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GENERALIZATION OF AN INEQUALITY FOR NONDECREASING SEQUENCES

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In 1981, A. Meir [**4**] proved the following theorem for nondecreasing sequences:

Theorem A. *Let* $a_0, a_1, \dots, a_{n-1}$ *and* $p_1, p_2, \dots, p_n$ *be nonnegative real numbers satisfying*

$$0 = a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1},$$
$$a_i - a_{i-1} \le p_i, \quad i = 1, \dots, n-1,$$

and

$$p_1 \le p_2 \le \cdots \le p_n. \tag{1}$$

If r *and* s *are real numbers with* $r \ge 1$ *and* $s \ge 2r+1$, *then*

$$\left[(s+1)\sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2}\right]^{1/(s+1)} \le \left[(r+1)\sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2}\right]^{1/(r+1)}. \tag{2}$$

In 1986, G.V. Milovanović and I.Ž. Milovanović [**5**] presented an interesting refinement of Theorem A. Their result states:

Theorem B. *If the numbers* a_i, $i = 0, 1, \dots, n-1$, p_i, $i = 1, 2, \dots, n$,

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r and s satisfy the assumptions of Theorem A*, then*

$$(3) \qquad \begin{aligned} 0 &\le \frac{(s+1)(s-r)}{8}\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) \\ &\le \left[(r+1)\sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2}\right]^{(s+1)/(r+1)} \\ &\quad - (s+1)\sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2}. \end{aligned}$$

Inspired by the proof of Theorem B, J.E. Pečarić [**6**] established recently a remarkable extension of Meir's result. He showed that the conclusion of Theorem A remains valid if assumption (1) is replaced by "$p_i \le p_n$, $i = 1, \dots, n-1$". It is natural to ask whether the same extension holds for Theorem B, too. By using a different approach than the one given in [**5**] we shall give an affirmative answer to this question.

Theorem. *Let $a_0, a_1, \dots, a_{n-1}$ and $p_1, p_2, \dots, p_n$ be nonnegative real numbers satisfying*

$$0 = a_0 \le a_1 \le a_2 \le \cdots \le a_{n-1},$$
$$a_i - a_{i-1} \le p_i, \quad i = 1, 2, \dots, n-1,$$

and

$$p_i \le p_n, \quad i = 1, 2, \dots, n-1.$$

If r and s are real numbers with $r \ge 1$ and $s \ge 2r+1$, then (3) *is valid.*

Proof. Using the identity

$$\sum_{i=1}^{n-1} a_i^{s-1}(p_{i+1}^2 - p_i^2) = \sum_{i=1}^{n-1}(a_i^{s-1} - a_{i-1}^{s-1})(p_n^2 - p_i^2),$$

we conclude from $p_n^2 \ge p_i^2$ and $a_i^{s-1} \ge a_{i-1}^{s-1}$, $i = 1, \dots, n-1$, the validity of the lefthand inequality of (3).

To prove the second inequality of (3) we assume (without loss of generality) that $a_1 > 0$. Next we define for positive real numbers x and y and for real parameters u and v the mean-value family $E(u, v; x, y)$ by

$$E(u,v;x,y) = \left[\frac{v}{u}\frac{x^u - y^u}{x^v - y^v}\right]^{1/(u-v)}, \quad u \neq v, uv \neq 0, x \neq y,$$
$$E(u,v;x,x) = \lim_{y \to x} E(u,v;x,y) = x.$$

(A detailed analysis of E can be found in [**1, 2, 3, 7**].) Since $E(u, v; x, y)$ is increasing in u and v (see [**7**]) we obtain for $i \in \{1, \ldots, n-1\}$:

$$E(r+1, 1; a_{i-1}, a_i) \leq E(2r, r; a_{i-1}, a_i),$$

which implies

$$\begin{aligned} a_i^{r+1} - a_{i-1}^{r+1} &\leq \frac{r+1}{2}(a_i - a_{i-1})(a_i^r + a_{i-1}^r) \\ &\leq \frac{r+1}{2} p_i (a_i^r + a_{i-1}^r). \end{aligned}$$

Hence, we get for $j \in \{1, \ldots, n-1\}$:

$$\begin{aligned} a_j^{r+1} &= \sum_{i=1}^{j} (a_i^{r+1} - a_{i-1}^{r+1}) \\ &\leq \frac{r+1}{2} \sum_{i=1}^{j} p_i (a_i^r + a_{i-1}^r) \\ &= (r+1)\left[A_j - \frac{1}{2} p_{j+1} a_j^r\right] \\ &= (r+1)\left[A_{j-1} + \frac{1}{2} p_j a_j^r\right] \end{aligned}$$

where

$$A_j = \sum_{i=1}^{j} a_i^r \frac{p_i + p_{i+1}}{2}.$$

This implies

$$\frac{1}{2}(A_{j-1}+A_j) \geq \frac{1}{r+1} a_j^{r+1}\left[1+\frac{r+1}{4a_j}(p_{j+1}-p_j)\right]. \tag{4}$$

Let $t=(s+1)/(r+1)$; since $t \geq 2$ we obtain

$$E(t,1;A_{j-1},A_j) \geq E(2,1;A_{j-1},A_j),$$

which leads to

$$\frac{A_j^t - A_{j-1}^t}{t(A_j-A_{j-1})} \geq \left(\frac{A_j+A_{j-1}}{2}\right)^{t-1}. \tag{5}$$

We consider two cases.

Case 1. $1+((r+1)/(4a_j))(p_{j+1}-p_j)>0$. Then we conclude from (4) and (5),

$$(r+1)^t(A_j^t-A_{j-1}^t) \geq t(r+1)(A_j-A_{j-1})a_j^{(r+1)(t-1)} \cdot \left[1+\frac{r+1}{4a_j}(p_{j+1}-p_j)\right]^{t-1}. \tag{6}$$

Applying Bernoulli's inequality

$$(1+x)^\alpha \geq 1+\alpha x, \quad x>-1, \quad \alpha \geq 1$$

with $x=((r+1)/(4a_j))(p_{j+1}-p_j)$ and $\alpha = t-1 = (s-r)/(r+1)$, we get from (6):

$$(r+1)^t(A_j^t-A_{j-1}^t) \geq t(r+1)(A_j-A_{j-1})a_j^{(r+1)(t-1)} \cdot \left[1+(t-1)\frac{r+1}{4a_j}(p_{j+1}-p_j)\right]. \tag{7}$$

Case 2. $1+((r+1)/(4a_j))(p_{j+1}-p_j) \leq 0$. Since $t-1 \geq 1$ we conclude that the expression on the righthand side of (7) is nonpositive, so that inequality (7) holds also in case 2.

From (7) we obtain

$$(r+1)^t(A_j^t - A_{j-1}^t) \geq \frac{s+1}{2}a_j^s(p_j + p_{j+1}) + \frac{(s+1)(s-r)}{8}a_j^{s-1}(p_{j+1}^2 - p_j^2).$$

Summing for $j = 1, 2, \dots, n-1$ yields the second inequality of (3). □

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