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EXPONENTIAL DICHOTOMIES IN LINEAR SYSTEMS WITH A SMALL PARAMETER

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1. Introduction. Using Melnikov's technique in bifurcation theory, we investigate the theory of exponential dichotomy in linear systems depending on a real parameter. We generalize a well-known result on the exponential dichotomy in K.J. Palmer [**8**]. It is well known that the theory of exponential dichotomy plays very important roles in studying nonautonomous dynamical systems and has received much attention. For example, on the stability theory, we refer to Coppel [**2**]; on the existence of almost periodic solutions, to Fink [**3**]; on the theory of topological equivalence, to Palmer [**7**]; on the bifurcation theory and chaos, to Meyer and Sell [**5, 6**], Palmer [**8, 9**] and Battelli and Palmer [**1**]. About the theory of exponential dichotomy, we refer to Sacker and Sell [**10**] and Coppel [**2**].

We consider a linear differential equation

$$\dot{x} = A(t)x \tag{1}$$

where $x \in R^n$ and $A(t)$ is an $n \times n$ continuous bounded matrix defined on R. We say that the linear differential equation (1) admits an exponential dichotomy with constants K and α on an interval J if there exist a projection P and the fundamental matrix, denoted by $X(t)$, of equation (1) satisfying

$$\begin{aligned} |X(t)PX^{-1}(s)| &\le Ke^{-\alpha(t-s)}, \qquad t \ge s \\ |X(t)(I-P)X^{-1}(s)| &\le Ke^{-\alpha(s-t)}, \qquad s \ge t \end{aligned}$$

for $t, s \in J$. In particular, when $A(t) = A$ is a constant matrix, equation (1) possesses an exponential dichotomy on R if and only if the real parts of the eigenvalues of the matrix A are different from zero. We are only interested in exponential dichotomies on $J = R$, R^+ and R^-.

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In studying bifurcation theory of ordinary differential equations, we often want to consider a linear differential equation depending on a real small parameter

$$\dot{x} = A(t, \varepsilon)x \tag{2}$$

where $x \in R^n$, ε is a real small parameter and $A(t, \varepsilon)$ is an $n \times n$ continuous bounded matrix defined on $R \times (-l_0, l_0)$, ($(-l_0, l_0)$ stands for an interval from $-l_0$ to l_0), and we want to study the following problem. Suppose when $\varepsilon = 0$ the linear equation

$$\dot{x} = A(t, 0)x \tag{3}$$

has an exponential dichotomy on both R^+ and R^- (but not an exponential dichotomy on R), then under what conditions can we guarantee the linear equation (2) admits an exponential dichotomy on R when ε is sufficiently small and nonzero? Palmer proved the following theorem in [**8**].

Theorem A. *Suppose equation* (3) *has an exponential dichotomy on both R^+ and R^-, and the sum of the dimensions of stable and unstable subspaces of equation* (3) *is n. Also, the derivatives A_ε, $A_{\varepsilon\varepsilon}$ exist and are continuous bounded on $R \times (-l_0, l_0)$. Moreover, suppose that equation* (3) *has* (*up to a scalar multiple*) *a unique nontrivial solution $\varphi(t)$ bounded on R.*

Then the equation adjoint to equation (3) *has* (*up to a scalar multiple*) *a unique nontrivial solution $\psi(t)$ bounded on R and if*

$$\int_{-\infty}^{\infty} \psi^*(t) A_\varepsilon(t, 0)\varphi(t)\, dt \neq 0 \tag{4}$$

(*$*$ denotes transpose*), *the equation* (2) *has for $\varepsilon \neq 0$ sufficiently small an exponential dichotomy on R. If, however,* (4) *does not hold the inhomogeneous equation*

$$\dot{x} = A(t, 0)x + A_\varepsilon(t, 0)\varphi(t) \tag{5}$$

has a solution $w(t)$ bounded on R and then if

$$\int_{-\infty}^{\infty} \psi^*(t)\{A_{\varepsilon\varepsilon}(t, 0)\varphi(t) + 2A_\varepsilon(t, 0)w(t)\}\, dt \neq 0 \tag{6}$$

the equation (2) *admits for* $\varepsilon \neq 0$ *sufficiently small an exponential dichotomy on* R.

It is natural to ask if (6) does not hold, what condition can guarantee that linear equation (1) admits for $\varepsilon \neq 0$ sufficiently small an exponential dichotomy on R? The main purpose of this paper is to solve the above problem. We obtain a general result on this problem.

2. Main result and proof. Before giving the main result of this paper, we need the following lemmas.

Lemma 1. *Suppose the matrix* $A(t,0)$ *is bounded on* R *and linear equation* (3)

$$\dot{x} = A(t,0)x$$

admits an exponential dichotomy on both R^+ *and* R^- *and* (*up to a scalar multiple*) *unique bounded solution on* R, *then for a continuous bounded function* $f(t)$, *the equation*

$$\dot{x} = A(t,0)x + f(t) \tag{6}$$

has bounded solutions on R *if and only if*

$$\int_{-\infty}^{\infty} \psi^*(t) f(t)\, dt = 0 \tag{7}$$

holds for all bounded solutions $\psi(t)$ *on* R *of the equation adjoint to equation* (3). *Moreover, if* (7) *holds, then equation* (6) *has a unique bounded solution* $x(t)$ *satisfying* $\varphi^*(0)x(0) = 0$ *where* $\varphi(t)$ *is a bounded solution of equation* (3).

The first part of this lemma is due to Palmer [**9**], the second part is due to Gruendler [**4**].

Lemma 2. *Suppose the matrix* $A(t,0)$ *satisfies the conditions of Lemma* 1. *Moreover, we assume the sum of dimensions of stable and unstable subspaces of equation* (3) *is* n. *If, for* $\varepsilon \neq 0$ *sufficiently small and any continuous bounded function* $f(t)$, *the equation*

$$\dot{x} = A(t,\varepsilon)x + f(t)$$

has a bounded solution on R, then for $\varepsilon \neq 0$ sufficiently small the linear equation

$$\dot{x} = A(t,\varepsilon)x$$

possesses an exponential dichotomy on R.

Proof. It follows from the assumption of Lemma 2 and roughness of exponential dichotomy that, for $\varepsilon \neq 0$ sufficiently small, linear equation (2) admits an exponential dichotomy on both R^+ and R^-, and the sum of dimensions of stable and unstable subspaces of equation (2) is n. If we can show for $\varepsilon \neq 0$ sufficiently small equation (2) has no bounded solution on R, it follows from the proof of Proposition 8.2 in Coppel [**2**, p. 69] that equation (2) possesses an exponential dichotomy on R for $\varepsilon \neq 0$ sufficiently small. Suppose for $\varepsilon \neq 0$ sufficiently small, equation (2) has a solution bounded on R. Then the equation adjoint to equation (2) also has a solution $\psi(t,\varepsilon)$ bounded on R (refer to Zeng [**11**] or Palmer [**9**]). From the assumption of the Lemma, we see the equation

$$\dot{x} = A(t,\varepsilon)x + \psi(t,\varepsilon)$$

has a solution bounded on R for $\varepsilon \neq 0$ sufficiently small. It follows from Lemma 1 that

$$\int_{-\infty}^{\infty} \psi^*(t,\varepsilon)\psi(t,\varepsilon)\,dt = 0$$

which results in a contradiction. The proof of Lemma 2 is complete. □

For convenience, we first introduce some notations. Let $w_0(t) = \varphi(t)$. Assuming $w_1(t), w_2(t), \dots, w_{k-1}(t)$ are bounded on R and, assuming equation $(9)_k$ has a bounded solution, we define $w_k(t)$ to be the unique solution, bounded on R and satisfying $\varphi^*(0)w_k(0) = 0$, of equation $(9)_k$

$$(9)_k \qquad \dot{x} = A(t,0)x + \sum_{m=1}^{k} \binom{k}{m} A_\varepsilon^{(m)}(t,0)w_{k-m}(t),$$

where $A_\varepsilon^{(m)}(t,0)$ denotes $(\partial^m/\partial\varepsilon^m)A(t,0)$. Obviously, $w_1(t)$ is the unique solution, bounded on R and satisfying $\varphi * (0)w_1(0) = 0$, of the equation

$$(10) \qquad \dot{x} = A(t,0)x + A_\varepsilon^{(1)}(t,0)\varphi(t);$$

and $w_2(t)$ is the unique solution, bounded on R and satisfying $\varphi^*(0)w_2(0) = 0$, of equation

$$\dot{x} = A(t,0)x + 2A_\varepsilon^{(1)}(t,0)w_1(t) + A_\varepsilon^{(2)}(t,0)\varphi(t). \tag{11}$$

The main result of this paper is as follows:

Theorem 1. *Suppose the linear equation* (3) *admits an exponential dichotomy on both* R^+ *and* R^- *with constants* K *and* α *and unique* (*up to a scalar multiple*) *solution* $\varphi(t)$ *bounded on* R, *and the sum of dimensions of stable and unstable subspaces is* n; $A_\varepsilon^{(i)}(t,\varepsilon)$ *is uniformly bounded in* $(t,\varepsilon) \in R \times (-l_0, l_0)$, $i = 0, 1, 2, \ldots, N$. *Let* $\psi(t)$ *be the unique* (*up to a scalar multiple*) *bounded solution on* R *of the equation adjoint to equation* (3). *If*

$$\sum_{m=1}^{i} \int_{-\infty}^{\infty} \psi^*(t) \binom{i}{m} A_\varepsilon^{(m)}(t,0) w_{i-m}(t)\,dt = 0, \tag{12$_i$}$$

$$i = 1, 2, \ldots, N-1 \textit{ and}$$

$$\sum_{m=1}^{N} \int_{-\infty}^{\infty} \psi^*(t) \binom{N}{m} A_\varepsilon^{(m)}(t,0) w_{n-m}(t)\,dt \neq 0, \tag{13}$$

then for $\varepsilon \neq 0$ *sufficiently small equation* (2)

$$\dot{x} = A(t,\varepsilon)x$$

admits an exponential dichotomy on R, *where* $N \geq 1$ *is an integer.*

Remark. If $(12)_i$ holds, it follows from Lemma 1 that, for all $k \leq N-1$, equation (9) has a unique bounded solution $w_k(t)$ satisfying $\varphi^*(0)w_k(0) = 0$. For example, when $N = 3$, equation $(12)_i$ becomes

$$\int_{-\infty}^{\infty} \psi^*(t) A_\varepsilon^{(1)}(t,0)\varphi(t)\,dt = 0 \tag{14}$$

$$\int_{-\infty}^{\infty} \psi^*(t)[2A_\varepsilon^{(1)}(t,0)w_1(t) + A_\varepsilon^{(2)}(t,0)\varphi(t)]\,dt = 0. \tag{15}$$

From (14) and Lemma 1, we see equation $(9)_1$, that is, equation (10), admits a unique bounded solution $w_1(t)$ satisfying $\varphi^*(0)w_1(0) = 0$. It follows from (15) and Lemma 1 that equation $(9)_2$, that is, equation (11), has a unique bounded solution $w_2(t)$ satisfying $\varphi^*(0)w_2(0) = 0$.

Proof of Theorem 1. For $h > 0$ an integer and any continuous bounded function $f(t)$, we consider the equation

$$\dot{x} = A(t,\varepsilon)x + \varepsilon^{h+N} f(t). \tag{16}$$

Let $x = y + \alpha\varphi(t)$ where α is a new parameter. Then equation (16) reads

$$\dot{y} = A(t,\varepsilon)y + \alpha[A(t,\varepsilon) - A(t,0)]\varphi(t) + \varepsilon^{N+h} f(t).$$

We write the above equation as

$$\dot{y} = A(t,0)y + [A(t,\varepsilon) - A(t,0)][y + \alpha\varphi(t)] + \varepsilon^{N+h} f(t). \tag{17}$$

Using Liapunov-Schmidt method, we see that the we only need to solve the following two equations

$$\begin{aligned} \dot{y} = A(t,0)y + [A(t,\varepsilon) - A(t,0)][y + \alpha\varphi(t)] + \varepsilon^{N+h} f(t) \\ - \bar{\bar{\theta}}(y(\cdot),\alpha,\varepsilon)\cdot\psi(t), \end{aligned} \tag{18}$$

$$\begin{aligned} \bar{\bar{\theta}}(y(\cdot),\alpha,\varepsilon) = \int_{-\infty}^{\infty} \psi^*(s)\{[A(s,\varepsilon) - A(s,0)][y(s) + \alpha\varphi(s)] \\ + \varepsilon^{N+h} f(s)\}\,ds = 0. \end{aligned} \tag{19}$$

By the same proof as one of the propositions in Battelli [**1**, p. 282], we can prove there exist $\varepsilon_0 > 0$ and $\Delta_0 > 0$ such that when $|\varepsilon| \leq \varepsilon_0$ equation (18) has a unique bounded solution $y(t,\alpha,\varepsilon)$ satisfying $\varphi * (0)y(0,\alpha,\varepsilon) = 0$ and

$$|y(t,\alpha,\varepsilon)| \leq \Delta_0.$$

Substituting $y = y(t,\alpha,\varepsilon)$ into (19), we obtain an equation

$$\begin{aligned} \bar{\theta}(\alpha,\varepsilon) &= \bar{\bar{\theta}}(y(\cdot,\alpha,\varepsilon),\alpha,\varepsilon) \\ &= \int_{-\infty}^{\infty} \psi^*(s)\{[A(s,\varepsilon) - A(s,0)][y(s,\alpha,\varepsilon) + \alpha\varphi(s)] \\ &\qquad + \varepsilon^{N+h} f(s)\}\,ds = 0. \end{aligned} \tag{20}$$

We let $\varepsilon = 0$ in equation (18); equation (18) becomes

$$\dot{y} = A(t,0)y \tag{21}$$

which has a unique bounded solution $y(t) = 0$ satisfying $\varphi^*(0)y(0) = 0$. Thus, by uniqueness, $y(t,\alpha,0) = 0$. Now we will solve equation (20).

We first prove, by using induction, $y_\varepsilon^{(k)}(t,\alpha,0) = \alpha \cdot w_k(t)$, $k = 1,2,\dots,N-1$. Now $y(t,\alpha,\varepsilon)$ satisfies

$$\begin{aligned}\dot{y}(t,\alpha,\varepsilon) = A(t,0)y(t,\alpha,\varepsilon) + [A(t,\varepsilon) - A(t,0)][y(t,\alpha,\varepsilon) + \alpha\cdot\varphi(t)]\\ + \varepsilon^{N+h} f(t) - \int_{-\infty}^{\infty} \psi^*(s)\{[A(s,\varepsilon) - A(s,0)]\\ [y(s,\alpha,\varepsilon) + \alpha\varphi(s)] + \varepsilon^{N+h} f(s)\}\, ds \cdot \psi(t).\end{aligned} \tag{22}$$

Differentiating both sides of the above equation with respect to ε and setting $\varepsilon = 0$ and noting that $y(t,\alpha,0) = 0$, we have

$$\begin{aligned}\dot{y}_\varepsilon^{(1)}(t,\alpha,0) = A(t,0)y_\varepsilon^{(1)}(t,\alpha,0) + \alpha\cdot A_\varepsilon^{(1)}(t,0)\varphi(t)\\ - \alpha\int_{-\infty}^{\infty} \psi^*(s)A_\varepsilon^{(1)}(s,0)\varphi(s)\, ds\cdot\psi(t).\end{aligned}$$

From condition $(12)_i$ with $i = 1$, we know

$$\int_{-\infty}^{\infty} \psi^*(s)A_\varepsilon^{(1)}(s,0)\varphi(s)\, ds = 0,$$

hence

$$\dot{y}_\varepsilon^{(1)}(t,\alpha,0) = A(t,0)y_\varepsilon^{(1)}(t,\alpha,0) + \alpha A_\varepsilon(t,0)\varphi(t). \tag{23}$$

It follows from the uniqueness of solutions satisfying $\varphi^*(0)y(0) = 0$ that $y_\varepsilon^{(1)}(t,\alpha,0) = \alpha\cdot w_1(t)$. Thus, for $k = 1$, $y_\varepsilon^{(k)}(t,\alpha,0) = \alpha w_k(t)$ is true. We suppose for all $k \le N-2$, $y_\varepsilon^{(k)}(t,\alpha,0) = \alpha w_k(t)$. Now we want to prove $y_\varepsilon^{(N-1)}(t,\alpha,0) = \alpha w_{N-1}(t)$. Differentiating both sides of equation (22) $N-1$ times with respect to ε and setting $\varepsilon = 0$, we have

$$\begin{aligned}\dot{y}_\varepsilon^{(N-1)}(t,\alpha,0) = A(t,0)y_\varepsilon^{(N-1)}(t,\alpha,0) \\ + \sum_{m=1}^{N-2}\binom{N-1}{m}A_\varepsilon^{(m)}(t,0)y_\varepsilon^{(N-m-1)}(t,\alpha,0) \\ + \alpha A_\varepsilon^{(N-1)}(t,0)\varphi(t) \\ - \int_{-\infty}^{\infty}\psi^*(s)\bigg\{\sum_{m=1}^{N-2}\binom{N-1}{m}A_\varepsilon^{(m)}(s,0)y_\varepsilon^{(N-m-1)}(s,\alpha,0) \\ + \alpha A_\varepsilon^{(N-1)}(s,0)\varphi(s)\bigg\}\,ds\cdot\psi(t).\end{aligned}$$

Since $y_\varepsilon^{(k)}(t,\alpha,0) = \alpha w_k(t)$ for all $k \le N-2$, we have

$$\begin{aligned}\dot{y}_\varepsilon^{(N-1)}(t,\alpha,0) = A(t,0)y_\varepsilon^{(N-1)}(t,\alpha,0) \\ + \alpha\sum_{m=1}^{N-1}\binom{N-2}{m}A_\varepsilon^{(m)}(t,0)\cdot w_{N-1-m}(t) \\ - \alpha\int_{-\infty}^{\infty}\psi^*(s)\sum_{m=1}^{N-1}\binom{N-1}{m}A_\varepsilon^{(m)}(s,0) \\ \cdot w_{N-m-1}(s)\,ds\cdot\psi(t).\end{aligned}$$

From condition $(12)_{N-1}$, we see

$$\int_{-\infty}^{\infty}\psi^*(s)\sum_{m=1}^{N-1}\binom{N-1}{m}A_\varepsilon^{(m)}(s,0)\cdot w_{N-m-1}(s)\,ds = 0.$$

Thus,

$$\begin{aligned}\dot{y}_\varepsilon^{(N-1)}(t,\alpha,0) = A(t,0)y_\varepsilon^{(N-1)}(t,\alpha,0) \\ + \alpha\sum_{m=1}^{N-1}\binom{N-1}{m}A_\varepsilon^{(m)}(t,0)w_{N-1-m}(t).\end{aligned}$$

By uniqueness, we have $y_\varepsilon^{(N-1)}(t,\alpha,0) = \alpha w_{N-1}(t)$. Therefore $y_\varepsilon^{(k)}(t,\alpha,0) = \alpha w_k(t)$, $k = 1,2,\dots,N-1$, by induction. We have

$$\bar{\theta}(\alpha,0) = 0,$$

and for all $i \le N-1$,

$$\begin{aligned}\bar{\theta}_\varepsilon^{(i)}(\alpha,0) &= \sum_{m=1}^{i-1}\int_{-\infty}^{\infty}\psi^*(t)\binom{i}{m}A_\varepsilon^{(m)}(t,0)y_\varepsilon^{(i-m)}(t,\alpha,0)\,dt \\ &\quad + \int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(i)}(t,0)\alpha\varphi(t)\,dt \\ &= \alpha\sum_{m=1}^{i-1}\binom{i}{m}\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(m)}(t,0)w_{i-m}(t)\,dt \\ &\quad + \alpha\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(i)}(t,0)w_0(t)\,dt \\ &= \alpha\sum_{m=1}^{i}\binom{i}{m}\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(m)}(t,0)w_{i-m}(t)\,dt \\ &= 0 \qquad \text{(from condition } (12)_i).\end{aligned}$$

Now we define a function $\theta(\alpha,\varepsilon)$ by

$$\theta(\alpha,\varepsilon) = \begin{cases} \frac{\bar{\theta}(\alpha,\varepsilon)}{\varepsilon^N}, & \varepsilon \neq 0 \\ \frac{1}{N!}\bar{\theta}_\varepsilon^{(N)}(\alpha,0), & \varepsilon = 0; \end{cases}$$

then $\theta(\alpha,\varepsilon)$ is continuous and differentiable in ε and α.

$$\begin{aligned}\theta(\alpha,0) &= \frac{1}{N!}\bar{\theta}_\varepsilon^{(N)}(\alpha,0) \\ &= \frac{1}{N!}\Big\{\sum_{m=1}^{N-1}\binom{N}{m}\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(m)}(t,0)y_\varepsilon^{(N-m)}(t,\alpha,0)\,dt \\ &\qquad + \alpha\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(N)}(t,0)\varphi(t)\,dt\Big\} \\ &= \frac{\alpha}{N!}\Big\{\sum_{m=1}^{N-1}\binom{N}{m}\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(m)}(t,0)w_{N-m}(t)\,dt \\ &\qquad + \int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(N)}(t,0)w_0(t)\,dt\Big\} \\ &= \frac{\alpha}{N!}\sum_{m=1}^{N}\binom{N}{m}\int_{-\infty}^{\infty}\psi^*(t)A_\varepsilon^{(m)}(t,0)w_{N-m}(t)\,dt,\end{aligned}$$

hence $\theta(0,0) = 0$ and

$$\theta_\alpha(0,0) = \frac{1}{N!} \sum_{m=1}^{N} \binom{N}{m} \int_{-\infty}^{\infty} \psi^*(t) A_\varepsilon^{(m)}(t,0) w_{N-m}(t)\, dt$$
$$\neq 0 \qquad \text{(from condition (13))}.$$

It follows from the implicit function theorem that there is a continuous function $\alpha(\varepsilon)$ such that $\alpha(0) = 0$ and for ε sufficiently small

$$\theta(\alpha(\varepsilon), \varepsilon) = 0.$$

Thus, for $\varepsilon \neq 0$ sufficiently small,

$$\bar{\theta}(\alpha(\varepsilon), \varepsilon) = 0.$$

Hence, for $\varepsilon \neq 0$ sufficiently small, equation (17) has a bounded solution $y = y(t, \alpha(\varepsilon), \varepsilon)$ and the equation

$$\dot{x} = A(t, \varepsilon)x + f(t)$$

has a bounded solution $(y(t, \alpha(\varepsilon), \varepsilon) + \alpha(\varepsilon)\varphi(t))/\varepsilon^{N+h}$. It follows from Lemma 2 that for $\varepsilon \neq 0$ sufficiently small, equation (2) possesses an exponential dichotomy on R. The proof of Theorem 1 is complete. □

Obviously, Theorem 1 is a generalization of Palmer's result in [**8**]. In fact, taking $N = 3$ in Theorem 1, we have the following result.

Theorem 2. *Suppose all conditions of Theorem* A *are satisfied. In addition, we assume* $A_{\varepsilon\varepsilon\varepsilon}(t, \varepsilon)$ *is uniformly bounded in* $(t, \varepsilon) \in R \times (-l_0, l_0)$. *If*

$$\int_{-\infty}^{\infty} \psi^*(t) A_\varepsilon(t, 0) \varphi(t)\, dt \neq 0, \tag{24}$$

then equation (2) *has for* $\varepsilon \neq 0$ *sufficiently small an exponential dichotomy on* R. *If* (24) *does not hold the inhomogeneous equation*

$$\dot{x} = A(t, 0)x + A_\varepsilon(t, 0)\varphi(t) \tag{25}$$

has a unique bounded solution $w_1(t)$ *on* R *satisfying* $\varphi^*(0)w_1(0) = 0$ *and if*

$$\int_{-\infty}^{\infty} \psi^*(t)\{A_{\varepsilon\varepsilon}(t,0)\varphi(t) + 2A_\varepsilon(t,0)w_1(t)\}\, dt \neq 0, \tag{26}$$

then equation (2) *has for* $\varepsilon \neq 0$ *sufficiently small an exponential dichotomy on* R*. If* (24) *and* (26) *don't hold, the equation*

$$\dot{x} = A(t,0)x + 2A_\varepsilon(t,0)w_1(t) + A_{\varepsilon\varepsilon}(t,0)\varphi(t)$$

has a unique bounded solution $w_2(t)$ *satisfying* $\varphi^*(0)w_2(0) = 0$*, and if*

$$\int_{-\infty}^{\infty} \psi^*(t)\{3A_\varepsilon(t,0)w_2(t) + 3A_{\varepsilon\varepsilon}(t,0)w_1(t) + A_{\varepsilon\varepsilon\varepsilon}(t,0)\varphi(t)\}\, dt \neq 0,$$

then equation (2) *admits for* $\varepsilon \neq 0$ *sufficiently small an exponential dichotomy on* R.

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