**2**]. Also, see [**9**]. Our approach is through transmutation operators [**3, 7**] and is essentially independent of the Bers-Vekua theory. There are also analogous function theories for certain singular elliptic and hyperbolic differential equations [**6**].

Received by the editors on January 1, 1993, and in revised form on December 8, 1993.
1991 AMS (MOS) *Subject Classifications.* Primary 30G30, Secondary 35C10, 35J05.
Key words and phrases. Yukawa equation, Helmholtz equation, heat equation, transmutations, function theory, right-regular, asymptotics.

The plan of this paper is as follows: A summary of the transmutation operators needed will be given in Section 2 along with the heat polynomials and associated heat functions which give rise to the basic solutions and associated functions for the Yukawa and Helmholtz equations. Section 3 will contain the basic functions, associated functions, generating functions, recurrence relations and generalized Cauchy-Riemann equations for the Yukawa equation. Similar material for the Helmholtz equation will be given in Section 4. Section 5 will present representation theorems for solutions of the Yukawa and Helmholtz equations with analytic data. Representation theorems for Yukawa and Helmholtz solutions with entire data will be treated in Section 6. Section 7 will contain Fourier transform criteria for Yukawa and Helmholtz solutions with entire data. The final section gives a summary of results.

2. Transmutation operators. It has been shown [**7**] that, if $w(x,t)$ is a solution of

$$w_t = w_{xx} - \mu^2 w \tag{2.1}$$

for $-\infty < x < \infty$, $t > 0$, with $w(x,0) = \phi(x)$, μ^2 a positive constant, then under appropriate assumptions on $\phi(x)$,

$$u(x,y) = \frac{y}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-y^2/4s} w(x,s)\, ds = T_1 w(x,t) \tag{2.2}$$

is a solution of the Dirichlet problem for the Yukawa equation

$$\begin{gathered} u_{xx} + u_{yy} = \mu^2 u, \quad -\infty < x < \infty,\ y > 0, \\ u(x,0) = \phi(x). \end{gathered} \tag{2.3}$$

Similarly,

$$v(x,y) = -\frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-y^2/4s} w(x,s)\, ds = T_2 w(x,t) \tag{2.4}$$

is a solution of the Neumann problem:

$$\begin{gathered} v_{xx} + v_{yy} = \mu^2 v, \quad -\infty < x < \infty, y > 0; \\ v_y(x,0) = \phi(x). \end{gathered} \tag{2.5}$$

If $h(x,t)$ is a solution of the heat equation

$$h_t = h_{xx}, \tag{2.6}$$

then $w(x,t) = e^{-\mu^2 t} h(x,t)$ is a solution of (2.1). In this case,

$$T_1 w(x,t) = \frac{y}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-y^2/4s} e^{-\mu^2 s} h(x,s)\, ds \tag{2.7}$$

$$T_2 w(x,t) = -\frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-y^2/4s} e^{-\mu^2 s} h(x,s)\, ds. \tag{2.8}$$

Hence, we have related solutions of the Yukawa equation to solutions of the heat equation.

In this paper we shall be mainly concerned with particular solutions of the heat equation, namely the heat polynomials,

$$h_n(x,t) = n! \cdot \sum_{j=0}^{[n/2]} \frac{x^{n-2j} t^j}{(n-2j)!j!}, \quad n = 0, 1, 2, \dots \tag{2.9}$$

and the associated heat functions,

$$H_n(x,t) = k(x,t) h_n(x/t, -1/t) = (-2)^n \frac{\partial^n}{\partial x^n} k(x,t) \tag{2.10}$$

where $k(x,t) = (1/\sqrt{4\pi t}) e^{-x^2/4t}$ is the fundamental solution of (2.6).

For reasons which will become clear later, we shall also need transmutations relating the damped wave equation

$$u_{tt} = u_{xx} - \mu^2 u \tag{2.11}$$

to the heat equation. If $h(x,t)$ is a solution of (2.6) for $-\infty < x < \infty$, $t > 0$, $h(x,0) = \phi(x)$, then

$$u(x,t) = t\sqrt{\pi} \mathcal{L}_s^{-1} \{ s^{-3/2} e^{-\mu^2/4s} h(x, 1/4s) \}_{s \to t^2} = T_3 h(x,t) \tag{2.12}$$

is a solution of (2.11) for $-\infty < x < \infty$, $t > 0$, with $u(x,0) = \phi(x)$, $u_t(x,0) = 0$. Here $\mathcal{L}_s^{-1}\{\cdots\}_{s \to t^2}$ denotes the inverse Laplace transform with t^2 the variable of inversion. Similarly,

$$v(x,t) = (1/2)\sqrt{\pi} \mathcal{L}_s^{-1} \{ s^{-3/2} e^{-\mu^2/4s} h(x, 1/4s) \}_{s \to t^2} = T_4 h(x,t) \tag{2.13}$$

is a solution of (2.11) for $-\infty < x < \infty$, $t > 0$, with $u(x,0) = 0$, $u_t(x,0) = \phi(x)$.

3. The Yukawa equation. In this section we shall begin the development of an analogous function theory for the Yukawa equation by defining some basic solutions and associated functions. We begin by transforming the heat polynomials of Rosenbloom and Widder [**13**] using the transmutations T_1 and T_2. We define

$$\begin{aligned} u_n^+(x,y) &= T_1[e^{-\mu^2 t} h_n(x,t)] \\ &= \frac{n!y}{\sqrt{4\pi}} \int_0^\infty e^{-y^2/4s} e^{-\mu^2 s} \sum_{j=0}^{[n/2]} \frac{x^{n-2j} s^{j-3/2}}{(n-2j)!j!}\, ds \\ &= x^n e^{-\mu y} + \frac{2\cdot n!y}{\sqrt{\pi}} \sum_{j=1}^{[n/2]} \frac{x^{n-2j}}{(n-2j)!j!4^j} \left(\frac{2y}{\mu}\right)^{j-1/2} K_{j-1/2}(\mu y). \end{aligned}$$

Using the fact that $K_{j-1/2}(\mu y) = (\sqrt{\pi}/2)e^{-\mu y}(\mu y)^{1/2-j}\Theta_{j-1}(\mu y)$, $j \ge 1$, where Θ_{j-1} is the reverse Bessel polynomial (see [**10**]), we have

$$u_n^+(x,y) = x^n e^{-\mu y} + n!\mu y e^{-\mu y} \sum_{j=1}^{[n/2]} \frac{x^{n-2j}\Theta_{j-1}(\mu y)}{(n-2j)!j!(2\mu^2)^j}.$$

Since the integral above connected with T_1 is equally meaningful if $\mu < 0$, we define

$$u_n^-(x,y) = x^n e^{\mu y} - n!\mu y e^{\mu y} \sum_{j=1}^{[n/2]} \frac{x^{n-2j}\Theta_{j-1}(-\mu y)}{(n-2j)!j!(2\mu^2)^j}.$$

Finally, we define our basic solutions as

$$\begin{aligned} u_n(x,y) &= [u_n^+(x,y) + u_n^-(x,y)]/2 \\ &= x^n \cosh \mu y + n!\mu y \sum_{j=1}^{[n/2]} \frac{x^{n-2j}}{(n-2j)!j!(2\mu^2)^j} \\ &\qquad \times \left\{ \frac{e^{-\mu y}\Theta_{j-1}(\mu y) - e^{\mu y}\Theta_{j-1}(-\mu y)}{2} \right\}. \end{aligned} \tag{3.1}$$

A few examples of these solution functions are

$$\begin{aligned}
u_0 &= \cosh\mu y, \qquad u_1 = x\cosh\mu y,\\
u_2 &= x^2\cosh\mu y - \frac{y}{\mu}\sinh\mu y,\\
u_3 &= x^3\cosh\mu y - \frac{3xy}{\mu}\sinh\mu y,\\
u_4 &= x^4\cosh\mu y - \frac{6x^2y}{\mu}\sinh\mu y - \frac{3y}{\mu^3}\sinh\mu y + \frac{3y^2}{\mu^2}\cosh\mu y.
\end{aligned}$$

It is clear that the $u_n(x,y)$, $n = 0,1,2,\dots$ are solutions of the Yukawa equation satisfying the condition $u_n(x,0) = x^n$.

Next, we transform the heat polynomials with the transform T_2:

$$\begin{aligned}
v_n^+(x,y) &= T_2[e^{-\mu^2 t}h_n(x,t)]\\
&= -\frac{n!}{\sqrt{\pi}}\int_0^\infty e^{-y^2/4s}e^{-\mu^2 s}\cdot\sum_{j=0}^{[n/2]}\frac{x^{n-2j}s^{j-1/2}}{(n-2j)!j!}\,ds\\
&= -\frac{2\cdot n!}{\sqrt{\pi}}\sum_{j=0}^{[n/2]}\frac{x^{n-2j}}{(n-2j)!j!}\left(\frac{y}{2\mu^2}\right)^{j+1/2}K_{j+1/2}(\mu y)\\
&= -\frac{n!}{\mu}\cdot\sum_{j=0}^{[n/2]}\frac{x^{n-2j}}{(n-2j)!j!(2\mu^2)^j}\cdot e^{-\mu y}\Theta_j(\mu y).
\end{aligned}$$

Similarly, replacing μ by $-\mu$, we have

$$v_n^-(x,y) = \frac{n!}{\mu}\cdot\sum_{j=0}^{[n/2]}\frac{x^{n-2j}}{(n-2j)!j!(2\mu^2)^j}\cdot e^{\mu y}\Theta_j(-\mu y).$$

Hence, it follows that

$$\begin{aligned}
v_n(x,y) &= \frac{v_n^+(x,y)+v_n^-(x,y)}{2}\\
&= \frac{n!}{\mu}\cdot\sum_{j=0}^{[n/2]}\frac{x^{n-2j}}{(n-2j)!j!(2\mu^2)^j}\\
&\qquad\cdot\left(\frac{e^{\mu y}\Theta_j(-\mu y) - e^{-\mu y}\Theta_j(\mu y)}{2}\right).
\end{aligned} \tag{3.2}$$

A few examples of these functions are:

$$v_0 = \frac{1}{\mu}\sinh \mu y, \qquad v_1 = \frac{x}{\mu}\sinh \mu y,$$
$$v_2 = \frac{x^2}{\mu}\sinh \mu y + \frac{1}{\mu^3}\sinh \mu y - \frac{y}{\mu^2}\cosh \mu y$$
$$v_3 = \frac{x^3}{\mu}\sinh \mu y + \frac{3x}{\mu^3}\sinh \mu y - \frac{3xy}{\mu^2}\cosh \mu y.$$

The functions $v_n(x,y)$, $n = 0,1,2,\dots$ are solutions of the Yukawa equation satisfying the condition $\partial u_n(x,0)/\partial y = x^n$.

To find generating functions for the solutions $u_n(x,y)$ and $v_n(x,y)$, we transform the generating function for the heat polynomials [**13**] which is given by the relation

$$e^{ax+a^2t} = \sum_{n=0}^{\infty} h_n(x,t)\frac{a^n}{n!}. \tag{3.3}$$

Hence, we have

$$T_1[e^{-\mu^2 t}e^{ax+a^2t}] = e^{ax}e^{-\sqrt{\mu^2-a^2}\cdot y}.$$

But this is the generating function for $\mu_n^+(x,y)$. Observing that $u_n^-(x,y) = u_n^+(x,-y)$, we find

$$e^{ax}\cosh(\sqrt{\mu^2-a^2}\cdot y) = \sum_{n=0}^{\infty} u_n(x,y)\frac{a^n}{n!} \tag{3.4}$$

is a generating function for $u_n(x,y)$. In a similar way, we find that

$$T_2[e^{-\mu^2 t}e^{ax+a^2t}] = -\frac{e^{ax}e^{-\sqrt{\mu^2-a^2}\cdot y}}{\sqrt{\mu^2-a^2}}$$

is a generating function for $v_n^+(x,y)$. However, $v_n^-(x,y) = -v_n^+(x,-y)$. Therefore, we get

$$\frac{e^{ax}\sinh(\sqrt{\mu^2-a^2}\cdot y)}{\sqrt{\mu^2-a^2}} = \sum_{n=0}^{\infty} v_n(x,y)\frac{a^n}{n!} \tag{3.5}$$

as a generating function for the $v_n(x,y)$.

Using these generating functions, it is easy to establish the following recurrence relations:

$$\frac{\partial u_n(x,y)}{\partial x} = n u_{n-1}(x,y) \tag{3.6}$$

$$\frac{\partial u_n(x,y)}{\partial y} = \mu^2 v_n(x,y) - n(n-1) v_{n-2}(x,y) \tag{3.7}$$

$$\frac{\partial v_n(x,y)}{\partial x} = n v_{n-1}(x,y) \tag{3.8}$$

$$\frac{\partial v_n(x,y)}{\partial y} = u_n(x,y). \tag{3.9}$$

It is possible to show, with the right combination of the u_n and the v_n, that we have conjugate pairs of panharmonic functions which satisfy generalized Cauchy-Riemann equations (see [**8**]). Let $u(x,y) = u_n(x,y)$, $v(x,y) = -\mu v_n(x,y) + n v_{n-1}(x,y)$. Then, using the above recurrence relations, we can show that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \mu u, \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\mu v. \tag{3.10}$$

Hence, $u + iv$ is right-regular in the sense of Duffin [**8**].

It is important to show that our function theory reduces to the analogous function theory for Laplace's equation (see [**5**]). This follows from the generating functions

$$e^{ax} \cos ay = \sum_{n=0}^{\infty} \operatorname{Re}(z^n) \frac{a^n}{n!} \tag{3.11}$$

$$\frac{e^{ax} \sin ay}{a} = \sum_{n=0}^{\infty} \left\{ \frac{\operatorname{Im}(z^n)}{n+1} \right\} \frac{a^n}{n!} \tag{3.12}$$

which are obviously limits of our generating functions for the basic solutions of the Yukawa equation as μ approaches zero.

To develop associated functions for the Yukawa equation, we must first obtain fundamental solutions by transforming the fundamental solution for the heat equation. We have

(3.13)
$$T_1\left[e^{-\mu^2 t}\cdot\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}\right]=\frac{y}{4\pi}\int_0^\infty s^{-2}e^{-\mu^2 s}e^{-(x^2+y^2)/4s}\,ds = \frac{\mu y}{\pi}\cdot\frac{K_1(\mu r)}{r}$$

and

(3.14)
$$T_2\left[e^{-\mu^2 t}\cdot\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}\right]=-\frac{1}{2\pi}\int_0^\infty s^{-1}e^{-\mu^2 s}e^{-(x^2+y^2)/4s}\,ds = -\frac{1}{\pi}\cdot K_0(\mu r).$$

The associated functions are then determined by the differentiation relations

$$U_n(x,y)=\frac{(-2)^n\mu y}{\pi}\cdot\frac{\partial^n}{\partial x^n}\frac{K_1(\mu r)}{r} \tag{3.15}$$

and

$$V_n(x,y)=-\frac{(-2)^n}{\pi}\frac{\partial^n}{\partial x^n}K_0(\mu r). \tag{3.16}$$

To be more explicit, we can transform the functions

$$H_n(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}t^{j-n}}{(n-2j)!j!}$$

and this leads to the sum forms:

(3.17)
$$U_n(x,y)=T_1[e^{-\mu^2 t}H_n(x,t)] = \frac{yn!}{2\pi}\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}(2\mu)^{n-j+1}}{(n-2j)!j!r^{n-j+1}}K_{n-j+1}(\mu r)$$

and

(3.18)

$$\begin{aligned} V_n(x,y) &= T_2[e^{-\mu^2 t} H_n(x,t)] \\ &= -\frac{n!}{\pi} \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j} (2\mu)^{n-j}}{(n-2j)!j!r^{n-j}} K_{n-j}(\mu r). \end{aligned}$$

To determine the generating functions for these types of solutions, we transform the generating function for the associated heat functions. Since

$$k(x-2a,t) = \sum_{n=0}^{\infty} H_n(x,t) \frac{a^n}{n!}, \tag{3.19}$$

it follows that

(3.20)

$$\begin{aligned} T_1[e^{-\mu^2 t} k(x-2a,t)] &= \frac{\mu y}{\pi} \cdot \frac{K_1(\mu\sqrt{(x-2a)^2+y^2})}{\sqrt{(x-2a)^2+y^2}} \\ &= \sum_{n=0}^{\infty} U_n(x,y) \frac{a^n}{n!} \end{aligned}$$

and

(3.21)

$$\begin{aligned} T_2[e^{-\mu^2 t} k(x-2a,t)] &= -\frac{1}{\pi} \cdot K_0(\mu\sqrt{(x-2a)^2+y^2}) \\ &= \sum_{n=0}^{\infty} V_n(x,y) \frac{a^n}{n!}. \end{aligned}$$

To show that the associated functions $U_n(x,y)$ and $V_n(x,y)$ reduce to the analogous associated functions for Laplace's equation, we use the asymptotic expansions for the Bessel functions $K_1(z)$ and $K_0(z)$, namely $K_1(z) \sim 1/z$ and $K_0(z) \sim -\ln(z)$ as $z \to 0$. Hence, as $\mu \to 0$, we have the asymptotic relations

$$\begin{aligned} \frac{\mu y}{\pi} \cdot \frac{K_1(\mu\sqrt{(x-a)^2+y^2})}{\sqrt{(x-a)^2+y^2}} &\sim \frac{1}{\pi} \cdot \frac{y}{(x-a)^2+y^2}, \\ -\frac{1}{\pi} K_0(\sqrt{(x-a)^2+y^2})) &\sim \frac{1}{2\pi} \cdot \log[(x-a)^2+y^2] \end{aligned}$$

except for a superfluous constant in the second case. These are the generating functions for the analogous associated functions for Laplace's equation found in [**5**].

We can obtain recurrence relations for the associated functions by using the formulas (3.15) and (3.16). These relations are given by the following equations:

(3.22)
$$\frac{\partial U_n}{\partial x} = \frac{(-2)^n \mu y}{\pi} \cdot \frac{\partial^{n+1}}{\partial x^{n+1}} \frac{K_1(\mu r)}{r} = -\frac{1}{2} U_{n+1}$$

(3.23)
$$\frac{\partial V_n}{\partial x} = \frac{(-2)^n}{\pi} \cdot \frac{\partial^{n+1}}{\partial x^{n+1}} K_0(\mu r) = -\frac{1}{2} V_{n+1}$$

(3.24)
$$\begin{aligned}\frac{\partial V_n}{\partial y} &= -\frac{(-2)^n}{\pi} \cdot \frac{\partial^n}{\partial x^n} \frac{\partial}{\partial y} K_0(\mu r) \\ &= \frac{(-2)^n}{\pi} \frac{\partial^n}{\partial x^n} \left(\frac{\mu y K_1(\mu r)}{r} \right) = U_n\end{aligned}$$

(3.25)
$$\frac{\partial U_n}{\partial y} = \frac{\partial^2 V_n}{\partial y^2} = \mu^2 V_n - \frac{\partial^2 V_n}{\partial x^2} = \mu^2 V_n - \frac{1}{4} V_{n+2}.$$

In this case, we also have pairs of panharmonic functions which are right-regular. Let $U(x,y) = U_n(x,y)$ and $V(x,y) = -\mu V_n(x,y) - V_{n+1}(x,y)/2$. Then it follows that

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} = \mu U \quad \text{and} \quad \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = -\mu V. \tag{3.26}$$

4. The Helmholtz equation. The counterparts of the transmutations for the Helmholtz equation would appear to be

$$\begin{aligned}T_1 w(x,t) &= \frac{y}{\sqrt{4\pi}} \int_0^\infty s^{-3/2} e^{-y^2/4s} e^{\mu^2 s} h(x,s)\, ds \\ T_2 w(x,t) &= -\frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-y^2/4s} e^{\mu^2 s} h(x,s)\, ds.\end{aligned}$$

Unfortunately, these integrals fail to converge. Hence, we must take a different approach for building up the basic functions for this equation.

In the Helmholtz equation, we replace x by ix and y by t, and the equation then becomes the damped wave equation given by

$$u_{tt} = u_{xx} - \mu^2 u. \tag{4.1}$$

We relate the solutions of this equation to those of the heat equation using the transmutations T_3 and T_4 in equations (2.12) and (2.13).

We begin by transforming constant multiples c_n of the heat polynomials where the c_n are to be determined. We find that

$$\begin{aligned}\tilde{u}_n(ix,t) &= c_n t\sqrt{\pi}\mathcal{L}_s^{-1}\left\{s^{-1/2}e^{-\mu^2/4s}h_n(x,1/4s)\right\}_{s\to t^2} \\ &= c_n t\sqrt{\pi}n!\sum_{j=0}^{[n/2]}\frac{x^{n-2j}}{(n-2j)!j!4^j}\mathcal{L}_s^{-1}\{s^{-j-1/2}e^{-\mu^2/4s}\}_{s\to t^2} \\ &= c_n\sqrt{\frac{\mu\pi}{2}}n!\sum_{j=0}^{[n/2]}\frac{x^{n-2j}t^{j+1/2}}{(n-2j)!j!(2\mu)^j}\cdot J_{j-1/2}(\mu t)\end{aligned} \tag{4.2}$$

from which it follows that

$$\tilde{u}_n(x,y) = c_n\sqrt{\frac{\mu\pi}{2}}n!\sum_{j=0}^{[n/2]}\frac{(-ix)^{n-2j}y^{j+1/2}}{(n-2j)!j!(2\mu)^j}J_{j-1/2}(\mu y).$$

In order to satisfy the initial condition $\tilde{u}_n(x,0) = x^n$, we must choose the constant c_n so that $c_n(-i)^n = 1$. With this choice for c_n, we obtain

$$\tilde{u}_n(x,y) = \sqrt{\frac{\mu\pi}{2}}n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}y^{j+1/2}}{(n-2j)!j!(2\mu)^j}\cdot J_{j-1/2}(\mu y). \tag{4.3}$$

Let us recall the following relation between the Bessel functions and the reverse Bessel polynomials for $j \geq 1$:

$$J_{j-1/2}(\mu y) = \frac{1}{\sqrt{2\pi}i(\mu y)^{j-1/2}}[e^{i\mu y}\Theta_{j-1}(-i\mu y) - e^{-i\mu y}\Theta_{j-1}(i\mu y)].$$

Using this in (4.3), we find that

$$\begin{aligned}\tilde{u}_n(x,y) &= x^n\cos\mu y \\ &+\mu y n!\sum_{j=1}^{[n/2]}\frac{(-1)^j x^{n-2j}}{(n-2j)!j!(2\mu^2)^j}\left\{\frac{e^{i\mu y}\Theta_{j-1}(-i\mu y) - e^{-i\mu y}\Theta_{j-1}(i\mu y)}{2i}\right\}.\end{aligned} \tag{4.4}$$

A few examples of these functions are

$$\tilde{u}_0 = \cos \mu y, \qquad \tilde{u}_1 = x \cos \mu y, \qquad \tilde{u}_2 = x^2 \cos \mu y - \frac{y}{\mu} \sin \mu y$$
$$\tilde{u}_3 = x^3 \cos \mu y - \frac{3xy}{\mu} \sin \mu y$$
$$\tilde{u}_4 = x^4 \cos \mu y - \frac{6x^2 y}{\mu} \sin \mu y + \frac{3y}{\mu^3}(\sin \mu y - \mu y \cos \mu y).$$

A similar calculation involving the transformation of the heat polynomials using T_4 results in

$$\tilde{v}_n(x,y) = \frac{n!}{\mu} \cdot \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}}{(n-2j)!j!(2\mu^2)^j} \cdot \left\{ \frac{e^{i\mu y}\Theta_j(-i\mu y) - e^{-i\mu y}\Theta_j(i\mu y)}{2i} \right\}. \tag{4.5}$$

A few examples of these solution functions are given as follows:

$$\tilde{v}_0 = \frac{\sin \mu y}{\mu}, \qquad \tilde{v}_1 = \frac{x \sin \mu y}{\mu}$$
$$\tilde{v}_2 = \frac{x^2 \sin \mu y}{\mu} - \frac{\sin \mu y}{\mu^3} + \frac{y \cos \mu y}{\mu^2}$$
$$\tilde{v}_3 = \frac{x^3 \sin \mu y}{\mu} - \frac{3x \sin \mu y}{\mu^3} + \frac{3xy \cos \mu y}{\mu^2}.$$

The $\tilde{v}_n(x,y)$ satisfy the initial condition

$$\frac{\partial \tilde{v}_n(x,0)}{\partial y} = x^n. \tag{4.6}$$

We can obtain the generating functions for $\tilde{u}_n$ and $\tilde{v}_n$ by observing that these two functions can be obtained from u_n and v_n, respectively, by replacing μ by $i\mu$. Therefore,

$$\tilde{u}(x,y,a) = \sum_{n=0}^{\infty} \tilde{u}_n(x,y) \frac{a^n}{n!} = e^{ax} \cos \sqrt{\mu^2 + a^2} \cdot y \tag{4.7}$$

$$\tilde{v}(x,y,a) = \sum_{n=0}^{\infty} \bar{v}_n(x,y) \frac{a^n}{n!} = e^{ax} \cdot \frac{\sin \sqrt{\mu^2 + a^2} \cdot y}{\sqrt{\mu^2 + a^2}}. \tag{4.8}$$

The following recurrence relations can be obtained from these generating functions by the usual differentiation procedures:

$$\frac{\partial \tilde{u}_n(x,y)}{\partial x} = n\tilde{u}_{n-1}(x,y) \tag{4.9}$$

$$\frac{\partial \tilde{u}_n(x,y)}{\partial y} = -\mu^2\tilde{v}_n(x,y) - n(n-1)\tilde{v}_{n-2}(x,y) \tag{4.10}$$

$$\frac{\partial \tilde{v}_n(x,y)}{\partial x} = n\tilde{v}_{n-1}(x,y) \tag{4.11}$$

$$\frac{\partial \tilde{v}_n(x,y)}{\partial y} = \tilde{u}_n(x,y). \tag{4.12}$$

If we let $\tilde{u}(x,y) = \tilde{u}_n(x,y)$ and $\tilde{v}(x,y) = -i\mu\tilde{v}_n(x,y) + n\tilde{v}_{n-1}$, then it follows from the above relations that $\partial\tilde{u}/\partial x - \partial\tilde{v}/\partial y = i\mu\tilde{u}$ and $\partial\tilde{u}/\partial y + \partial\tilde{v}/\partial x = -i\mu\tilde{v}$. Therefore, $\tilde{u} + i\tilde{v}$ is right-regular.

Before defining the associated functions, we must first construct the fundamental solutions for the Helmholtz equation. Since $(\mu y/\pi)\cdot K_1(\mu r)/r$ is a solution of the Yukawa equation, it is clear that $(i\mu y/\pi)\cdot K_1(i\mu r)/r$ and $(-i\mu y/\pi)\cdot K_1(-i\mu r)/r$ are solutions of the Helmholtz equation. The objective is to combine these solutions to obtain the proper asymptotic forms as μ approaches zero. Since $i\mu K_1(i\mu y) = (-i\mu\pi/2)[J_1(\mu r) - iY_1(\mu r)]$ and $-i\mu K_1(-i\mu y) = (i\mu\pi/2)[J_1(\mu r) + iY_1(\mu r)]$, we find

$$\frac{i\mu y}{2\pi}\cdot\frac{K_1(i\mu r)}{r} - \frac{i\mu y}{2\pi}\cdot\frac{K_1(-i\mu r)}{r} = -\frac{\mu y}{2}\cdot\frac{Y_1(\mu r)}{r}$$

and $Y_1(\mu r) \sim 2/(\mu\pi r)$ as μ approaches zero. Hence, $-(\mu y/2)\cdot Y_1(\mu r)/r \sim y/(\pi(x^2+y^2))$ which is the fundamental solution for Laplace's equation. Therefore, we define the fundamental solution of the Helmholtz equation as $-(\mu y/2)\cdot Y_1(\mu r)$ and the associated functions by

$$\widetilde{U}_n(x,y) = -(-2)^n\frac{\partial^n}{\partial x^n}\left[\frac{\mu y}{2}Y_1(\mu r)\right]. \tag{4.13}$$

A similar calculation gives us the fundamental solution $Y_0(\mu r)/2$ and the other set of associated functions, namely

$$\tilde{V}_n(x,y) = (-2)^n\frac{\partial^n}{\partial x^n}\left[\frac{Y_0(\mu r)}{2}\right]. \tag{4.14}$$

More explicit versions of $\widetilde{U}_n$ and $\widetilde{V}_n$ can be found by substituting $i\mu$ and $-i\mu$ for μ in the formulas (3.17) and (3.18) for U_n and V_n and using the following properties of K_v when v is an integer: $K_v(\pm i\mu y) = (1/2)\pi(\mp i)^{v+1}[J_v(\mu r) \mp iY_v(\mu r)]$. The results are

$$\widetilde{U}_n(x,y) = -\frac{n!y}{4} \cdot \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}(2\mu)^{n-j+1}}{(n-2j)!j!r^{n-j+1}} \cdot Y_{n-j+1}(\mu r) \tag{4.15}$$

and

$$\widetilde{V}_n(x,y) = \frac{n!}{2} \cdot \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j}(2\mu)^{n-j}}{(n-2j)!j!r^{n-j}} \cdot Y_{n-j}(\mu r). \tag{4.16}$$

By analogy with Section 3, the generating functions for the $\widetilde{U}_n$ and $\widetilde{V}_n$ are

$$\begin{aligned} \widetilde{U}(x,y,a) &= \sum_{n=0}^{\infty} \widetilde{U}_n(x,y)\frac{a^n}{n!} \\ &= -\frac{\mu y}{2} \cdot \frac{Y_1(\mu\sqrt{(x-2a)^2+y^2)}}{\sqrt{(x-2a)^2+y^2)}} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \widetilde{V}(x,y,a) &= \sum_{n=0}^{\infty} \widetilde{V}_n(x,y)\frac{a^n}{n!} \\ &= \frac{1}{2}Y_0(\mu\sqrt{(x-a)^2+y^2}). \end{aligned} \tag{4.18}$$

Recurrence relations for the $\widetilde{U}_n$ and $\widetilde{V}_n$ can be found as in Section 3, namely,

$$\frac{\partial \widetilde{U}_n}{\partial x} = -\frac{\widetilde{U}_{n+1}}{2} \tag{4.19}$$

$$\frac{\partial \widetilde{U}_n}{\partial y} = -\mu^2 \widetilde{V}_n - \frac{\widetilde{V}_{n+2}}{4} \tag{4.20}$$

$$\frac{\partial \widetilde{V}_n}{\partial x} = -\frac{\widetilde{V}_{n+1}}{2} \tag{4.21}$$

$$\frac{\partial \widetilde{V}_n}{\partial y} = \widetilde{U}_n. \tag{4.22}$$

Once again, if we let $\widetilde{U}(x,y) = \widetilde{U}_n(x,y)$ and $\widetilde{V}(x,y) = -i\mu\widetilde{V}_n(x,y) - \widetilde{V}_{n+1}(x,y)/2$, then the validation, as usual, of the equations $\partial\widetilde{U}/\partial x - \partial\widetilde{V}/\partial y = i\mu\widetilde{U}$ and $\partial\widetilde{U}/\partial y + \partial\widetilde{V}/\partial x = -i\mu\widetilde{U}$ verify that the function $\widetilde{U} + i\widetilde{V}$ is right-regular.

5. Solutions with analytic data. In the case of Laplace's equation, the power series in z converging inside a disk centered at the origin is the classical case of solutions with analytic data. Outside such a disk, one considers power series in z^{-1}. In the cases of the Yukawa and Helmholtz equations, the counterparts are series of basic solutions for the interior problem and series of associated functions for the exterior problem. In this section we shall indicate that the results for Laplace's equation have their counterparts for the Yukawa and Helmholtz equations. There are eight different cases to consider. However, for the sake of brevity, we shall treat only two of them because the approach is the same. We shall show that each class of functions has its analogous set for the Laplace case and that these functions are asymptotic to the corresponding set for the Laplace equation. Furthermore, we shall show that the regions of convergence for the series considered depend on the coefficients in the series and not on the value of μ in the partial differential equation. Therefore, we can assume that μ is very small and use the asymptotic forms in the Laplace case to determine the region of convergence. As representative cases, we shall consider series of the forms $\sum_{n=0}^{\infty} a_n u_n(x,y)$ and $\sum_{n=0}^{\infty} b_n \widetilde{V}_{n+1}(x,y)$.

Before proceeding with these cases, we need to rewrite the special function solutions of the Yukawa and Helmholtz equations in more suitable forms which will permit showing that the convergence results are independent of the parameter μ. The required forms are given in the following with the notation indicating their dependence on the parameter μ:

(5.1)

$$u_n(x,y;\mu) = x^n\left[\cosh\mu y + \mu y n! \sum_{j=1}^{[n/2]} \frac{e^{-\mu y}\Theta_{j-1}(\mu y) - e^{\mu y}\Theta_{j-1}(-\mu y)}{2^{j+1}(n-2j)!j!(\mu x)^{2j}}\right]$$

(5.2)

$$v_n(x,y;\mu) = x^n y\left[\frac{n!}{\mu y}\sum_{j=0}^{[n/2]} \frac{e^{\mu y}\Theta_j(-\mu y) - e^{-\mu y}\Theta_j(\mu y)}{2^{j+1}(n-2j)!j!(\mu x)^{2j}}\right]$$

(5.3)

$$U_n(x,y;\mu) = \frac{1}{r^{n+1}}\left[\frac{\mu y n!}{\pi}\sum_{j=0}^{[n/2]} \frac{(-1)^j(\mu x)^{n-2j}2^{n-j}}{(n-2j)!j!(\mu r)^{-j}}K_{n-j+1}(\mu r)\right]$$

(5.4)

$$V_n(x,y;\mu) = \frac{1}{r^n}\left[\frac{-n!}{\pi}\sum_{j=0}^{[n/2]} \frac{(-1)^j(\mu x)^{n-2j}2^{n-j}}{(n-2j)!j!(\mu r)^{-j}}K_{n-j}(\mu r)\right]$$

(5.5)

$$\tilde{u}_n(x,y;\mu) = x^n\Bigg[\cos\mu y + \mu y n! i$$

$$\sum_{j=1}^{[n/2]} \frac{(-1)^j\{e^{-i\mu y}\Theta_{j-1}(i\mu y) - e^{i\mu y}\Theta_{j-1}(-i\mu y)\}}{2^{j+1}(n-2j)!j!(\mu x)^{2j}}\Bigg]$$

(5.6)

$$\tilde{v}_n(x,y;\mu) = x^n y\left[\frac{n!i}{\mu y}\sum_{j=0}^{[n/2]} \frac{(-1)^j\{e^{-i\mu y}\Theta_j(i\mu y) - e^{i\mu y}\Theta_j(-i\mu y)}{2^{j+1}(n-2j)!j!(\mu x)^{2j}}\right]$$

(5.7)

$$\widetilde{U}_n(x,y;\mu) = \frac{1}{r^{n+1}}\left[-\frac{\mu y n!}{2}\sum_{j=0}^{[n/2]} \frac{(-1)^j(\mu x)^{n-2j}2^{n-j}}{(n-2j)!j!(\mu r)^{-j}}Y_{n-j+1}(\mu r)\right]$$

(5.8)

$$\widetilde{V}_n(x,y;\mu) = \frac{1}{r^n}\left[\frac{n!}{2}\sum_{j=0}^{[n/2]} \frac{(-1)^j(\mu x)^{n-2j}2^{n-j}}{(n-2j)!j!(\mu r)^{-j}} Y_{n-j}(\mu r)\right].$$

Lemma 5.1. *Assume that the series* $\sum_{n=0}^{\infty} a_n u_n(x,y;\varepsilon)$ *converges for* $\rho = \sqrt{x^2+y^2} < \sigma$, *where* $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\sigma$ *for some* $\varepsilon > 0$. *Then, if* $\mu > 0$, *the series* $\sum' = \sum_{n=0}^{\infty} a_n u_n(x,y;\mu)$ *also converges for* $\rho < \sigma$.

Proof. Introduce the dilation of coordinates $\mu x = \varepsilon\xi$, $\mu y = \varepsilon\eta$. Then, by equation (5.1), the sum $\sum'$ becomes $\sum_{n=0}^{\infty} a_n(\varepsilon/\mu)^n u_n(\xi,n;\varepsilon)$, and this converges for $\rho' = \sqrt{\xi^2+\eta^2} < \sigma'$ where $\limsup_{n\to\infty}(\varepsilon/\mu)|a_n|^{1/n} = (\varepsilon/\mu)/\sigma$ or where $\sigma' = (\mu/\varepsilon)\sigma$. Thus, the region of convergence for the sum $\sum'$ is $\rho' = \sqrt{(\mu x/\varepsilon)^2 + (\mu y/\varepsilon)^2} = (\mu/\varepsilon)\rho$ and this last term is $< (\mu/\varepsilon)\sigma$. Hence, it follows that $\rho < \sigma$. $\square$

Lemma 5.2. *Assume that the series* $\sum_{n=0}^{\infty} b_n \widetilde{V}_n(x,y;\varepsilon)$ *converges for* $\rho = \sqrt{x^2+y^2} > \sigma$, *where* $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma$, *for some* $\varepsilon > 0$. *Then, if* $\mu > 0$, *the series* $\sum' = \sum_{n=0}^{\infty} b_n \widetilde{V}_n(x,y;\mu)$ *also converges for* $\rho > \sigma$.

Proof. Introduce the dilation of coordinates $\mu x = \varepsilon\xi$, $\mu y = \varepsilon\eta$ as in Lemma 5.1. Then, by the equation (5.8), the sum $\sum'$ becomes $\sum_{n=0}^{\infty} b_n(\mu/\varepsilon)^n \widetilde{V}_n(\xi,\eta;\varepsilon)$ and this converges for $\rho' = \sqrt{\xi^2+\eta^2} > \sigma'$ where $\sigma' = \limsup_{n\to\infty}(\mu/\varepsilon)2n|b_n|^{1/n}/e = (\mu/\varepsilon)\sigma$. Therefore, the region of convergence for $\sum'$ is $\rho' = \sqrt{(\mu x/\varepsilon)^2 + (\mu y/\varepsilon)^2} = (\mu/\varepsilon)\rho > (\mu/\varepsilon)\sigma$ or $\rho > \sigma$. $\square$

Theorem 5.1. *Let* a_n *be real,* $n = 0,1,2,\dots$, *and suppose that* $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\sigma$. *Then the series* $u(x,y) = \sum_{n=0}^{\infty} a_n u_n(x,y;\mu)$ *converges to a solution of the Yukawa equation for* $r = \sqrt{x^2+y^2} < \sigma$ *but does not converge everywhere in any including disk. Furthermore,* $u(x,0) = \phi(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $\phi(x)$ *is analytic for* $|x| < \sigma$.

Proof. As indicated in Lemma 5.1, we can assume that the parameter μ is very small. With this in mind, we can show that the function $u_n(x,y;\mu)$ is asymptotic to $\mathrm{Re}\,(z^n)$ as μ approaches zero. The generating function for the solutions $u_n(x,y;\mu)$ as given by (3.4) is

$$g(x,y;\mu,a) = e^{ax}\cosh(\sqrt{\mu^2-a^2}y) = e^{ax}\sum_{k=0}^{\infty}\frac{(\mu^2-a^2)^k y^{2k}}{(2k)!}.$$

This is an entire function of (μ,a) for fixed values of (x,y). As such, it has continuous partial derivatives. Therefore,

$$\begin{aligned}\lim_{\mu\to 0} u_n(x,y;\mu) &= \lim_{\mu\to 0}\frac{\partial^n g(x,y;\mu,a)}{\partial a^n}\bigg|_{a=0} = \frac{\partial^n}{\partial a^n}g(x,y;0,a)\bigg|_{a=0} \\ &= \frac{\partial^n}{\partial a^n}e^{ax}\cos ay\bigg|_{a=0} = \mathrm{Re}\,(z^n).\end{aligned} \tag{5.9}$$

It is well known that the series $\sum_{n=0}^{\infty} a_n r^n \cos n\theta$ converges absolutely if $r<\sigma$ and uniformly if $r\le\sigma'<\sigma$ for any fixed σ'. Also, this series diverges if $\theta=0$ and $r>\sigma$. Using Taylor series expansions, it is not difficult to show that the second term in the asymptotic expansion as $\mu\to 0$ for $u_n(x,y;\mu)$ is

$$R_n(x,y;\mu) = \frac{\mu^2 y^2}{2} n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}y^{2j}}{(n-2j)!(2j+1)!}. \tag{5.10}$$

To obtain a bound for $|R_n|$, we compare this series with the series

$$\mathrm{Re}\,(z^n) = n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}y^{2j}}{(n-2j)!(2j)!} \tag{5.11}$$

which attains its maximum r^n when $y=0$ and $x=r$. We can rewrite

$$n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}y^{2j}}{(n-2j)!(2j+1)!} = n!\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}[y/(2j+1)^{1/2j}]^{2j}}{(n-2j)!(2j)!} \tag{5.12}$$

which also attains its maximum when $y=0$ and $x=r$. Hence, $|R_n(x,y;\mu)|\le \mu^2 r^{n+2}/2$. As a result, we see that

$$u_n(x,y;\mu) - \mathrm{Re}\,(z^n) = 0(\mu^2 r^{n+2}/2). \tag{5.13}$$

It is clear from this that the u_n has essentially the same growth as $\mathrm{Re}\,(z^n)$ and the series $\sum_{n=0}^{\infty} a_n u_n(x,y;\mu)$ converges absolutely when $r < \sigma$ and uniformly when $r \le \sigma' < \sigma$. By substitution $\sum_{n=0}^{\infty} a_n u_n(x,0;\mu) = \sum_{n=0}^{\infty} a_n x^n = \phi(x)$ is analytic for $|x| < \sigma$.

To show that $u(x,y)$ satisfies the Yukawa equation for $r < \sigma$, we must establish the uniform convergence of the series involving the first and second partial derivatives of the $u_n(x,y;\mu)$. This is easy to do by using the recurrence relations (3.6), (3.7) and the pair of second order differentiation formulas $\partial^2 u_n/\partial x^2 = n(n-1)u_{n-2}$ and $\partial^2 u_n/\partial y^2 = \mu^2 u_n - n(n-1)u_{n-2}$ that follow from these. In view of the fact that the formula (3.7) involves the v_n functions, one requires asymptotic forms analogous to (5.10) for these v_n to completely establish the uniform convergence of the series for $\partial u/\partial y$. But these forms can be obtained by using an argument quite similar to the one above for obtaining the asymptotic forms for the u_n. For this, one needs to call upon the formula (3.5). This completes the proof. □

Theorem 5.2. *Let b_n be real, $n = 0,1,2,\dots$, and suppose that $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e = \sigma$. Then the series $\widetilde{V}(x,y) = \sum_{n=0}^{\infty} b_n \widetilde{V}_{n+1}(x,y;\mu)$ converges to a solution of the Helmholtz equation for $r > \sigma$ but does not converge everywhere for $r > \sigma - \varepsilon$, $\varepsilon > 0$.*

Proof. We shall again assume that μ is very small and use asymptotic forms for the $\widetilde{V}_n(x,y;\mu)$ as $\mu \to 0$. The fundamental solution for this equation is $Y_0(\mu r)/2$ which we have shown to be asymptotic to $(2\pi)^{-1}\ln(x^2+y^2)$. This implies that

$$\widetilde{V}_n(x,y;\mu) \sim \frac{-2^n(n-1)!}{\pi}\mathrm{Re}\,\Big(\frac{z^n}{r^{2n}}\Big), \quad n = 1,2,3,\dots.$$

Using asymptotic expansions of the $Y_n(\mu r)$, $n = 1,2,3,\dots$, (see [**11**]), we can derive, for $n \ge 3$, the formula

$$\begin{aligned}\widetilde{V}_n(x,y;\mu) \sim &-\frac{n!}{2\pi}\sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j} 4^{n-j}(n-j-1)!}{(n-2j)!j!(r^2)^{n-j}}\\ &-\frac{\mu^2 r^2}{8\pi} n! \sum_{j=0}^{[n/2]} \frac{(-1)^j x^{n-2j} 4^{n-j}(n-j-2)!}{(n-2j)!j!(r^2)^{n-j}}.\end{aligned} \tag{5.14}$$

The first term in the right member of this is $-2^n(n-1)!\mathrm{Re}\,(z^n/r^{2n})/\pi$ which is bounded by $2^n(n-1)!r^{-n}/\pi$. Dividing out the factor $-2^n(n-1)!/(\pi r^n)$ from this term, we have

$$\text{(5.15)} \qquad \mathrm{Re}\,\left(\frac{z^n}{r^n}\right) = 2^{n-1}n\sum_{j=0}^{[n/2]}\frac{(-1)^j x^{n-2j}(n-j-1)!}{(n-2j)!j!r^{n-2j}4^j}.$$

This sum attains its maximum of 1 when $x = r$ and $y = 0$. If we let B_n denote the sum in the right member of (5.15) with these choices, we get

$$\text{(5.16)} \qquad B_n = 2^{n-1}n\sum_{j=0}^{[n/2]}\frac{(-1)^j(n-j-1)!}{(n-2j)!j!4^j} = 1.$$

To obtain a bound for the second term in the right member of (5.14), we compare that sum with the following sum:

$$\text{(5.17)} \quad C_n = -2^{n-1}n\sum_{j=0}^{[n/2]}\frac{(-1)^j(n-j-1)!}{(n-2j)!j!4^j(n-j-1)}, \quad n = 3, 4, 5, \dots .$$

We find that C_n is positive and that the sum

$$\text{(5.18)} \qquad B_n + C_n = 2^{n-1}n\sum_{j=0}^{[n/2]}\frac{(-1)^j(n-j-1)!}{(n-2j)!j!4^j}\left[1 - \frac{1}{n-j-1}\right]$$

approaches 1 as n tends to infinity. This establishes that C_n is bounded. The second term in the asymptotic expansion of $\widetilde{V}_n(x, y; \mu)$ is bounded by $\mu^2 r^2 2^n(n-1)!C_n/(4\pi r^n)$. Hence, it follows that

$$\text{(5.19)} \quad \widetilde{V}_n(x, y; \mu) + 2^n(n-1)!\frac{\mathrm{Re}\,(z^n/r^{2n})}{\pi} = O\left(\frac{2^n(n-1)!\mu^2}{r^{n-2}}\right). \qquad \square$$

We will now make a comparison of the series $\sum_{n=0}^{\infty} b_n\widetilde{V}_{n+1}(x, y; \mu)$ with the series $-\sum_{n=0}^{\infty} 2^{n+1}n!b_n\mathrm{Re}\,(z^{n+1}/r^{2n+2})/\pi$. Using Stirling's formula for $n!$, it can be shown that $\limsup_{n\to\infty}(2^n n!|b_n|)^{1/n} = \sigma$, and that this last series converges absolutely for $r > \sigma$ and uniformly for $r \geq \sigma' > \sigma$ for any fixed σ' (see [**5**]). It diverges for $y = 0$, $|x| < \sigma$. By comparison, the series $\sum_{n=0}^{\infty} b_{n+1} \widetilde{V}_n(x, y; \mu)$ exhibits this same convergence and divergence behavior under the same restrictions.

To show that the function $\widetilde{V}(x, y)$ satisfies the Helmholtz equation for $r > \sigma$, we must establish the uniform convergence of the series involving the first and second partial derivatives. This can be done by calling upon the recurrence relations (4.21) and (4.22) and the second derivative formulas that follow from these, namely, $\partial^2 \widetilde{V}_n/\partial x^2 = \widetilde{V}_{n+2}/4$ and $\partial^2 \widetilde{V}_n/\partial y^2 = -\mu^2 \widetilde{V}_n - \widetilde{V}_{n+2}/4$. Since the formula (4.22) involves the function $\widetilde{U}_n$, an asymptotic bound is also needed for this function. But this can be developed from the formula (5.7) by using arguments similar to those employed above.

6. Solutions corresponding to entire data. In this section we develop integral type formulas for the solutions of the Yukawa and Helmholtz equations in terms of solutions of the wave equation when the data is entire. For this purpose, we first derive representations of the Bessel polynomials and reverse Bessel polynomials in the form of Laplace transforms.

The Bessel polynomials can be written as

$$y_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} (n+k)! \frac{x^k}{2^k}, \quad n = 0, 1, 2, \ldots . \tag{6.1}$$

Replacing x by $1/s$, we have

$$\frac{1}{s^{n+1}} y_n\left(\frac{1}{s}\right) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^k} \frac{(n+k)!}{s^{n+k+1}}.$$

Taking the inverse Laplace transform with x the variable of inversion, we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}} y_n\left(\frac{1}{s}\right)\right\}_{s \to x} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{x^{n+k}}{2^k} = \frac{x^n}{n!}(1 + x/2)^n.$$

Hence,

$$\frac{1}{s^{n+1}} y_n\left(\frac{1}{s}\right) = \frac{1}{n!} \int_0^{\infty} e^{-s\sigma} [\sigma(1 + \sigma/2)]^n \, d\sigma.$$

Upon replacing s by $1/x$ (assuming that $x > 0$) and then making the replacement $\sigma \to 2x\sigma$ for the variable of integration, we obtain the following result:

$$y_n(x) = \frac{2^{n+1}}{n!} \int_0^\infty e^{-2\sigma} [\sigma(1 + x\sigma)]^n \, d\sigma. \tag{6.2}$$

This was derived by assuming $x > 0$, but it holds for all x. The reverse Bessel polynomial is defined as $\Theta_n(x) = x^n y_n(1/x)$, and this along with (6.2) implies that

$$\Theta_n(x) = \frac{2^{n+1}}{n!} \int_0^\infty e^{-2\sigma} [\sigma(x + \sigma)]^n \, d\sigma. \tag{6.3}$$

Using this formula we can express the functions $u_n(x, y)$ as

$$u_n(x,y) = x^n \cosh \mu y + \mu y n! \sum_{j=1}^{[n/2]} \frac{x^{n-2j}}{(n-2j)! j! (2\mu^2)^j} \times \frac{2^{j-1}}{(j-1)!} \int_0^\infty e^{-2\sigma} \sigma^{j-1} [e^{-\mu y}(\mu y + \sigma)^{j-1} - e^{\mu y}(-\mu y + \sigma)^{j-1}] \, d\sigma$$

which can be rearranged into the form

$$\begin{aligned} u_n(x,y) &= x^n \cosh \mu y \\ &+ \mu y \int_0^\infty e^{-2\sigma} \bigg[\frac{e^{-\mu y}}{2\mu^2} \sum_{j=1}^{[n/2]} \frac{n!}{(n-2j)! j! (j-1)!} \left\{ \frac{\sigma(\mu y + \sigma)}{\mu^2} \right\}^{j-1} x^{n-2j} \\ &\qquad - \frac{e^{\mu y}}{2\mu^2} \sum_{j=1}^{[n/2]} \frac{n!}{(n-2j)! j! (j-1)!} \left\{ \frac{\sigma(-\mu y + \sigma)}{\mu^2} \right\}^{j-1} x^{n-2j} \bigg] \, d\sigma. \end{aligned} \tag{6.4}$$

It was shown in [**5**] that the wave polynomials $w_n(x,t)$ can be expressed as

$$w_n(x,t) = \frac{(x+t)^n + (x-t)^n}{2} = t\sqrt{\pi} \mathcal{L}^{-1} \left\{ s^{-1/2} h_n(x, 1/4s) \right\}_{s \to t^2} \tag{6.5}$$

or

(6.6)
$$w_n(x,\sqrt{\tau}) = \sqrt{\pi\tau}\mathcal{L}^{-1}\left\{s^{-1/2}h_n(x,1/4s)\right\}_{s\to\tau}$$

where $h_n(x,t)$, as usual, denotes a heat polynomial.

For $n \geq 2$, we define $S_n(x,t)$ by the sum:

$$S_n(x,t) = \sum_{j=1}^{[n/2]} \frac{n!}{(n-2j)!j!(j-1)!} t^{j-1}x^{n-2j}$$

which, after a change of index of summation $k = j-1$ becomes

$$S_n(x,t) = \sum_{k=0}^{[(n-2)/2]} \frac{n!}{(n-2-2k)!(k+1)!k!} t^{k-1}x^{n-2-2k}. \tag{6.7}$$

Then

$$t\cdot S_n(x,t) = n(n-1)\sum_{k=1}^{[(n-2)/2]} \frac{(n-2)!2^{2k}}{(n-2-2k)!k!(k+1)!} x^{n-2-2k}\frac{t^{k+1}}{2^{2k}}.$$

Taking the Laplace transform of this on the t variable, we get

$$\begin{aligned}\mathcal{L}\{tS_n(x,t)\} &= \frac{n(n-1)}{s^2}2^{n-2}\sum_{k=0}^{[(n-2)/2]} \frac{(n-2)!}{(n-2-2k)!k!}\frac{(x/2)^{n-2-2k}}{(4s)^k} \\ &= \frac{n(n-1)2^{n-2}}{s^2}h_{n-2}\left(\frac{x}{2},\frac{1}{4s}\right).\end{aligned}$$

Using (6.6), we can invert this transform to obtain

$$\begin{aligned}tS_n(x,t) &= n(n-1)2^{n-2}\left[\frac{t^{1/2}}{\Gamma(3/2)} * \left\{\frac{1}{\sqrt{\pi t}}w_{n-2}\left(\frac{x}{2},\sqrt{t}\right)\right\}\right] \\ &= \frac{2^{n-1}n(n-1)}{\pi}\int_0^1 (1-\sigma)^{1/2}\frac{w_{n-2}(x/2,\sqrt{t\sigma})}{\sqrt{\sigma}}\,d\sigma.\end{aligned}$$

After the change of variables $\sigma = t\lambda$, we have

$$(6.8)\quad S_n(x,t) = \frac{n(n-1)2^{n-1}}{\pi}\int_0^1 (1-\lambda)^{1/2} w_{n-2}\bigg(\frac{x}{2}, \sqrt{t\lambda}\bigg)\lambda^{-1/2}\,d\lambda.$$

From (6.7) and (6.8), and using the fact that $2^{n-2}w_{n-2}(x/2,\xi) = w_{n-2}(x, 2\xi)$, we deduce

(6.9)

$$u_n(x,y) = x^n\cosh\mu y + \frac{n(n-1)y}{\pi\mu}\int_0^\infty e^{-2\sigma} \times \bigg\{\int_0^1 (1-\lambda)^{1/2}\lambda^{-1/2}[e^{-\mu y}w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(\mu y+\sigma)\lambda}) - e^{\mu y}w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(-\mu y+\sigma)\lambda})]\,d\lambda\bigg\}\,d\sigma.$$

Suppose we require a solution of the Yukawa equation with data $u(x,0) = \phi(x) = \sum_{n=0}^\infty a_n x^n$ with ϕ entire of appropriate growth. Then

$$\begin{aligned} u(x,y) &= \sum_{n=0}^\infty a_n x^n\cosh\mu y \\ &\quad + \sum_{n=0}^\infty \frac{n(n-1)a_n y}{\mu\pi}\int_0^\infty e^{-2\sigma}\{(1-\lambda)^{1/2}\lambda^{-1/2} \\ &\qquad \times [e^{-\mu y}w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(\mu y+\sigma)\lambda}) \\ &\qquad\quad - e^{\mu y}w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(-\mu y+\sigma)\lambda})]\,d\lambda\}\,d\sigma \\ &= \phi(x)\cosh\mu y + \int_0^\infty e^{-2\sigma}\bigg\{\int_0^1 (1-\lambda)^{1/2}\lambda^{-1/2}\frac{y}{\mu\pi} \\ &\quad \times \bigg[e^{-\mu y}\sum_{n=0}^\infty n(n-1)a_n w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(\mu y+\sigma)\lambda}) \\ &\quad - e^{\mu y}\sum_{n=0}^\infty n(n-1)a_n w_{n-2}(x, 2\mu^{-1}\sqrt{\sigma(-\mu y+\sigma)\lambda})\bigg]\,d\lambda\bigg\}\,d\sigma. \end{aligned}$$

But $\sum_{n=0}^\infty n(n-1)a_n w_{n-2}(x,t) = [\phi''(x+t)+\phi''(x-t)]/2 = w(x,t)$ is a solution of the wave equation corresponding to the data $\phi''(x) =$

$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$. Therefore,

$$u(x,y) = \phi(x)\cosh\mu y + \int_0^\infty e^{-2\sigma}\bigg\{\int_0^1 (1-\lambda)^{1/2}\lambda^{-1/2}\frac{y}{\mu\pi} \times [e^{-\mu y}w(x, 2\mu^{-1}\sqrt{\sigma(\mu y+\sigma)\lambda}) - e^{\mu y}w(x, 2\mu^{-1}\sqrt{\sigma(-\mu y+\sigma)\lambda})]\bigg\}\,d\lambda\,d\sigma. \tag{6.10}$$

By carrying out analogous manipulations on the functions $v_n(x,y)$, we can also show that

$$v_n(x,y) = \frac{1}{\mu\pi}\int_0^\infty\int_0^1 e^{-2\sigma}\lambda^{-1/2}(1-\lambda)^{-1/2} \times [e^{\mu y}w_n(x, 2\mu^{-1}\sqrt{\lambda\sigma(-\mu y+\sigma)}) - e^{-\mu y}w_n(x, 2\mu^{-1}\sqrt{\lambda\sigma(-\mu y+\sigma)})]\,d\lambda\,d\sigma \tag{6.11}$$

and a solution $v(x,y)$ of the Yukawa equation satisfying the conditions $v(x,0)+0$ and $v_y(x,0)=\phi(x)$, with $\phi(x)$ entire and of appropriate growth, is

$$v(x,y) = \frac{1}{\mu\pi}\int_0^\infty\int_0^1 e^{-2\sigma}\lambda^{-1/2}(1-\lambda)^{-1/2} \times [e^{\mu y}w(x, 2\mu^{-1}\sqrt{\lambda\sigma(-\mu y+\sigma)}) - e^{-\mu y}w(x, 2\mu^{-1}\sqrt{\lambda\sigma(-\mu y+\sigma)})]\,d\lambda\,d\sigma \tag{6.12}$$

where the function $w(x,t)$ here is given by $[\phi(x+t)+\phi(x-t)]/2$. To construct integral formulas for the functions $\tilde{u}_n(x,y)$ and $\tilde{v}_n(x,y)$ and the solutions of the Helmholtz equation corresponding, respectively, to the conditions $\tilde{u}(x,0)=\phi(x)$, $\tilde{u}_y(x,0)=0$ or $\tilde{v}(x,0)=0$, $\tilde{v}_y(x,0)=\phi(x)$, replace μ by $i\mu$ in the above formulas for $u_n(x,y)$, $u(x,y)$, $v_n(x,y)$ and $v(x,y)$.

7. Fourier transform criteria. In [**13**], Rosenbloom and Widder developed a Fourier transform criteria for expansions of solutions of the heat equation in terms of associated heat polynomials. The authors [**5**] have shown that there are analogous criteria for the wave and Laplace equations. We again find that this is possible for the solutions of the Yukawa and Helmholtz equations.

To establish these, we must first find Fourier representations of the corresponding fundamental solutions. Starting with the formula 15 on page 306 of [**12**], it is easy to show that

$$\frac{-K_0(\mu r)}{\mu} = -\frac{1}{\pi}\int_\mu^\infty \frac{e^{-yt}\cos x\sqrt{t^2-\mu^2}}{\sqrt{t^2-\mu^2}}\,dt.$$

Introducing the change of variables $s=\sqrt{t^2-\mu^2}$, we find that

$$\begin{aligned}(7.1)\qquad -\frac{K_0(\mu r)}{\pi} &= -\frac{1}{\pi}\int_0^\infty \frac{e^{-y\sqrt{s^2+\mu^2}}\cos xs}{\sqrt{s^2+\mu^2}}\,ds\\ &= -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-y\sqrt{s^2+\mu^2}}e^{isx}}{\sqrt{s^2+\mu^2}}\,ds.\end{aligned}$$

From the formula $-\pi^{-1}\partial K_0(\mu r)/\partial y = \mu y K_1(\mu r)/(\pi r)$, we get

$$(7.2)\qquad \frac{\mu y}{\pi}\frac{K_1(\mu r)}{r} = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-y\sqrt{s^2+\mu^2}}e^{isx}\,ds.$$

For the case of the Helmholtz equation, we begin with formula 3, page 296 of [**12**], for $v=0$. The result is

$$H_0^{(2)}(\mu r) = \frac{2i}{\pi}\int_0^\infty \frac{e^{-\mu y(t+i)}\cos \mu x\sqrt{t^2+2it}}{\sqrt{t^2+2it}}\,dt.$$

After the change of variables, $\tau = t+i$, we obtain

$$H_0^{(2)}(\mu r) = \frac{2i}{\pi}\int_i^{i+\infty} \frac{e^{-\mu y\tau}\cos \mu x\sqrt{\tau^2+1}}{\sqrt{\tau^2+1}}\,d\tau.$$

The further change of variables $s=\mu\sqrt{\tau^2+1}$ then leads to the formula

$$H_0^{(2)}(\mu r) = \frac{2i}{\pi}\int_C \frac{e^{-y\sqrt{s^2-\mu^2}}\cos xs}{\sqrt{s^2-\mu^2}}\,ds$$

where the integration is over the contour C given by

$$s = \xi + i\eta = \mu t\biggl(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{t^2}}\biggr)^{1/2} + i\mu t\biggl(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{t^2}}\biggr)^{1/2}, \quad 0 \le t < \infty.$$

From the relation $H_0^{(1)}(\mu r) = \overline{H_0^{(2)}(\mu r)}$, we get

$$H_0^{(1)}(\mu r) = -\frac{2i}{\pi}\int_{\bar{C}} \frac{e^{-g\sqrt{s^2-\mu^2}}\cos xs}{\sqrt{s^2-\mu^2}}\, ds$$

where $\overline{C}$ is the reflection of C in the real axis. Then from the fact that $Y_0(\mu r) = [H_0^{(1)}(\mu r) + H_0^{(2)}(\mu r)]/2i$, we deduce the formula

$$\begin{aligned}\frac{Y_0(\mu r)}{2} &= -\frac{1}{2\pi}\int_C + \int_{\overline{C}} \biggl(\frac{e^{-y\sqrt{s^2-\mu^2}}\cos xs}{\sqrt{s^2-\mu^2}}\biggr)\, ds \\ &= -\frac{1}{4\pi}\int_C + \int_{\overline{C}} \biggl(\frac{e^{-y\sqrt{x^2-\mu^2}}}{\sqrt{s^2-\mu^2}}(e^{ixs} + e^{-ixs})\biggr)\, ds.\end{aligned}$$

Finally, for the second term in the integral, we make the replacement $s \to -s$ to get

$$\frac{Y_0(\mu r)}{2} = -\frac{1}{4\pi}\int_\Gamma + \int_{\overline{\Gamma}} \biggl(\frac{e^{-y\sqrt{s^2-\mu^2}}}{\sqrt{s^2-\mu^2}} \cdot e^{isx}\biggr)\, ds \tag{7.3}$$

where the contour Γ is given by

$$\begin{aligned}s &= \xi + i\eta \\ &= \mu t\biggl(\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{t^2}}\biggr)^{1/2} \\ &\quad + i\mu t\biggl(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{t^2}}\biggr)^{1/2}, \quad -\infty < t < \infty,\end{aligned}$$

and $\overline{\Gamma}$ is the reflection of Γ in the real axis. Then, from the differentiation formula $(1/2)\partial Y_0(\mu r)/\partial y = -\mu y Y_1(\mu r)/(2r)$, we obtain a companion formula to (7.3), namely

$$-\frac{\mu y}{2} \cdot \frac{Y_1(\mu r)}{r} = \frac{1}{4\pi}\int_\Gamma + \int_{\overline{\Gamma}} (e^{-y\sqrt{s^2-\mu^2}} \cdot e^{isx})\, ds. \tag{7.4}$$

Theorem 7.1. *The series* $\sum_{n=0}^{\infty} b_n U_n(x,y)$ *converges for* $y > \sigma \geq 0$ *if and only if* $\sum_{n=0}^{\infty} b_n U_n(x,y) = (1/2\pi)\int_{-\infty}^{\infty} e^{ixs} e^{-y\sqrt{s^2+\mu^2}}\psi(s)\,ds$ *where* $\psi(z)$ *is an entire function of growth* $(1,\sigma)$ *and* $b_n = \psi^{(n)}(0)/\{n!(-2i)^n\}$.

Proof. If the series converges, then $\limsup_{n\to\infty} 2n|b_n|^{1/n}/e \leq \sigma$ which implies that $\psi(z) \in (1,\sigma)$. Conversely, from the formula (7.2) and the definition of the U_n, it follows that

$$U_n(x,y) = (-2)^n \frac{\partial^n}{\partial x^n}\frac{\mu y}{\pi}\frac{K_1(\mu r)}{r} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-2is)^n e^{isx} e^{-y\sqrt{s^2+\mu^2}}\,ds.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} b_n U_n(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\bigg(\sum_{n=0}^{\infty} b_n(-2is)^n\bigg)e^{ixs}e^{-y\sqrt{s^2+\mu^2}}\,ds\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}e^{-y\sqrt{s^2+\mu^2}}\psi(s)\,ds\end{aligned}$$

where the last integral follows from the definition of $\psi(z)$ and provided that the term-by-term integration is valid. This is guaranteed by the facts that $\psi(z)$ has growth $(1,\sigma)$ and $y > \sigma$. □

Theorem 7.2. *The series* $\sum_{n=0}^{\infty} b_n V_n(x,y)$ *converges for* $y > \sigma \geq 0$ *if and only if*

$$\sum_{n=0}^{\infty} b_n V_n(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixs}e^{-y\sqrt{s^2+\mu^2}}}{\sqrt{s^2+\mu^2}}\psi(s)\,ds$$

where $\psi(z)$ *is an entire function of growth* $(1,\sigma)$ *and* $b_n = \psi^{(n)}(0)/\{n!(-2i)^n\}$.

Proof. The proof of this is the same as the proof of Theorem 7.1 but with $-K_0(\mu r)/\pi$ replacing $\mu y K_1(\mu y)/(\pi r)$. □

Theorem 7.3. *The series* $\sum_{n=0}^{\infty} b_n \widetilde{U}_n(x,y)$ *converges for* $y > \sigma \geq 0$ *if and only if* $\sum_{n=0}^{\infty} b_n \widetilde{U}_n(x,y) = (1/4\pi)\int_{\Gamma} + \int_{\overline{\Gamma}} e^{ixs}e^{-y\sqrt{s^2-\mu^2}}\psi(s)\,ds$

where $\psi(z)$ is an entire function of growth $(1,\sigma)$ and $b_n = \psi^{(n)}(0) /\{n!(-2i)^n\}$.

Proof. The proof of this is the same as the proof of Theorem 7.1, but with $-\mu y Y_1(\mu r)/(2r)$ replacing $\mu y K_1(\mu r)/(\pi r)$. □

Theorem 7.4. *The series $\sum_{n=0}^{\infty} b_n \widetilde{V}_n(x,y)$ converges for $y > \sigma \geq 0$ if and only if*

$$\sum_{n=0}^{\infty} b_n \widetilde{V}_n(x,y) = -\frac{1}{4\pi}\int_{\Gamma} + \int_{\overline{\Gamma}} \frac{e^{ixs}e^{-y\sqrt{s^2-\mu^2}}}{\sqrt{s^2-\mu^2}}\psi(s)\,ds$$

where $\psi(z)$ is an entire function of growth $(1,\sigma)$ and $b_n = \psi^{(n)}(0)/\{n!(-2i)^n\}$.

Proof. Again, this follows the proof of Theorem 7.3 with $Y_0(\mu r)/2$ replacing $-\mu y Y_1(\mu r)/(2r)$. □

In Theorem 7.3 if μ approaches zero, then the contours Γ and $\overline{\Gamma}$ collapse to the real axis so that the fundamental solution becomes

$$\frac{1}{\pi}\frac{y}{x^2+y^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixs}e^{-y|s|}\,ds$$

and thus reduces to Theorem 4.3 of [**5**]. In the case of Theorem 7.4, the integral

$$-\frac{1}{4\pi}\int_{\Gamma} + \int_{\overline{\Gamma}} \frac{e^{ixs}e^{-y\sqrt{s^2-\mu^2}}}{\sqrt{s^2-\mu^2}}\,ds \quad \text{reduces to} \quad -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixs}e^{-y|s|}}{|s|}\,ds$$

when μ approaches zero. This integral must be taken in the generalized sense (see [**15**]) and has the evaluation

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ixs}e^{-y|s|}}{|s|}\,ds = \frac{1}{2\pi}\log(x^2+y^2) + \frac{\gamma}{\pi}$$

where γ is the Euler constant. This is essentially the fundamental solution for Laplace's equation which gives rise to the associated functions

$$V_n(x,y) = -2^n(n-1)!\mathrm{Re}\,(z^n/r^{2n})/\pi, \quad n = 1,2,3,\dots .$$

8. Summary of results. In this paper, we have established, by means of transmutations, function theories for the Yukawa and Helmholtz equations. These are analogues of the corresponding classical function theory for the Laplace equation. It is useful to summarize how the elements of these theories match up. Corresponding to the conjugate polynomial sets $\{\operatorname{Re} z^n\}_{n=0}^{\infty}$ and $\{\operatorname{Im} z^n\}_{n=0}^{\infty}$ associated with the Laplace equation, conjugate polynomial sets $\{u_n(x,y)\}, \{v_n(x,y)\}$ and $\{\tilde{u}_n(x,y)\}, \{\tilde{v}_n(x,y)\}$ involving Bessel polynomials were constructed, respectively, for the Yukawa and the Helmholtz equations. The generating functions which were obtained have a strong resemblance to those for the Laplace case. Suitable linear combinations of $u_n(x,y)$ and $v_n(x,y)$ satisfy general Cauchy-Riemann type conditions as do appropriate linear combinations of the $\tilde{u}_n(x,y)$ and $\tilde{v}_n(x,y)$. If the coefficients a_n in the series $\sum_{n=0}^{\infty} a_n x^n$ satisfy the condition $\limsup_{n\to\infty} |a_n|^{1/n} = 1/\sigma$, then both of the series $\sum_{n=0}^{\infty} a_n u_n(x,y)$ and $\sum_{n=0}^{\infty} a_n v_n(x,y)$ converge to a solution of the Yukawa equation in the disk $x^2+y^2<\sigma^2$. Similarly, each of the series $\sum_{n=0}^{\infty} a_n \tilde{u}_n(x,y)$ and $\sum_{n=0}^{\infty} a_v \tilde{v}_n(x,y)$ converge to a solution of the Helmholtz equation in that same disk. This is in agreement with Laplace polynomial expansions. Associated function sets $\{U_n(x,y)\}$ and $\{V_n(x,y)\}$ which permit series expansions in regions exterior to disks were also constructed for the Yukawa equation as were the similar sets $\{\widetilde{U}_n(x,y)\}$ and $\{\widetilde{V}_n(x,y)\}$ for the Helmholtz equation (these are analogues of the function sets $\{\operatorname{Re} z^{-n}\}$ and $\{\operatorname{Im} z^{-n}\}$ for the Laplace equation). Generating functions for the various associated sets were developed along with appropriate Cauchy-Riemann type equations. If the data associated with a Yukawa or Helmholtz equation is entire, then the solution of that problem was shown to be represented by an integral that involves a solution of the wave equation. Finally, Fourier transform criteria were obtained for defining expansions of solutions of Yukawa and Helmholtz equations in terms of the different sets of associated functions. Once again, these are analogues of criteria for expansions of solutions of the Laplace equation in terms of the sets $\{\operatorname{Re} z^{-n}\}$ and $\{\operatorname{Im} z^{-n}\}$. In summary, we have matched up element by element the building blocks for the function theory for the Yukawa and Helmholtz equations with the corresponding ones for the analytic function theory associated with the Laplace equation.

REFERENCES

1. P.W. Berg and J.L. McGregor, *Elementary partial differential equations*, Holden-Day, Inc., San Francisco, 1966.

2. L. Bers, *An outline of the theory of pseudoanalytic functions*, Bull. Amer. Math. Soc. **62** (1956), 291–331.

3. L.R. Bragg and J.W. Dettman, *An operator calculus for related differential equations*, J. Math. Anal. Appl. **22** (1968), 261–271.

4. ———, *Expansion of solutions of certain hyperbolic and elliptic problems in terms of Jacobi polynomials*, Duke Math. J. **36** (1969), 129–144.

5. ———, *Analogous function theories for the heat, wave, and Laplace equations*, Rocky Mountain J. Math. **13** (1983), 191–214.

6. ———, *Analogous function theories for certain singular partial differential equations*, Rocky Mountain J. Math. **14** (1984), 383–396.

7. ———, *A class of related Dirichlet and initial value problems*, Proc. Amer. Math. Soc. **21** (1969), 50–56.

8. R.J. Duffin, *Yukawan potential theory*, J. Math. Anal. Appl. **35** (1971), 105–130.

9. ———, *Hilbert transforms in Yukawan potential theory*, Proc. Nat. Acad. Sci. **69** (1972), 3677–3679.

10. E. Grosswald, *Bessel polynomials*, Springer Verlag, Berlin, Heidelberg, 1978.

11. F.W. Olver, *Asymptotics and special functions*, Academic Press, New York, 1974.

12. G.E. Roberts and H. Kaufman, *Table of Laplace transforms*, Saunders, Philadelphia, 1966.

13. P.C. Rosenbloom and D.V. Widder, *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc. **92** (1959), 220–266.

14. D.V. Widder, *Some analogies from classical analysis in the theory of heat conduction*, Arch. Rational Mech. Anal. **21** (1966), 108–120.

15. A.H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill Book Co., New York, 1965.

Department of Mathematical Sciences, Oakland University, Rochester, Michigan 48309-4401