ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

SUPERHARMONIC FUNCTIONS IN HÖLDER DOMAINS

DAVID A. STEGENGA AND DAVID C. ULLRICH

1. Introduction. We will call the open set $\Omega \subset \mathbf{R}^m$ an L^p domain if $\int_\Omega u^p\,dx < \infty$ for every function $u \geq 0$ which is superharmonic in Ω; we will say that Ω is an LP domain if Ω is an L^p domain for some $p > 0$.

Armitage [**3**] has shown that every domain of "bounded curvature" is an L^p domain for $p < m/(m-1)$. A more recent result due to Maeda and Suzuki states that every Lipschitz domain is an LP domain; this is proved in [**15**], with explicit bounds on p. Our main result (Theorem 1) states that Ω is LP under much weaker hypotheses; namely, that Ω be a Hölder domain . The converse statement is also true for finitely connected planar domains. See [**21**] for a restricted version of this result for positive harmonic functions and the necessity of the Hölder condition for simply connected planar domains. See [**16, 17**] for the result for finitely connected planar domains and [**1**] for some results with sharp bounds on p. See also [**12**] where closely related questions are studied.

In order to discuss Hölder domains we consider proper open connected subdomains Ω of Euclidean m-space R^m, for $m \geq 2$. Following [**9**] we define the quasi-hyperbolic metric k_Ω in Ω by

$$k_\Omega(x_1, x_2) = \inf_\gamma \int_\gamma \frac{ds}{\delta_\Omega(x)} \tag{1.1}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in Ω. Here we denote by $\delta_\Omega(x)$ the Euclidean distance between x and $\partial\Omega$ and ds is integration with respect to arc length.

Fix a point $x_0 \in \Omega$. We say that Ω is a Hölder domain if

$$k_\Omega(x_0, x) \leq c_1 \log \frac{\delta_\Omega(x_0)}{\delta_\Omega(x)} + c_2, \qquad x \in \Omega \tag{1.2}$$

Received by the editors on January 19, 1994 and in revised form on July 5, 1994.
Key words and phrases. Superharmonic functions, John domains, Hölder domains, NTA domains.

holds for some finite constants c_1, c_2. See [**19**] where this terminology is introduced and the relation to uniform domains, NTA domains and John domains is discussed.

Briefly, the terminology is motivated by a result of Becker and Pommerenke [**6**] where it is proved that simply connected planar Hölder domains are precisely the conformal images of the disk under Hölder continuous Riemann mappings. Gehring and Martio established an R^m version of this result by replacing the Riemann mapping function with a k-quasiconformal mapping. See [**10**], where domains satisfying (1.2) are said to satisfy a quasi-hyperbolic boundary condition.

On the other hand, Hölder domains are closely related to John domains. We fix a point $x_0 \in \Omega$, then we say that Ω is a *John domain* provided that for each $x_1 \in \Omega$ there is an arc γ joining x_0 to x_1 in Ω along which

$$\delta_\Omega(x) \geq \alpha l(\gamma(x, x_1)), \qquad x \in \gamma. \tag{1.4}$$

Here α is a positive constant, $\gamma(x, x_1)$ is the portion of γ joining x to x_1 and $l(\gamma(x, x_1))$ is its arc length. It is elementary that John domains are Hölder domains. But the thickness condition (1.4), which can be visualized as a twisted cone condition, does not hold in general for Hölder domains. In [**18**] an example of a Hölder domain is constructed which contains a sequence of tubes of width $\varepsilon_n > 0$ and length $\varepsilon_n \log \varepsilon_n^{-1}$, where ε_n tends to zero. Thus (1.4) is violated and hence Hölder domains are not necessarily John domains. See also the example in [**6**].

Theorem 1. *Let $\Omega \subset \mathbf{R}^m$, and let B_0 be a ball for which the ball with the same center but twice the radius is contained in Ω. If Ω is a Hölder domain, then there exist $p = p(\Omega) > 0$ and $M = M(\Omega, B_0) < \infty$ such that*

$$\int_\Omega u^p \, dx \leq M(\min_{B_0} u)^p \tag{1.5}$$

for every function $u \geq 0$ which is superharmonic on Ω.

Theorem 1 answers a question posed by Armitage in [**5**]: Suppose that for any $\zeta \in \partial\Omega$ there exists an open cone $\Gamma_\zeta \subset \Omega$ with vertex at ζ, such that Γ_ζ has opening α and height h, uniformly for $\zeta \in \partial\Omega$; does it follow that Ω is LP? We shall see below (Example 1 in Section 4) that the "cone condition" in precisely this form does not imply LP; however a slightly stronger version of the same condition will suffice: If we suppose that for each $x \in \Omega$ there exists a cone as above with vertex at x, then Ω is a John domain, hence LP.

In fact, Example 1 gives a bounded simply connected domain in the plane which satisfies an interior cone condition at every point of the boundary but which is not LP, while Masumoto [**16**] has shown that if a domain in the plane bounded by finitely many Jordan curves satisfies an interior cone condition at every boundary point then the domain must be LP. One may see that a plane domain as in Masumoto's theorem must in fact satisfy a cone condition at every point of the interior, which implies that the domain must be a John domain, as above; hence this result follows from Theorem 1. (Another result in [**16**], the fact that a plane domain bounded by finitely many quasicircles is LP, can also be derived from Theorem 1: It was shown in [**14**] that a domain bounded by one quasicircle is NTA, which implies that a domain bounded by finitely many quasicircles is a John domain).

We shall see that the hypotheses of Theorem 1 are not necessary; there exist infinitely connected planar domains which are LP but not Hölder (Example 2 in Section 4). The Hölder condition is nonetheless fairly sharp; as we show in Theorem 2, an LP domain cannot have a cusp. (Note that $\dim(\partial\Omega) < m$ if Ω is Hölder; see [**18**]. We do not know whether there exists an LP domain Ω with $\dim(\partial\Omega) = m$.)

2. The proof of Theorem 1. A few preliminary definitions:

We will let $\mathfrak{D}_n$ denote the set of *dyadic cubes of length* 2^{-n} in $\mathbf{R}^m$: $Q \in \mathfrak{D}_n$ if there exists $k \in \mathbf{Z}^m$ such that $Q = \{x \in \mathbf{R}^m : k_j 2^{-n} \le x_j \le (k_j+1)2^{-n}, 1 \le j \le m\}$. We set $\mathfrak{D} = \cup_{n=-\infty}^{\infty} \mathfrak{D}_n$. Given Q_1, $Q_2 \in \mathfrak{D}$ will say that Q_1 and Q_2 are *adjacent* if at least one of the two cubes has a face contained in the other; in this case we will write $Q_1 \sim Q_2$. We say Q_1 and Q_2 are *essentially disjoint* if $Q_1 \cap Q_2$ has empty interior, and we set $\sigma(Q) = 2^{-n}$ for $Q \in \mathfrak{D}_n$.

Now suppose Ω is a bounded open set in $\mathbf{R}^m$. A *Whitney decomposition* for Ω is a family of dyadic cubes $W = W(\Omega) \subset \mathfrak{D}$ such that

(i) $\Omega = \cup_{Q \in W} Q$,

(ii) the cubes in W are pairwise essentially disjoint,

and finally

(iii) for any $Q \in W$ we have

$$5m^{1/2}\sigma(Q) < d(Q, \partial\Omega) \le 20m^{1/2}\sigma(Q). \tag{2.1}$$

Here $d(Q, \partial\Omega)$ denotes the (Euclidean) distance from Q to $\partial\Omega$, the boundary of Ω. Any Ω has such a decomposition [**22**, p. 167]; the constants in (2.1) have been chosen so as to facilitate the proof of Lemma 1 below.

The present section is devoted to the proof of Theorem 1. The plan is to obtain an upper bound for the integral of u^p over an arbitrary Whitney cube Q in terms of the average of u over a fixed cube Q_0 and the quasihyperbolic distance from Q_0 to Q, and then to use the exponential integrability of the quasihyperbolic distance on Hölder domains [**20**]. The next three lemmas hold for any domain whatsoever; although the first is extremely elementary, we nonetheless include an explicit proof because the precise value of the constant is of some importance in the proof of Lemma 1.

Lemma 0. *If Q_1, $Q_2 \in W(\Omega)$ and $Q_1 \sim Q_2$ then $\sigma(Q_2) \le 4\sigma(Q_1)$.*

Proof. Set $\sigma_j = \sigma(Q_j)$, $d_j = d(Q_j, \partial\Omega)$. The fact that the diameter of Q_1 is $m^{1/2}\sigma_1$ shows that

$$d_2 \le d_1 + m^{1/2}\sigma_1 \le 21m^{1/2}\sigma_1,$$

so that $\sigma_2 \le (21/5)\sigma_1$. This implies $\sigma_2 \le 4\sigma_1$, since each σ_j is a power of two. □

Let $|E|$ denote the Lebesgue measure of the set E; if u is integrable on E, set $u_E = |E|^{-1}\int_E u\,dx$. The following lemma may be regarded as a version of Harnack's inequality for positive superharmonic functions. We thank Professor S. Gardiner for bringing to our attention the fact that this result is essentially in [**4**, Lemma 3.2].

Lemma 1. *There exists a constant η, depending only on the dimension m, with the following property: If Ω is an open set in $\mathbf{R}^m$,*

$u \geq 0$ is superharmonic in Ω, and Q_1, Q_2 are two adjacent cubes in a Whitney decomposition of Ω, then

$$u_{Q_2} \leq \eta u_{Q_1}. \tag{2.2}$$

Proof. Let x_0 denote the center of Q_1, and set B_j equal to the Euclidean ball with center x_0 and radius r_j, with $r_1 = \sigma(Q_1)/2$ and $r_2 = 5m^{1/2}\sigma(Q_1)$. It follows from (1.1) and Lemma 0 that $B_1 \subset Q_1$ and $Q_2 \subset B_2 \subset \Omega$. Since $u \geq 0$, this shows

$$u_{Q_2} \leq \frac{|B_2|}{|Q_2|} u_{B_2} \quad \text{and} \quad u_{B_1} \leq \frac{|Q_1|}{|B_1|} u_{Q_1}.$$

But

$$u_{B_2} \leq u_{B_1},$$

because u is superharmonic and $r_2 > r_1$. This gives (2.2) (with $\eta = 40^m m^{m/2} \geq |Q_1||B_2|/|Q_2||B_1|$). □

Note. Throughout the paper the letter η will denote the constant appearing in (2.2), while the letter c will indicate the traditional "constant, the value of which may vary from line to line." We note that the value of p which will be obtained in Theorem 1 depends on the value of η, so that even if there were no loss elsewhere in the argument, to obtain optimal bounds on p by the present method would require optimal bounds on η, a goal which seems fairly ambitious.

Lemma 2. *Let W be a Whitney decomposition of Ω and assume that $Q_0 \in W$ is a cube with x_0 in its interior. Then there is a constant $c_3 < \infty$ satisfying the following: if $u \geq 0$ is a superharmonic function on Ω and $Q \in W$ then*

$$u_Q \leq u_{Q_0} e^{c_3 k_\Omega(x_0, x_Q)} \tag{2.3}$$

where x_Q is the center of Q.

Proof. Let $u \geq 0$ be a superharmonic function on Ω and $Q \in W$ with center x_Q. Since the quasihyperbolic distance between the centers of any two adjacent Whitney cubes is comparable to 1, it follows that there is an absolute constant c and a chain of adjacent cubes $Q_0, Q_1, \dots, Q_n$ with $n \le c k_\Omega(x_0, x_Q)$ and such that $Q_n = Q$ (see [**19**, Lemma 9]). So, by Lemma 1,

$$u_Q \le \eta^n u_{Q_0} \le u_{Q_0} e^{c_3 k_\Omega(x_0, x_Q)},$$

where $c_3 = c \log \eta$. □

Proof of Theorem 1. Suppose Ω is a Hölder domain , and let W be a Whitney decomposition; we may suppose that $\max\{\sigma(Q) : Q \in W\} = 1$. Suppose $u \ge 0$ is superharmonic in Ω. Note that

$$(u^p)_Q \le (u_Q)^p \tag{2.4}$$

if $0 < p \le 1$, by Hölder's inequality.

Suppose that $0 < p \le 1$ and that $Q \in W$; then by (2.3) and (2.4) we have that

$$\int_Q u^p\,dx = |Q|(u^p)_Q \le |Q|(u_{Q_0})^p e^{pc_3 k_\Omega(x_0, x_Q)}. \tag{2.5}$$

Thus, by (2.5)

$$\begin{aligned}\int_\Omega u^p\,dx &= \sum_{Q\in W} \int_Q u^p\,dx \\ &\le (u_{Q_0})^p \sum_{Q \in W} |Q| e^{pc_3 k_\Omega(x_0, x_Q)} \\ &\le c(u_{Q_0})^p \int_\Omega e^{pc_3 k_\Omega(x_0, x)}\,dx.\end{aligned} \tag{2.6}$$

[**20**, Theorem A] shows that if $p = p(c_1, \eta) > 0$ is small enough then the integral above is finite.

Finally, let B_0 be a ball contained in Ω, and let B_1 be a ball with the same center x_1 as B_0 and radius $r_1 = \delta_\Omega(x_1)$, so $B_0 \subset B_1$. By the definition of W, there is an absolute constant ν and a Whitney cube $Q \subset B_1$ for which $|B_1| \le \nu |Q|$. If $u \ge 0$ is a superharmonic function, then it follows from Lemma 2 that

$$u_{Q_0} \le c_Q u_Q \le c_Q \nu u_{B_1} \le c_Q \nu u_{B_0}.$$

By using Lemma 1 and the super-mean-value property it is easy to see that u_{B_0} is smaller than some absolute constant times the minimum value of u on B_0. Hence (2.3) follows from the above and (2.6). ◻

Various other interesting inequalities follow by the same sort of argument. One may easily show by the method above that

$$\int_\Omega u(x)\delta_\Omega(x)^q\,dx < \infty \tag{2.7}$$

and

$$\int_\Omega \delta_\Omega(x)^{-r}\,dx < \infty \tag{2.8}$$

for some q, $r > 0$. (Note that (2.7) and (2.8) imply Theorem 1 by Hölder's inequality, as in [**15**].)

3. Domains with cusps. As mentioned earlier, there is a Hölder domain (in fact a Jordan domain in $\mathbf{C}$) which is not a John domain, although Theorem 1 shows it must be an LP domain. Nonetheless Theorem 2 shows that the John condition is sharp in a fairly strong sense:

Definition. Suppose $\Omega \subset \mathbf{R}^m$ is a bounded and connected domain with Whitney decomposition W, and let $2^{-\nu}$ denote the maximal side length of a cube in W. Suppose $\{N_n\}$ is a sequence of natural numbers. We will say that Ω is an $\{N_n\}$-*John domain* if the following holds: Whenever $Q \in W$ and $\sigma(Q) = 2^{-n} < 2^{-\nu}$ then there exists a chain of Whitney cubes $Q = Q_1 \sim Q_2 \sim \cdots \sim Q_K$ with $\sigma(Q_K) > \sigma(Q)$ and $K \le N_n$.

(Thus the John domains are precisely the $\{N_n\}$-John domains with $\sup N_n < \infty$.)

Theorem 2. *Suppose* $\lim_{n\to\infty} N_n = \infty$. *There exists a bounded domain* $\Omega \subset \mathbf{R}^m$ *with connected boundary which is an* $\{N_n\}$*–John domain but not* LP.

This will follow immediately from the following:

Proposition 1. *Let* $\Omega = \{(x,y) \in \mathbf{R} \times \mathbf{R}^{m-1}\colon\ 0 < x < 1, |y| < f(x)\}$*, where* $f : [0,1] \longrightarrow [0,1]$ *is a continuously differentiable function such that* $f(0) = 0$*,* $f'(0) = 0$*, and* $f'(x)$ *is increasing on* $[0,1]$*. Then* Ω *is not an LP domain. In fact, if* $\delta_\Omega(x,y)$ *denotes the distance from* $(x,y) \in \Omega$ *to* $\partial\Omega$ *then there exists a positive harmonic function* u *in* Ω *such that* $\int_\Omega u^p \delta_\Omega^q \,dx\,dy = \infty$ *for every* $p > 0$ *and* $q \geq 0$*.*

Given a sequence $N_n \to \infty$ it is clear that we may choose f as in the proposition in such a way that Ω will be an $\{N_n\}$–John domain, giving Theorem 2. (The proposition also gives an example of a domain bounded by the graph of a Hölder function of order $\alpha < 1$ which is not LP; in fact if $\lim_{\delta\to 0} \delta^{-1}\omega(\delta) = 0$ we may obtain a domain (locally) bounded by the graph of a function with modulus of continuity ω which is not LP.)

Proof of Proposition 1. Choose $x_1 \in (0,1)$ such that $f(x_1) = 1 - x$; having chosen x_j, choose $x_{j+1} \in (0,1)$ such that $f(x_{j+1}) = x_j - x_{j+1}$. Then the sequence x_j decreases to zero, while the fact that $f'(0) = 0$ shows that

$$\lim_{j\to\infty} \frac{x_j - x_{j+1}}{x_{j-1} - x_j} = 1. \tag{3.1}$$

Let u be a strictly positive harmonic function in Ω such that $u(z_n) \to 0$ for any sequence $\{z_n\} \subset \Omega$ with $z_n \to \zeta \in \partial\Omega$, $\zeta \neq (0,0)$. (In other words, u is a kernel function with pole at the origin; such a function may be obtained as a limit of suitably normalized harmonic functions continuous on $\overline{\Omega}$ which vanish everywhere on the boundary except in a small neighborhood of the origin.)

Let $c_j = \max\{u(x,y) : (x,y) \in \Omega, x \geq x_j\}$, and set $\Omega_j = \{(x,y) \in \Omega : x_j \leq x \leq x_{j-2}\} (j \geq 2)$, and $P_j = \{(x,y) \in \Omega : x = x_j\}$. Note that in fact $c_j = \max\{u(x,y) : (x,y) \in P_j\}$, by the maximum principle. It follows from (3.1) that Ω_j is essentially a cylinder of radius $x_{j-1} - x_j$ and height $2(x_{j-1} - x_j)$. In particular, it is clear by comparison with the cylinder $\{x_j < x < x_{j-2}, |y| < f(x_{j-2})\}$ that there exists a constant $0 < \alpha < 1$ such that if $z \in P_{j-1}$ then the harmonic measure of $P_j \cup P_{j-2}$ at z, relative to Ω_j, does not exceed α. Since u is no larger than c_j on $P_j \cup P_{j-2}$ and vanishes elsewhere on the boundary of Ω_j it follows that

$u(z) \le \alpha c_j$ for $z \in P_{j-1}$. This shows that $c_{j-1} \le \alpha c_j$, so that

$$c_j \ge c_1 \beta^{j-1}, \qquad j \ge 2, \tag{3.2}$$

where $\beta = \alpha^{-1} > 1$.

It is not hard to see that the maximum of u on the "vertical slice" P_j must be achieved at the center: $c_j = u(x_j, 0)$. It follows from Harnack's inequality that

$$u(z) \ge c\beta^j, \qquad z \in B_j, \tag{3.3}$$

where B_j is the ball with center $(x_j, 0)$ and radius $(x_j - x_{j-1})/2$.

Now note that $\delta_\Omega(z) \approx (x_{j-1} - x_j)$ for $z \in B_j$. It follows from (3.3) that

$$\int_\Omega u(x,y)^p \delta_\Omega(x,y)^q \, dx \, dy \ge c \sum_{j=1}^{\infty} \beta^{jp} (x_j - x_{j-1})^{m+p}. \tag{3.4}$$

But $\beta^p > 1$, so $\lim\limits_{j \to \infty} \beta^{jp} (x_{j-1} - x_j)^{m+p} = \infty$, by (3.1). □

Similar results hold for domains which look like twisted versions of the domain in Theorem 2. For example:

Theorem 3. *Suppose that* $\lim_{n\to\infty} N_n = \infty$*, and let* $Q_1, Q_2, \ldots$ *be a sequence of pairwise essentially disjoint dyadic cubes such that* $\sigma(Q_j) = 2^{-n}$ *for* $M_n \le j < M_{n+1}$*, where* $M_0 = 1$ *and* $M_{n+1} = M_n + N_n$*. Suppose that* $Q_j \sim Q_k$ *if and only if* $|j-k| = 1$*, and that there exists* $J < \infty$ *such that* $Q_j \cap Q_k = \varnothing$ *for* $|j-k| \ge J$*. Let* Ω *be the interior of the union of the* Q_j*. Then* Ω *is not an* LP *domain.*

The proof is essentially the same as the proof of Theorem 2, with $Q_j \cap Q_{j+1}$ in place of P_j; the inequality corresponding to (3.3) follows from Lemma 3. We define $D_k = \cup_{j=k-J}^{k+J} Q_j$, $k \ge J$.

Lemma 3. *Suppose* $\{Q_j\}$ *is a sequence of cubes as in Theorem* $2'$*, and set* $P_j = Q_j \cap Q_{j+1}$*. Suppose* $u \ge 0$ *is continuous on* $\overline{D}_k$*,* *harmonic in D_k, and vanishes everywhere on the boundary of D_k except on $P_{k-J-1} \cup P_{k+J}$. Then*

$$u(p_k) \geq c \max\{u(z) : z \in P_k\}, \tag{3.5}$$

if p_k is the center of P_k.

The lemma may be deduced from the "boundary Harnack principle" for Lipschitz domains [**2, 8, 14**] together with the Carleson-Hunt-Wheeden lemma [**7, 13** Lemma 2.1, **14**, Lemma 4.1]. (Note that up to a change of scale there are only finitely many possibilities for D_k, so that these results for Lipschitz domains hold in D_k with constants independent of k.)

Finally, another interesting extension to Proposition 1 can be obtained by replacing the smoothness and convexity assumptions by a Lipshitz condition. This yields a more natural cusp condition but its proof requires a subtle fact about the geometry of Hölder domains.

Theorem 4. *Let Ω be defined as in Proposition* 1*, except that we* only *require that $f(x)$ be Lipschitz continuous, $f(x) > 0$ for $x > 0$ and $f(x)/x \to 0$ as $x \to 0$. Then the conclusion of Proposition* 1 *holds.*

Proof. Consider a Whitney decomposition of Ω into dyadic cubes. By taking the lower vertices of the Whitney cubes with bottom side on the real axis we determine a sequence $\{x_n\}$, with $0 < x_n < 1$, which tends to zero (and replaces the sequence in the proposition). Next, we define c_j, Ω_j and P_j as before. Since Ω_j is a Lipshitz domain we can use the boundary Harnack principle again along with standard facts about Lipschitz domains to obtain a similar harmonic measure estimate and thus (3.3) holds. Now the proof must change since (3.1) is no longer true; it's easy to construct examples where $f'(x) = 0$ on (relatively) large intervals near the origin.

We want to prove that the series in (3.4) diverges and so we need only prove that there does not exist $\alpha < 1$ with

$$x_j - x_{j+1} < M\alpha^j \quad \text{for all } j = 1, 2, \ldots . \tag{3.6}$$

To obtain a contradiction suppose there is such an α. By the construction and the fact that f is Lipschitz it follows that $\delta_j = \delta_\Omega((x_j, 0))$ is comparable to $x_{j-1} - x_j$. Now (3.6) implies that there are constants c, c_1, c_2 satisfying:

$$k_\Omega((1,0),(x_j,0)) < cj < c_1 \log(1/\delta_j) + c_2, \qquad j = 1, 2, \dots, \tag{3.7}$$

where k_Ω is the quasihyperbolic metric on Ω. It can be shown that (3.7) implies that Ω is a Hölder domain and hence cannot contain a cusp. A version of this last fact (for $m = 2$) is Corollary 6 in [**18**] and a more general version follows from the results in [**19**], see Theorem 3 or Lemma 3 in that paper.

Following a suggestion from the referee we give a direct proof using the full strength of (3.6). It follows from (3.6) that

$$x_j = \sum_{k=j}^{\infty} (x_k - x_{k+1}) \le M(1-\alpha)^{-1}\alpha^j$$

for $j = 1, 2, \dots$ and hence

$$j < c \log \frac{1}{x_j}, \qquad j = 1, 2, \dots. \tag{3.8}$$

On the other hand, the remaining hypothesis on $f(x)$ implies that $\delta_\Omega(x) < \varepsilon x$ for all small x. Hence,

$$\frac{1}{2\varepsilon} \log \frac{1}{x_j} = \int_{x_j}^{\sqrt{x_j}} \frac{dx}{\varepsilon x} < \int_{x_j}^{1} \frac{dx}{\delta_\Omega(x)} = k_\Omega((x_j,0),(1,0)) < cj$$

and thus,

$$\log \frac{1}{x_j} < 2c_3 \varepsilon j \quad \text{for all sufficiently large } j$$

where c_3 is an absolute constant and ε is an arbitrary positive number. Combining the above inequality with (3.8) we get a contradiction which proves the theorem. □

4. Some examples. We describe four examples in this section. For Example 1, we take Ω to be the simply connected planar domain obtained from the open unit disk by removing a Jordan arc which is contained in the disk except for one endpoint which is on the boundary.

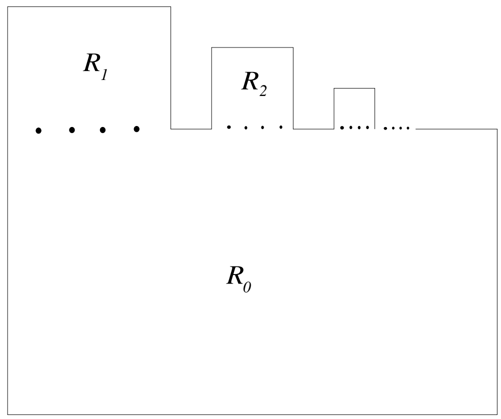

FIGURE 1. A non-Hölder LP domain.

It is easy to see that such a curve can be constructed with the property that for every $\zeta \in \partial\Omega$ there exists an open sector $S_\zeta \subset \Omega$ with vertex at ζ, and with a fixed opening and height. (Of course, if we supposed the same condition at every point of Ω, instead of just on the boundary, then Ω would be a John domain.) Furthermore, this can be done in such a way that Ω is *not* LP. One simply constructs a curve which is nice on one "side" but has a cusp on the other. If such a domain were to be LP then it would have to be a Hölder domain by Theorem 3 in [**21**]. But Ω would have a cusp at one of its prime ends, and hence cannot be a Hölder domain, by Corollary 6 [**18**].

For Example 2 (see Figure 1 above), we start with a simply connected John domain which is an LP domain by Theorem 1. Then, we remove an infinite discrete sequence of interior points so that Ω is no longer a Hölder domain. We force the quasihyperbolic distance between the center of the *room* R_n and the center of the fixed room R_0 below to be too large compared to the diameter of the room R_n $\delta_\Omega(z_n)$ where z_n is the center for (1.2) not to hold. On the other hand, the discrete sequence is a removable set for any positive superharmonic function on Ω, see for example [**11**, Chapter 7]. Hence Ω is an LP domain.

While Figure 2 gives a very simple example of an LP domain which does not satisfy the Hölder condition, there are at least two natural questions left unanswered by this example: First, Example 2 leaves open the question of the existence of a non-Hölder LP domain which is

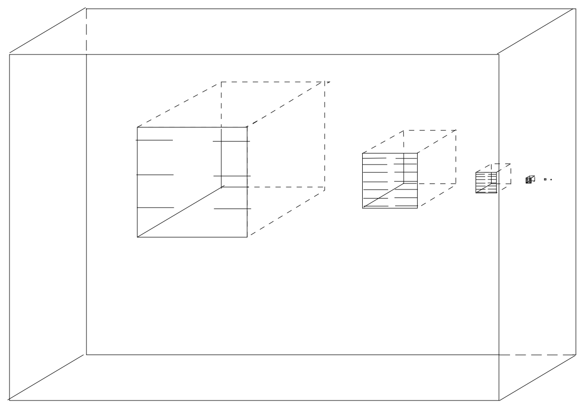

FIGURE 2. A non-Hölder LP domain in space which is topologically a ball.

regular for the Dirichlet problem. Second, the proof that the domain in Example 2 is LP uses the curious fact that there exists a Hölder domain Ω_1 such that every positive superharmonic function in Ω extends to another such function in $\vee_1$; thus it might be the case that every LP domain is Hölder "modulo a removable set," as in Example 2.

We shall see in a moment that Example 2 can be modified to give a non-Hölder LP domain which is regular for the Dirichlet problem. We wish to point out that such an example *cannot* be obtained by simply starting with a Hölder domain and removing a "removable" set, as was done in Example 2; that is to say that an example answering the first of the two objections to Example 2 mentioned above must necessarily answer the second as well. In fact, it is clear that if K is a compact subset of a bounded open set R and $R\backslash K$ is regular for the Dirichlet problem, then there exists a positive superharmonic function in $R\backslash K$ which does not extend to a superharmonic function in R.

It is nonetheless very easy to modify Example 2 to obtain a non-Hölder LP domain which is also regular for the Dirichlet problem: We obtain Example 3 by beginning with the domain constructed in Example 2 and removing a countable family of pairwise disjoint horizontal line segments, with one segment centered at each of the points which were deleted in Example 2. The resulting domain is regular for the Dirichlet problem because every component of the boundary contains more than one point, and it is clear that the Hölder condition fails, exactly as in Example 2.

It is also clear that if the removed segments are too long (so that the distance between two adjacent segments is much shorter than the length of the segment in question) then the rectangles R_n will be almost separated from R_0, so that the domain will *not* be LP. (One might consider this a "room with Venetian blinds" example, in analogy with the traditional "rooms with corridors" examples in potential theory).

On the other hand, it seems clear that the fact that we have removed segments instead of points should have no effect on the (global) integrability of superharmonic functions, *if* the removed segments are short enough, and in fact Example 3 will be LP in this case: For $n \geq 1$, let Q_n denote the reflection of R_n in the upper edge of R_0. Proposition 2 below will show that

$$\int_{R_n} u^p\,dx \leq M \int_{Q_n} u^p\,dx, \quad n = 1, 2, \dots,$$

if the removed segments are short enough, so that

$$\sum_{n=0}^{\infty} \int_{R_n} u^p\,dx \leq (1+M) \int_{R_0} u^p\,dx < \infty\,.$$

We will need some notation: We define $S = \{x + iy \in \mathbf{C} : |x| < 1, |y| < 2\}$, and if F is a finite subset of the open interval $(-1, 1)$ and $\delta > 0$ we define $K(F, \delta) = \cup_{x\in F}[x - \delta, x + \delta]$. Note that if F is fixed then $K(F, \delta) \subset S$ for all sufficiently small δ; for each such δ we set $R = R(F, \delta) = S \setminus K(F, \delta)$. Finally, $R_+ = \{x + iy \in S : y > 0\}$ and $R_- = \{x + iy \in S : y < 0\}$.

Proposition 2. *There exist constants $M < \infty$ and $p > 0$ with the following property. If F is any finite subset of $(-1, 1)$ then there exists $\delta_F > 0$ such that*

$$\int_{R_+} u^p\,dx \leq M \int_{R_-} u^p\,dx$$

whenever $0 < \delta < \delta_F$ and $u > 0$ is superharmonic in $R = R(F, \delta)$.

Proof. Let D_+ and D_- denote the closed discs of radius $1/2$ with centers i and $-i$. Suppose $u \geq 0$ is superharmonic in $R(F, \delta)$ and set

$t = \min_{z\in D_+} u(z)$. Now a square is certainly a John domain, so that Theorem 1 shows that there exists $p > 0$ and $M_1 < \infty$ with

$$\int_{R_+} u^p \, dx \leq M_1 t^p \, .$$

Thus we are done if we can demonstrate the existence of a constant $c > 0$ (depending on F) such that $u(z) \geq ct$ for all $z \in D_-$, whenever δ is small enough. This follows by a harmonic measure argument:

Given a domain Ω, regular for the Dirichlet problem, and a Borel set $E \subset \partial\Omega$, the notation $\omega(z, E, \Omega)$ will refer to the harmonic measure of E relative to Ω. We define $\omega(z) = \omega(z, \partial D_+, R \setminus D+)$. If $z \in \partial D_+$ then $u(z) \geq t\omega(z)$; since u is superharmonic it follows that $u \geq t\omega$ in $R \setminus D_+$.

Thus we need only show that there exists a constant $c > 0$ such that if δ is small enough then

$$\omega(z) \geq c, \qquad z \in D_- \, . \tag{4.5}$$

This seems quite clear if we interpret $\omega(z)$ as the probability that the first place a Brownian path starting at z hits the boundary of $R \setminus D_+$ is at a point of ∂D_+; surely our Brownian traveler will most likely not notice that K is there, if δ is small enough! For the skeptical reader we include a sketch of a proof without Brownian motion:

Let $F = \{x_1, \dots, x_N\}$, and denote $I_j = [x_j - \delta, x_j + \delta]$, so that $K = \cup_{j=1}^N I_j$. The fact that $\omega(z, E, \Omega)$ increases with Ω shows that

$$\begin{aligned} \omega(z) &= 1 - \omega(z, \partial S, R \setminus D+) - \sum_{j=1}^N \omega(z, I_j, R \setminus D+) \\ &\geq 1 - \omega(z, \partial S, S \setminus D+) - \sum_{j=1}^N \omega(z, I_j, S \setminus I_j) \, . \end{aligned}$$

Note that the term $\omega(z, \partial S, S \setminus D+)$ does not depend on δ; certainly

$$\sup_{z\in D_-} \omega(z, \partial S, S \setminus D+) = \gamma < 1,$$

since D_- is a compact subset of $S \setminus D_+$. And it is clear that for each j we have $\lim_{\delta\to 0} \omega(z, I_j, S \setminus I_j) = 0$ uniformly for $z \in D_-$ (it is easy to give explicit examples of subharmonic functions in the unit disc showing this); this gives (4.5) with $c = (1-\gamma)/2$, if δ is sufficiently small.

Finally, our last example is a domain in R^3. As noted in the introduction, a simply connected planar domain is LP if and only if it is Hölder. Now we have proved that a Hölder domain in 3-space is LP and hence it is natural to ask whether the converse holds for domains which are topologically equivalent to a ball in R^3. This might seem plausible due to the result of Gehring and Martio which proves that the Hölder continuous *quasiconformal* images of the ball are precisely the Hölder domains which are homeomorphic to a ball. However, a small modification of LP domain which is topologically equivalent to the ball and yet is not a Hölder domain. We start with a cube and then add an infinite sequence of smaller cubes to one side (which is depicted in the figure below). So far we have a John domain and hence it is LP. Finally, we remove an infinite sequence of intervals so as to violate the Hölder condition. Since a line segment is a polar set it is removable for the class of positive superharmonic functions, see Chapter 7 of Helms' book [**11**]. Hence, the domain is LP since the family of segments is a removable set (see [**11**, Theorem 7.6]) and clearly it is homeomorphic to a ball.

Another, perhaps simpler, example is to remove countably many segments of various lengths from the unit cube. The resulting set will be a cube with one side cluttered with "spikes" forming a "bed of nails." This example can be constructed in such a way as to be a non-Hölder domain which is homeomorphic to a ball and equal to a cube minus a removable set.

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Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822
E-mail address: davids@uhunix.uhcc.hawaii.edu

Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74078
E-mail address: ullrich@hardy.math.okstate.edu