ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 3, Summer 1995

ε-SPACES

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ABSTRACT. Consider a Tychonoff space, X, and the lattice-ordered group $C(X)$ of real-valued continuous functions. Within a certain category of l-groups, the "epicomplete epireflection" $\varepsilon C(X)$ of $C(X)$ looks enough like the l-group $B(X)$ of Baire functions on X to present the question: For what X is $\varepsilon C(X) = B(X)$? That equality means just that each homomorphism from $C(X)$ to an epicomplete target lifts to a homomorphism of $B(X)$ and is, we show, equivalent to this condition on the placement of X in its Stone-Čech compactification βX: If E is a Baire set of βX which misses X, then there are zero-sets $Z_1, Z_2, \dots$ of βX for which $E \subseteq \cup_n Z_n \subseteq \beta X - X$. We call such an X an "ε-space" and examine these spaces, rather inconclusively.

Algebra to topology. In the first two sections we present a synopsis of the theory in [**1, 2, 3**] to motivate the question, "$\varepsilon C(X) = B(X)$?" and to make the topological reduction described in the abstract. The reader who finds the definition of ε-spaces in the abstract sufficiently compelling can, for the most part, just skip to Section 3.

1. Epicompleteness. $\mathcal{W}$ is the category of Archimedean l-groups with distinguished weak order unit and morphisms the l-homomorphisms which preserve unit. Each $C(X)$, with unit the constant function 1, is an object of $\mathcal{W}$, the $\mathcal{W}$-morphisms between $C(X)$'s are exactly the homomorphisms described in Chapter 10 of [**7**] and, in many other ways, the category $\mathcal{W}$ generalizes (significantly) the theory of $C(X)$ in [**7**]; see, e.g., [**8**]. The discussion of this section takes place "in $\mathcal{W}$."

An epimorphism (or just "epic") is a homomorphism $\alpha : A \to B$ for which $\gamma\alpha = \delta\alpha$ (with γ, δ homomorphisms) implies $\gamma = \delta$. [**1**] contains an explicit description of the epics, but we can skip this.

Received by the editors on July 15, 1993, and in revised form on October 13, 1993.
1991 *Mathematics Subject Classification.* 06F20, 46E25, 54D40.

An object is called *epicomplete* if it has no proper epic extension, and an *epicompletion* of G is an epic extension of G to an epicomplete object. Let EC denote the class of epicomplete objects.

Proposition 1.1 [**1, 2**]. *The following are equivalent.*

(a) $G \in EC$.

(b) *G is σ-complete and σ-laterally complete, and divisible.*

(c) *G "is" a vector lattice of real-valued measurable functions modulo an abstract σ-ideal of null functions.*

Proposition 1.2 [**11, 2**]. *To each G, there corresponds an epicompletion $G \overset{\varepsilon_G}{\hookrightarrow} \varepsilon G$ such that, given $G \overset{\varphi}{\to} E$, with $E \in EC$, there is a unique $\varepsilon G \overset{\bar{\varphi}}{\to} E$ with $\bar{\varphi}\varepsilon_G = \varphi$.*

Proposition 1.3 [**3**]. *$\varepsilon C(X)$ is $B(\beta X)/N(C(X))$, where βX is Stone-Čech compactification, $B()$ denotes the l-group of Baire functions, and*

$$N(C(X)) = \Big\{ f \in B(\beta X) \mid \operatorname{coz} f \subseteq \bigcup_n (\beta g_n)^{-1}(\pm\infty), \\ \textit{for some } g_1, g_2, \ldots \in C(X) \Big\}$$

(where $\beta g : \beta X \to R \cup \{\pm\infty\}$ is the extension of $g \in C(X)$). And, $\varepsilon_{C(X)} : C(X) \to \varepsilon C(X)$ is defined by:

$$\varepsilon_{C(X)}(g) = g' + N(C(X)),$$

where

$$g' = \begin{cases} \beta g & \text{on } (\beta g)^{-1}(R) \\ 0, & \text{on } (\beta g)^{-1}(\pm\infty). \end{cases}$$

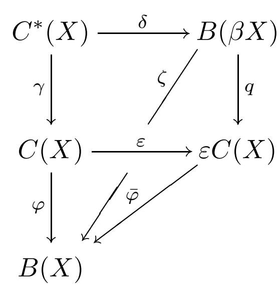
2. $B(X)$ versus $\varepsilon C(X)$. Let φ label the inclusion $C(X) \overset{\varphi}{\subseteq} B(X)$, and let us just write ε for the map $\varepsilon_{C(X)}$ of Propositions 1.2 and 1.3. We say that $B(X) = \varepsilon C(X)$ if these extensions of $C(X)$ are isomorphic over $C(X)$, i.e., if there is an isomorphism $\varepsilon C(X) \xrightarrow{\psi} B(X)$ with $\psi\varepsilon = \varphi$.

Theorem 2.1. *$C(X) \xrightarrow{\varphi} B(X)$ is an epicompletion of $C(X)$, and the unique $\varepsilon C(X) \xrightarrow{\bar{\varphi}} B(X)$ with $\bar{\varphi}\varepsilon = \varphi$ (of 1.4) is a surjection.*

$B(X) = \varepsilon C(X)$ if and only if φ is one-to-one, and that occurs if and only if whenever E is a Baire set of βX with $E \cap X = \phi$, there are zero-sets $Z_1, Z_2, \ldots$ of βX with each $Z_n \cap X = \phi$, with $E \subseteq \cup_n Z_n$.

Proof. This information will be extracted in steps from the following diagram

(2.2)



in which: $C^*(X)$ is the sub-l-group of $C(X)$ consisting of bounded functions, and γ is the indicated inclusion; δ is the composite $C^*(X) \simeq C(\beta X) \hookrightarrow B(\beta X)$; $q(f) = f + N(C(X))$ is the quotient map implicit in 1.3; $\zeta(f) = f \mid X$ is the restriction homomorphism.

By Proposition 1.1, $B(X) \in EC$ so there is a unique $\bar{\varphi}$ with $\bar{\varphi}\varepsilon = \varphi$. The first paragraph of Theorem 2.1 (and more) is included in

Proposition 2.3. *Diagram* (2.2) *has these features*:

$$\delta \textit{ is epic}; \quad \zeta \textit{ is onto}; \quad \zeta\delta = \varphi\gamma; \quad \varphi \textit{ is epic};$$
$$\varepsilon\gamma = q\gamma; \quad \bar{\varphi}q = \zeta; \quad \bar{\varphi} \textit{ is onto.}$$

Proof. δ epic: 5.3(b) of [**2**]. (This requires the characterization of epics in [**1**].)

ζ onto: Let $g \in B(X)$. We are to find $f \in B(X)$ with $f \mid X = g$. Fix $k \in N$. For each integer n, $g^{-1}([n/k, (n+1)/k))$ is a Baire set on X, which by 8.7 of [**4**] extends to a Baire set F_n^k of βX; by induction on n, we may and do assume the F_n^k, $n \in Z$, disjoint. Let χ_n^k be the characteristic function of F_n^k. Then $f^k \equiv \sum_n (n/k)\chi_n^k \in B(\beta X)$, and $|f^k(x) - g(x)| \leq 1/k$ for each $x \in X$. But (f^k) is uniformly Cauchy and thus converges uniformly to the desired $f \in B(\beta X)$.

$\zeta\delta = \varphi\gamma$. Obvious.

φ epic: ζ is onto, hence epic. $\zeta\delta$ is the composition of two epics, hence is epic. So $\varphi\gamma = \zeta\delta$ is epic. Thus, φ is epic, as a second factor of an epic.

$\varepsilon\gamma = q\gamma$. Reflection upon the definitions.

$\bar{\varphi}q = \zeta$. $\bar{\varphi}q\delta = \bar{\varphi}\varepsilon\gamma = \varphi\gamma$; since γ is epic, $\bar{\varphi}q = \zeta$.

$\bar{\varphi}$ onto: From ζ onto, and $\bar{\varphi}q = \delta$.

We now establish the second paragraph of Theorem 2.1.

If an isomorphism ψ witnesses $B(X) = \varepsilon C(X)$, of course $\psi = \bar{\varphi}$ (by uniqueness of $\bar{\varphi}$), so $\bar{\varphi}$ is one-to-one. Conversely, $\bar{\varphi}$ is already onto, so if one-to-one, $\bar{\varphi}$ is the desired ψ.

Finally, since $\zeta = \bar{\varphi}q$, it is visible in (2.2) that $\bar{\varphi}$ one-to-one means $f \in B(\beta X)$, $f \mid X = 0 \Rightarrow f + N(C(X)) = 0$. Now $f + N(C(X)) = 0$ means $\operatorname{coz} f \in N(C(X))$, i.e., $\operatorname{coz} f \subseteq \cup_n (\beta g_n)^{-1}(\infty)$ for some $g_1, g_2, \dots, \in C(X)$. But the sets of the form $(\beta g)^{-1}(\infty)$, $g \in C(X)$, are exactly the zero-sets of βX which miss X (by inverting functions). And the set $\operatorname{coz} f$ is a typical Baire set of βX which misses X. Thus, $\bar{\varphi}$ one-to-one is equivalent to the topological condition in Theorem 2.1.

The proof of Theorem 2.1 is concluded. $\square$

We summarize the situation:

Proposition 2.4. *An ε-space is a Tychonoff space X which satisfies the following equivalent conditions*:

(a) *Each $\mathcal{W}$-homomorphism $\varphi : C(X) \to E$, with $E \in EC$, has an* *extension* $\bar{\varphi} : B(X) \to E$;

(b) *Each Baire set of* βX *which misses* X *is contained in a "*$\mathcal{Z}_\sigma$*" of* βX *which misses* X (*i.e., the condition of Theorem* 2.1 *and the abstract*).

Here (a) is just the statement $\varepsilon C(X) = B(X)$, since $B(X)$ is an epicompletion of $C(X)$ by Proposition 2.1 and $\varepsilon C(X)$ is, up to isomorphism, the only epicompletion of $C(X)$ with the universal mapping property of (a) (or Proposition 1.2).

What spaces are ε-spaces? Various versions of this question occupy the rest of the paper; sometimes we will apply Proposition 2.4 (a) and sometimes 2.4 (b), and sometimes both. But we try to emphasize the topology.

For brevity in the sequel, we let $\mathcal{B}(Y)$ stand for the σ-field of Baire sets of the space Y (the σ-field generated by the zero-sets). "Space" always means "Tychonoff space."

3. Compact and pseudocompact spaces are ε. For, compact X has $\beta X = X$ and condition 2.4 (b) holds. (Alternatively, one may use Proposition 1.3 in which, when X is compact, $\beta X = X$ and $N(C(X)) = (0)$.)

If X is pseudocompact, then no nonvoid Baire set of βX misses X, since this is true of zero-sets [**7**, 6I], and each Baire set is the union of zero-sets [**4**; 8.2]. Thus condition 2.4(b) holds (just as with compact).

(Also, "compact are ε" implies "pseudocompact are ε," via Theorem 6.1 below.)

4. Absolute Baire spaces. If $X \in \mathcal{B}(\beta X)$, then X is a Baire set in any compactification, and is called "absolute Baire" [**4**; p. 79].

Theorem 4.1. *The following are equivalent about* X.

(a) X *is an* ε*-space and an absolute Baire space.*

(b) *There are cozero-sets* $C_1, C_2, \ldots$ *of* βX *with* $\cap_n C_n = X$.

(c) X *is Lindelöf and Čech-complete* (*i.e., a* G_δ *in* βX).

Proof. (a) $\Leftrightarrow$ (b). If $X \in \mathcal{B}(\beta X)$, then $\beta X - X \in \mathcal{B}(\beta X)$ and Proposition 2.4(b) is equivalent to Theorem 4.1(b), by complementing.

(b) $\Rightarrow$ (c). Such an X is clearly G_δ in βX and is Lindelöf by [**4**, 9.8].

(c) $\Rightarrow$ (b). We shall need the following (now and later); see [**5**, p. 165].

A space X is *normally placed* in its superspace K if each closed set in K which misses X is contained in a zero-set of K which misses X. (This definition, and Proposition 4.4 below, are due to Yu.M. Smirnov.)

Proposition 4.4. *These are equivalent about X.*

(a) *X is normally placed in βX.*

(b) *X is normally placed in some compactification of X.*

(c) *X is normally placed in every superspace.*

(d) *X is Lindelöf.*

To prove (c) $\Rightarrow$ (b) in Theorem 4.1, suppose $X = \cap_n G_n$ for open G_n in βX, and suppose X is Lindelöf. Then, for each n, there is a zero-set Z_n with $Z_n \cap X = \varnothing$ and $Z_n \supseteq \beta X - G_n$ (by condition 4.4(a)). Thus, $\beta X - X = \cup_n Z_n$ and condition 4.1(b) holds.

Corollary 4.5. *The space P of irrationals is an ε-space. The space Q of rationals is not an ε-space.*

Proof. P is Lindelöf and Čech-complete, and Q is not Čech-complete [**5**, p. 146]. □

We shall see in (5.4) below that a Lindelöf ε-space need not be absolute Baire.

5. P-spaces. Recall from [**7**; 4J] that a space is called a *P-space* if each G_δ is open, equivalently, each zero-set is open, or cozero. It follows easily that X is a P-space if and only if each Baire set is a zero-set, equivalently, $C(X) = B(X)$. Then, vacuously as it were, $B(X)$ has the universal mapping property of Proposition 2.4(a), so that

Theorem 5.1. *A P-space is an ε-space.*

On the other hand, the topologist would like to see a topological proof of Proposition 2.4(b), so here is an elegant one.

The following slight variant on the famous Loomis-Sikorsky (Stone-von Neumann) theorem is articulated in [**3**, III] (and probably elsewhere too), and proved just as in [**9**].

Given Y with its Baire field $\mathcal{B}(Y)$, let $Z\mathcal{M}$ be the σ-ideal in $\mathcal{B}(Y)$ generated by the nowhere dense zero-sets (the ideal of "zero-meager" sets).

Theorem 5.2. *If Y is basically disconnected (i.e., each cozero set has open closure), then for each $B \in \mathcal{B}(Y)$ there is a unique clopen C such that $(B - C) \cup (C - B) \in Z\mathcal{M}$ (and thus the composite Boolean homomorphism* $\operatorname{clop} Y \hookrightarrow \mathcal{B}(Y) \to \mathcal{B}(Y)/Z\mathcal{M}$ *is an isomorphism).*

We also need, from [**7**; 4K].

Proposition 5.3. *Each P-space is basically disconnected, and Y is basically disconnected if and only if βY is.*

Proof of Theorem 5.1. We now prove Theorem 5.1 again, by showing condition 2.4(b).

Let X be P, and let $B \in \mathcal{B}(\beta X)$ with $B \cap X = \varnothing$. Since βX is basically disconnected (5.3), there is clopen C in βX with $(B - C) \cup (C - B) \in Z\mathcal{M}$, i.e., there are nowhere dense zero-sets $Z_1, Z_2, \dots$ of βX with $(B - C) \cup (C - B) \subseteq \cup_n Z_n$. But, when X is P, if Z is a nowhere dense zero set of βX, then $Z \cap X$ is clopen and nowhere dense in X, thus void. Therefore, $Z_n \cap X = \varnothing$ for each n. Then

$$\begin{aligned} C \cap X = C - (\beta X - X) &\subseteq (C - B) \cap X \subseteq (\cup_n Z_n) \cap X \\ &= \cup_n (Z_n \cap X) = \varnothing, \end{aligned}$$

so that $C = \varnothing$ (being clopen). Thus,

$$B = B - C \subseteq \cup_n Z_n \subseteq \beta X - X. \qquad \square$$

Example 5.4. Let λD denote an uncountable discrete space D, with one point P adjoined, each of whose neighborhoods has countable complement in D. Then λD is a Lindelöf P-space (hence an ε-space) which is not absolute Baire since it is not G_δ in $\beta\lambda D$. (If λD were a G_δ in $\beta\lambda D$, there would be a perfect function f from λD onto a metric space [**4**, 9.5], and $f^{-1}(f(p))$ would be compact. But $f(p)$ is a G_δ, so $f^{-1}(f(p))$ is a G_δ, thus a neighborhood of p, and cannot be compact.)

6. ε is algebraic. We ask the reader to recall the Hewitt real-compactification υX from [**7**].

Theorem 6.1. *X is an ε-space if and only if υX is.*

Proof. Of course, $\upsilon X \subseteq \beta X$. For $B \in \mathcal{B}(\beta X)$ $B \cap X = \varnothing$ if and only if $B \cap \upsilon X = \varnothing$ (because that is true for zero-sets via [**7**, 8.7], and every Baire set is the union of zero-sets [**4**, 8.2]. Now Theorem 6.1 is easy. □

We note from [**8**] that a map $\varphi : C(X) \to C(Y)$ is a $\mathcal{W}$-homomorphism if and only if it is a ring homomorphism preserving identity, and from [**7**] that such φ has the form $\varphi(f) = f \circ \tau$, $f \in C(X)$, for a unique continuous $\upsilon X \overset{\tau}{\leftarrow} \upsilon Y$, and φ is an isomorphism if and only if τ is a homeomorphism.

The following is what the title of this section means.

Corollary 6.2. *Suppose $C(X)$ and $C(Y)$ are $\mathcal{W}$-isomorphic. Then X is an ε-space if and only if Y is.*

Proof. Suppose X is ε, $\varphi : C(X) \to C(Y)$ is an isomorphism, and $\upsilon X \overset{\tau}{\leftarrow} \upsilon Y$ is the homeomorphism with $\varphi(f) = f \circ \tau$. Evidently, υX is ε if and only if υY is, and the result follows by Theorem 6.1. □

Remarks 6.3. Corollary 6.2 implies Theorem 6.1, since $C(X)$ and $C(\upsilon X)$ are isomorphic.

One can argue Corollary 6.2 from condition 2.4(a) as well, the key being that an isomorphism $\varphi : C(X) \to C(Y)$ lifts to an isomorphism $\bar{\varphi} : B(X) \to B(Y)$. We omit the details.

7. ε-placement, and absolute ε-spaces. The notions (defined shortly) seem natural (perhaps in view of Proposition 4.4) and useful in the study of ε-spaces (but certainly raise more questions than they answer).

Definition 7.1. (a) Let $X \subseteq Y$. X is *ε-placed* in Y if, whenever $B \in \mathcal{B}(Y)$ and $B \cap X = \varnothing$, there are zero-sets $Z_1, Z_n, \ldots$ of Y, each with $Z_n \cap X = \varnothing$, and $B \subseteq \cup_n Z_n$. (Thus, X is an ε-space if and only if X is ε-placed in βX.)

(b) If X is ε-placed in each of its compactifications, X is called an *absolute ε-space.*

(c) If X is ε-placed in some compactification, X is called a *weak ε-space.*

Among many immediate questions concerning these notions, we articulate only a few:

Questions 7.2. (a) Must an ε-space X be absolute ε? Response: No (Example 7.3 below), but yes for Lindelöf X (Theorem 7.5 below).

(b) What are the absolute ε-spaces? Response: We do not know, but it seems plausible that they are ε-spaces X with υX Lindelöf. We have proved neither implication.

(c) If X is ε-placed in *some* compactification, must X be an ε-space? Response: Yes for σ-compact X (Theorem 7.8 below) but, in general, no (Example 7.4 below).

(d) What *are* the weak ε-spaces? Response: We do not know, but they include all "weakly pseudocompact" spaces, and thus every Hedgehog with uncountably many spines [**6**].

Example 7.3. An uncountable discrete space X (hence ε, by Theorem 5.1) with a compactification K in which X is not ε-placed.

Remark. It is not hard to see that, for discrete spaces $X \subseteq Y$, if K is a compactification of X in which X is not ε-placed, then $K + \beta(Y - X)$ is such a compactification of Y. Thus, one wants an example X with minimum cardinal, optimally ω_1. The construction below has X of cardinal c, and (superficially, at least) difficulties develop upon trying to reduce to ω_1. However, H. Zhou asserts that this is possible; see [**14**].

The construction which follows comes from page 172 of [**13**] and is attributed to M. Katětov. The (Boolean algebraic) purpose in [**13**] is to construct, in a compact K, a meager Baire set B not contained in any countable union of nowhere dense zero-sets. One notes that, for any such, $X \equiv K - B$ fails to be ε-placed in K.

Let $I = [0, 1]$, and let $B \in \mathcal{B}(I)$ with the features:

(a) For each countable $E \subseteq I$, $B \cup E$ is not F_σ;

(b) $C \equiv I - B$ is dense.

(One may take for B any Baire set in the Cantor set which is of "exact class 3." These exist by a theorem of Lebesgue, see page 207 of [**10**].)

Now an Alexandrov-doubling construction is made, as follows:

Let X be a disjoint copy of C, and let $K = I \cup X$, topologized like this:

(i) For each $x \in X$, $\{x\}$ is open;

(ii) For each $y \in I$, a basic open neighborhood of y is of the form $G \cup [(G \cap C)' - F]$, where G is an open neighborhood of $y \in I$, $(G \cap C)'$ denotes the copy of $G \cap C$ in X, and F is finite.

It is easily seen that K is a compactification of discrete X, and I inherits from K its original topology, that B is a Baire set of K with $B \cap X = \varnothing$, and that Z is a nowhere dense zero set of K if and only if Z is closed and $Z - B$ is countable.

Then, if we had zero-sets Z_n of K, each with $Z_n \cap X = \varnothing$, and $B \subseteq \cup_n Z_n$, then we would have $\cup_n Z_n = B \cup \cup_n (Z_n - B)$; here the left side is F_σ, while each Z_n is nowhere dense, hence $Z_n - B$ is countable, thus too $\cup_n (Z_n - B)$, so the right side is not F_σ (by (a) above). □

Example 7.4. A locally compact space which is not an ε-space.

(This substantiates the "no" in Question 7.2(c), since a locally compact space is always ε-placed in its one-point compactification. We note that the example below is not realcompact; a realcompact example has failed to occur to us.)

Let $I = [0,1]$, $\hat{Q} = Q \cap I$, $\hat{P} = P \cap I$; here Q is the rationals and P is the irrationals. We will use the fact that P is not σ-compact later.

Observe that $\hat{P}$ is a G_δ-subset of I, and hence belongs to $\mathcal{B}(I)$. Also observe that $\hat{P}$ is not σ-compact.

$\omega = \{0, 1, \dots\}$ with the discrete topology. As usual, put $\omega^* = \beta\omega \backslash \omega$. We will use the well-known and easily established fact that every nonempty G_δ subset of ω^* has nonempty interior in ω^* (see [**7**] or [**4**]).

Let $\pi : \omega \to \hat{Q}$ be a surjection, let $f = \beta\pi : \beta\omega \to I$ be its Stone-Čech extension, and let $g = f|\omega^*$. It is clear that g is surjective. For every $q \in \hat{Q}$, let S_q denote the interior of $g^{-1}(q)$ in ω^*. By what we just observed, each S_q is nonempty.

Put $X = \omega \cup \cup_{q \in \hat{Q}} S_q$. Notice that $\omega \subseteq X \subseteq \beta\omega$ so that $\beta X = \beta\omega$. Also, notice that X is locally compact, because its complement is a closed subspace of ω^*, and hence is compact. Let $B = f^{-1}[\hat{P}] = g^{-1}[\hat{P}]$. Observe that B is a Baire subset of $\beta\omega$ and that $B \subseteq \beta X \backslash X$.

The promised example is X. Observe that X is not pseudocompact because f maps X onto $\hat{Q}$. We claim that there is no countable collection $\mathcal{A}$ of zero-sets of $\beta\omega$ such that $B \subseteq \cup\mathcal{A} \subseteq \beta X \backslash X$. To the contrary, assume that such a family exists. Fix an arbitrary member $A \in \mathcal{A}$. Then A is a G_δ-subset of ω^*. Assume that there exists $q \in \hat{Q}$ such that $A \cap g^{-1}(q) \neq \varnothing$. Then $A \cap g^{-1}(q)$ is a nonempty G_δ-subset of ω^* and consequently has nonempty interior. As a consequence, A intersects the interior in ω^* of $g^{-1}(q)$. But this is impossible because this interior is in X and A is in $\beta X \backslash X$. We conclude that $A \cap g^{-1}(q) = \varnothing$. Since A, and in turn q, was arbitrary, we find

$$\left(\bigcup \mathcal{A}\right) \bigcap g^{-1}[\hat{Q}] = \varnothing,$$

i.e.,

$$\bigcup \mathcal{A} \subseteq g^{-1}[\hat{P}].$$

So it follows that

$$\bigcup \mathcal{A} = g^{-1}[\hat{P}]$$

because also $g^{-1}[\hat{P}] \subseteq \cup\mathcal{A}$. By the continuity of g, this now implies that $\hat{P}$ is σ-compact, which is a contradiction.

Theorem 7.5. *Let X be a Lindelöf ε-space. If $X \subseteq K$, and K is compact, then X is ε-placed in K. Thus, X is absolute ε.*

Lemma 7.6. *If $f : Y \to K$ is continuous and closed, and onto, if Lindelöf $X \subseteq f(Y)$, and if $f^{-1}X$ is ε-placed in Y, then X is ε-placed in K.*

Proof of Lemma 7.6. Let $B \in \mathcal{B}(K)$ with $B \cap X = \varnothing$. Then $f^{-1}B \in \mathcal{B}(Y)$ and $f^{-1}B \cap f^{-1}X = \varnothing$, so there are zero-sets Z_n of Y with $f^{-1}B \subseteq \cup_n Z_n$ and each $Z_n \cap f^{-1}X = \varnothing$. Then $B \subseteq f(f^{-1}B) \subseteq f(\cup_n Z_n) = \cup_n f(Z_n)$ and each $f(Z_n) \cap X = \varnothing$. Since f is closed, each $f(Z_n)$ is closed, and since X is Lindelöf, Proposition 4.4 provides a zero-set W_n of K with $f(Z_n) \subseteq W$, $W \cap X = \varnothing$. We then have $B \subseteq \cup_n W_n \subseteq K - X$, as desired. □

Proof of Theorem 7.5. Use Lemma 7.6 with $f : \beta X \to K$ the Stone-Čech extension of the inclusion $X \subseteq K$. □

Lemma 7.6 has a Corollary in another direction, which may be interesting. We ask the reader to recall, say from [**12**], the "absolute of a space," say $\pi_X : aX \to X$, and that, for the absolute $\pi_{\beta X} : a\beta X \to \beta X$, we have $aX = \pi_{\beta X}^{-1} X$, with $\pi_X = \pi_{\beta X}|X$. □

Corollary 7.7. *If X is Lindelöf and aX is an ε-space, then X is an ε-space.*

Proof. Use Lemma 7.6 with $Y = a\beta X$, $K = \beta X$, and $f = \pi_{\beta X}$. □

Thus, aQ is not an ε-space, by Corollary 7.7 and Corollary 4.5.

Theorem 7.8. *Let X be σ-compact, and ε-placed in some compact space. Then X is an ε-space, indeed, absolute ε.*

Lemma 7.9. *If $f : Y \to K$ is continuous with Y compact, if σ-compact $X \subseteq f(Y)$, and if X is ε-placed in K, then $f^{-1}X$ is ε-placed in Y.*

Proof of Lemma 7.9. Let $B \in \mathcal{B}(Y)$ with $B \cap f^{-1}X = \varnothing$. Since Y is compact, B is Lindelöf [**4**, 9.10], and thus so is $f(B)$. Of course, $f(B) \cap X = \varnothing$. Now,

$(*)$ *In a space K, if X is F_σ and Lindelöf $L \subseteq K - X$, then there is an $F \in \mathcal{B}(K)$ with $L \subseteq F \subseteq K - X$.*

Proof of $(*)$. Write $X = \cup_n X_n$ with X_n closed in K. Then, for each n, $L \cap X_n = \varnothing$, and by Proposition 4.4, there is a zero-set Z_n with $Z_n \supseteq X_n$, $Z_n \cap L = \varnothing$. Thus, $L \cap \cup_n Z_n = \varnothing$, and $\cup_n Z_n \supseteq \cup_n X_n = X$, and so $L \subseteq K - \cup_n Z_n \equiv F \subseteq K - X$. So $(*)$ is proved. □

We can apply $(*)$ to $L = f(B)$ and the $F_\sigma X$, finding $F \in \mathcal{B}(K)$ with $f(B) \subseteq F \subseteq K - X$. Since X is ε-placed in K, there are zero-sets $\mathcal{W}_n$ with $f(B) \subseteq F \subseteq \cup_n \mathcal{W}_n \subseteq K - X$, and thus $B \subseteq f^{-1}f(B) \subseteq f^{-1}F \subseteq f^{-1}(\cup_n \mathcal{W}_n) = \cup_n f^{-1}(\mathcal{W}_n) \subseteq f^{-1}(K - X) = Y - f^{-1}(X)$, and we have proved Lemma 7.9. □

Proof of Theorem 7.8. Let X be ε-placed in K, and use Lemma 7.9 with $f : \beta X$ to K the Stone-Čech extension of the inclusion $X \subseteq K$. □

Corollary 7.10. *If X is a σ-compact ε-space, then its absolute is an ε-space.*

Proof. As with Corollary 7.7, using Lemma 7.9. □

Combining Theorem 7.5 and Theorem 7.8 with Section 4, we find

Corollary 7.11. (a) *A Lindelöf Čech-complete space is absolute* ε*.*

(b) Q *is* ε*-placed in no compact space.*

8. Subspaces. What kinds of subspaces of ε-spaces are again ε? We hardly know the complete answer, but

Theorem 8.1. *If* X *is an* ε*-space, and* Z *is a* C^**-embedded zero-set in* X*, then* Z *is an* ε*-space.*

Remarks 8.2. (a) In Theorem 8.1, "zero" cannot be replaced by "closed"; see Proposition 8.7 below.

(b) In Theorem 8.1, "zero" cannot be replaced by "open": Example 7.4.

(c) Example 7.4 also shows that the euphonious statement

($*$) "An ε-placed subspace of an ε-space is an ε-space,"

is false: the space in Example 7.4 is ε-placed in its one-point compactification, which is an ε-space. Nonetheless, ($*$) may represent some truth, since Theorem 8.1 can be viewed as a weak version of it: a zero-set is always ε-placed, since its complement is cozero and thus the union of a sequence of zero-sets.

(d) We do not know if, in Theorem 8.1, the hypothesis "C^*-embedded" can be dropped.

Corollary 8.3. *A clopen subset of an* ε*-space is an* ε*-space.*

We set out to prove Theorem 8.1. For spaces $A \subseteq B$, we say that A is Baire-embedded in B if each Baire set of A is the intersection with A of a Baire set of B.

Lemma 8.4. *Let* $Y \subseteq K$*, with* $\overline{Y}$ *Baire-embedded in* K*. If* Y *is* ε*-placed in* K*, then* Y *is* ε*-placed in* Y*.*

Proof. Let $E \in \mathcal{B}(\overline{Y})$ with $E \cap Y = \varnothing$. By the hypothesis, there is an $E' \in \mathcal{B}(K)$ with $E' \cap \overline{Y} = E$, and thus $E' \cap Y = \varnothing$. Thus, we have zero-sets of K, Z_n, with $E' \subseteq \cup_n Z_n \subseteq K - Y$, and then

$$\begin{aligned} E = E' \cap \overline{Y} &\subseteq (\cup_n Z_n) \cap \overline{Y} \\ &= \cup_n (Z_n \cap \overline{Y}) \subseteq (K - Y) \cap \overline{Y} \\ &= \overline{Y} - Y, \end{aligned}$$

as desired. □

Remark 8.5. Conversely, it can be shown that, when K is compact, if X is ε-placed in $\overline{X}$, then X is ε-placed in K. We omit this proof.

Lemma 8.6. *Let X be an ε-space and $Z' \in \mathcal{B}(\beta X)$. Then $Z' \cap X$ is ε-placed in Z'.*

Proof. Let $E \in \mathcal{B}(Z')$ with $E \cap X = \varnothing$. Now Z' is Lindelöf [**4**, 9.10], thus z-embedded in βX [**4**, 9.11], thus Baire embedded (by an easy transfinite induction on the class of the Baire sets). So there is an $F \in \mathcal{B}(\beta X)$ with $F \cap Z' = E$ so $E \in \mathcal{B}(\beta X)$. Then, since $E \cap X = \varnothing$, there are zero-sets Z_n of βX with $E \subseteq \cup_n Z_n \subseteq \beta X - X$. Then

$$\begin{aligned} E = E \cap Z' &\subseteq (\cup_n Z_n) \cap Z' = \cup_n (Z_n \cap Z') \\ &\subseteq (\beta X - X) \cap Z = Z' - Z' \cap X. \qquad \square \end{aligned}$$

Proof of Theorem 8.1. Let Z be a C^*-embedded zero-set in the ε-space X. Since C^*-embedding implies "z-embedding," there is a zero-set Z' of βX with $Z' \cap X = Z$. Now $Z' \in \mathcal{B}(\beta X)$, so Lemma 8.6 says that Z is ε-placed in Z'. Now use Lemma 8.4 with $Y = Z$ and $K = Z'$: since Z' is compact, so is $\overline{Z}$ (closure in Z'), and thus $\overline{Z}$ is "Baire-embedded" in Z' (since it is C^*-embedded). So Lemma 8.4 says Z is ε-placed in $\overline{Z}$. This closure in Z' is also the closure in βX since Z' is closed. By C^*-embedding, $\overline{Z}$ is βZ, and we are done. □

Proposition 8.7. *Each space F is a C^*-embedded closed set in a pseudocompact* (*hence ε-*) *space $X(F)$.*

Proof. Let F be given, and let Y be any pseudocompact noncompact space. Then $Y \times \beta F$ is pseudocompact [**7**, 9.14]. Choose any $p \in$

$\beta Y - Y$, and let $X(F)$ be the following subspace of $(Y \cup \{p\}) \times \beta F$:

$$X(F) = (Y \times \beta F) \cup (\{p\} \times F).$$

Then, $X(F)$ is pseudocompact, since $X(F)$ contains densely the pseudocompact space $Y \times \beta F$, and the copy $\{p\} \times F$ of F in $X(F)$ is easily seen to be closed and C^*-embedded. □

9. Sums. For $\{X_\alpha \mid \alpha \in A\}$ a set of spaces, $\sum_{\alpha \in A} X_\alpha$ (called the *sum*) denotes the disjoint union with the topology: G is open if and only if $G \cap X_\alpha$ is open for each α.

Theorem 9.1. (a) *If $\sum_{\alpha \in A} X_\alpha$ is an ε-space, then each X_α is an ε-space.*

(b) *If A is countable, and each X_α is an ε-space, then $\sum_{\alpha \in A} X_\alpha$ is an ε-space.*

Proof. (a) By Corollary 8.3.

(b) Let $X = \sum X_n$, $n \in N$, suppose each X_n is an ε-space, and let $E \in \mathcal{B}(\beta X)$ with $E \cap X = \varnothing$. Since X_n is clopen in X, it follows that βX_n is the closure of X_n in βX and is clopen there. Thus, $Y \equiv \cup_n \beta X_n$ is a cozero set of βX, so $\beta X - Y$ is a zero-set.

Now $E_n \equiv E \cap \beta X_n \in \mathcal{B}(\beta X_n)$, and $E_n \cap X_n = \varnothing$, so there are zero-sets Z_k^n of βX_n with $E_n \subseteq \cup_k Z_k^n \subseteq \beta X_n - X_n$. Since βX_n is clopen in βX, each Z_k^n is a zero-set of βX as well. We now have

$$\begin{aligned} E &= (\cup_n E_n) \cup (E \cap (\beta X - Y)) \\ &\subseteq (\cup_n \cup_k Z_k^n) \cup (\beta X - Y) \\ &\subseteq \beta X - X, \end{aligned}$$

showing X is an ε-space. □

Remarks 9.2. (a) In Theorem 9.1, the hypothesis "A is countable" cannot be dropped. This is due to H. Zhou [**14**]: the sum of ω_1 unit intervals is not ε. This is quite complicated, and [**14**] contains numerous other results involving ordinals and set-theoretic assumptions.

(b) What about unions (which are not sums) of ε-spaces? Doubtless, there are positive results, but we know no interesting ones. Here are two negative results.

Example 9.3. A countable disjoint union of C^*-embedded closed ε-spaces which is not ε: the rationals Q.

Example 9.4. A disjoint union of two ε-spaces which is not ε: the space $X = \omega \cup \cup_{q \in \hat{Q}} S_q$ of Example 7.4.

Proof. We shall refer to Example 7.4. As seen there, X is not ε. By Theorem 4.1 or 5.1, ω is ε. We want to see that $\cup_{q \in \hat{Q}} S_q$ is ε. Now, in its inherited topology, this union is the sum (since each S_q is open, and thus too its complement equals $\cup_{r \neq q} S_r$), so by Theorem 9.1(b), we want to see that each S_q is ε. Indeed, each S_q is pseudocompact (thus ε by Section 3), as we now explain.

A space Y is called ω-bounded if each of its countable subsets has compact closure (in Y). Clearly, such a space is countably compact, thus pseudocompact. The following, well-known to aficionados, applies to each S_q.

Lemma 9.5. *In ω^*, the interior of a zero-set is ω-bounded.*

Proof. All operations shall refer to ω^*. Let $U = \operatorname{int} Z$, and let A be a countable subset of U. We show that $\overline{A} \subseteq U$ (which suffices, since ω^* is compact). Choose a cozero-set C for which $A \subseteq C \subseteq Z$ (possible, since the cozero-sets form a base, $A \subseteq \operatorname{int} Z$ and A is Lindelöf). Now C and $\omega^* - Z$ are disjoint cozero-sets in the F-space ω^* [**2**, 14.27], and thus have disjoint closures [**7**, 14.N]. So $\overline{C} \subseteq \operatorname{int} Z = U$, hence $\overline{A} \subseteq U$. □

10.The Sorgenfrey line. This space, S, is the real numbers with the topology generated from the open basis of all $[a, b)$'s. Note that, for a function f from S to another space, f may be viewed as defined on the usual real line R, and as such, for $p \in R$, f may or may not be continuous at p. If not, we say that f is R-discontinuous at p.

Our theorem here is the following special result. We find it interesting, and do not know about $S \times S$.

Theorem 10.1. *S is an ε-space.*

Proof. Let E be a Baire set of βS which misses S. By [**4**, 9.1], there are metrizable M with a Baire set F and continuous $g : \beta S \to M$ for which $E = g^{-1}(F)$. Let f be the restriction $g \mid S$, and let $\mathcal{D} = \{p \in R \mid f \text{ is } R\text{-discontinuous at } p\}$. By Proposition 10.2(c) below, $\mathcal{D}$ is *countable*: we write $\mathcal{D} = \{r_n \mid n \in N\}$.

Let $i : S \to R$ be the identity function, continuous in the direction indicated, with $\beta i : \beta S \to \beta R$ the Stone-Čech extension. Let $Z_0 = (\beta i)^{-1}(\beta R - R)$, and, fixing a metric ρ on M, let $Z_{nm} = (\beta i)^{-1}(\{r_n\}) \cap \{x \in \beta S \mid \rho(g(x), f(r_n)) \geq \frac{1}{m}\}$. For elementary reasons, all these Z's are zero-sets of βS which miss S. ($Z_{nm} \subseteq \beta S - S$ since $x \in S$ and $\beta i(x) = r_n$ imply $g(x) = f(r_n)$.)

We claim that $g^{-1}((M - g(S))$ is contained in the union of the Z's, whence the Z's cover E, as desired. To see that: if $x \notin Z_0$, then $\beta i(x) \in R$, and if also $x \notin \cup_{n,m} Z_{nm}$, then f is R-continuous at $\beta i(x)$, whence $g(x) = f((\beta i(x))$ (one sees readily); thus $g(x) \in g(S)$.

The following will conclude the proof of Theorem 10.1.

Proposition 10.2. (a) *S is hereditarily Lindelöf.*

(b) *If X is hereditarily Lindelöf, and $U \subseteq X$, then $E_u \equiv \{x \in U \mid U \cap G_x$ is countable for some open G_x containing $x\}$ is countable.*

(c) *If M is metrizable, and $g : S \to M$ is continuous, then the set $\mathcal{D}$ of R-discontinuity points of g is countable.*

Proof. (a) See [**5**, p. 141].

(b) For each $x \in E_u$, choose G_x *per* the definition. Then $\{G_x \cap U \mid x \in E_u\}$ is an open cover of E_u consisting of countable sets, and extraction of a countable subcover shows that E_u is countable.

(c) Let $w : R \to R$ be the usual oscillation function for $g : R \to M$, with respect to a fixed metric ρ on M, so $\mathcal{D} = \{y \mid w(y) > 0\} = \cup_{m \in N} \{y \mid w(y) \geq 1/m\}$. It suffices that, for each m, the set $U = \{y \mid$

$w(y) \geq 1/m\}$ is countable. In the space S, now use (a) and (b) to write $U = E_u \cup (U - E_u)$ with E_u countable. If $x \in U - E_u$, then for each $a > x$, $[x, a] \cap U$ is uncountable, so there is a $y \in (x, a) \cap U$. Since $w(y) \geq 1/m$, there are y_1 and $y_2 \in (x, a)$ with $\rho((g(y_1), g(y_2)) \geq 1/m$. Thus, on any S-neighborhood of x, g varies $\geq 1/m$ and cannot be S-continuous at x. This contradiction shows $U - E_u = \varnothing$, so that U is countable. □

(Proposition 10.2(b) is a variant of the Cantor-Bendixson theorem, alluded to in [**10**; p. 159]. Proposition 10.2(c) may well be known, but was not previously known to us.)

11. Concluding remarks. We collect together some of the issues we have left unresolved. (Neither their difficulty, nor their lasting importance, are clear to us, but they seem interesting.)

(1) What are the absolute ε-spaces? (υX Lindelöf?)

(2) What are the weak ε-spaces?

(3) Is there a real-compact locally compact non-ε-space?

(4) What kinds of subspaces of ε-spaces are again ε?

(5) When is the union of two ε-spaces again ε?

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