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RESONANT SINGULAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. Existence theory is developed for the "resonant" singular problem $(1/(pq))(py')' + \lambda_0 y = f(t, y, py')$ almost everywhere on $[0,1]$ with $\lim_{t\to 0^+} p(t)y'(t) = ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0$. Here λ_0 is the first eigenvalue of $(1/(pq))(pu')' + \lambda u = 0$ almost everywhere on $[0,1]$ with $\lim_{t\to 0^+} p(t)u'(t) = au(1) + b\lim_{t\to 1^-} p(t)u'(t) = 0$. We do *not* assume $\int_0^1 ds/p(s) < \infty$ in this paper.

1. Introduction. This paper presents existence results for the second order singular "resonant" boundary value problem

$$(1.1)\qquad \begin{cases} \frac{1}{pq}(py')' + \lambda_0 y = f(t,y,py'), & \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, & a > 0,\ b \ge 0 \end{cases}$$

where λ_0 is the *first* eigenvalue (described in more detail later) of

$$(1.2)\qquad \begin{cases} Lu = \lambda u, & \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)u'(t) = 0 \\ au(1) + b\lim_{t\to 1^-} p(t)u'(t) = 0, & a > 0,\ b \ge 0 \end{cases}$$

with $Lu = -(1/(pq))(pu')'$.

Throughout the paper $p \in C[0,1] \cap C^1(0,1)$ together with $p > 0$ on $(0,1)$; also q is measurable with $q > 0$ almost everywhere on $[0,1]$ and $\int_0^1 p(x)q(x)\,dx < \infty$.

Remark. We do *not* assume $\int_0^1 ds/p(s) < \infty$ in this paper.

Also $pqf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is an L^1-Caratheodory function. By this, we mean:

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Definition 1.1. (i) $t \to p(t)q(t)f(t,y,v)$ is measurable for all $(y,v) \in \mathbf{R}^2$.

(ii) $(y,v) \to p(t)q(t)f(t,y,v)$ is continuous for almost every $t \in [0,1]$;

(iii) for any $r > 0$ there exists $h_r \in L^1[0,1]$ such that $|p(t)q(t)f(t,y,v)| \le h_r(t)$ for almost every $t \in [0,1]$ and for all $|y| \le r$, $|v| \le r$.

For notational purposes, let w be a weight function. By $L^1_w[0,1]$ we mean the space of functions u such that $\int_0^1 w(t)|u(t)|\,dt < \infty$. $L^2_w[0,1]$ denotes the space of functions u such that $\int_0^1 w(t)|u(t)|^2\,dt < \infty$; also for $u, v \in L^2_w[0,1]$ define $\langle u, v\rangle = \int_0^1 w(t)u(t)\overline{v(t)}\,dt$. Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

We now state an existence principle [**5, 11**], which was established using fixed point methods.

Theorem 1.1. *Suppose that $pqf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is an L^1-Cartheodory function with*

$$p \in C[0,1] \cap C^1(0,1) \quad \textit{with } p > 0 \quad \textit{on } (0,1) \tag{1.3}$$

$$q \in L^1_p[0,1] \quad \textit{with } q > 0 \quad \textit{a.e. on } (0,1) \tag{1.4}$$

and

$$\begin{aligned} &\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\,dx\,ds < \infty \textit{ and} \\ &\int_0^1 \frac{1}{p(s)} \int_0^s h_r(x)\,dx\,ds < \infty \textit{ for any } r > 0; \\ &\textit{here } h_r \textit{ is as described in Definition } 1.1. \end{aligned} \tag{1.5}$$

In addition, assume that there is a constant M_0, independent of λ, with

$$||y||_* = \max\{\sup_{[0,1]} |y(t)|, \sup_{(0,1)} |p(t)y'(t)|\} \le M_0$$

for any solution y (here $y \in C[0,1] \cap C^1(0,1)$ with $py' \in AC[0,1]$) to

$$\begin{cases} \frac{1}{pq}(py')' = \lambda[f(t,y,py') - \lambda_0 y] \quad \textit{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, \qquad a > 0,\ b \ge 0 \end{cases} \tag{1.6$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then* (1.1) *has at least one solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $py' \in AC[0,1]$.

Next we gather together some results on the singular eigenvalue problem (1.2). Assume (1.3), (1.4) and

$$\int_0^1 \frac{1}{p(s)}\left(\int_0^s p(x)q(x)\,dx\right)^{1/2} ds < \infty \tag{1.7}$$

hold.

Remarks. (i) Notice [**11, 13**] that $\int_{1/2}^1 ds/p(s) < \infty$.

(ii) Notice [**11, 13**] that (1.7) implies $\int_0^1 p(s)q(s)(\int_s^1 dx/p(x))^2\,ds < \infty$.

(iii) Now $t = 0$ is a singular point in the limit circle case [**11, 13, 16**].

(iv) If $p(t) = t^{n-1}$, $n \geq 0$ and $q \equiv 1$, then (1.7) is satisfied if $n < 4$.

Let

$$D(L) = \biggl\{ w \in C[0,1] : w, pw' \in AC[0,1] \text{ with } \frac{1}{pq}(pw')' \in L^2_{pq}[0,1] \\ \text{and } \lim_{t\to 0^+} p(t)w'(t) = aw(1) + b \lim_{t\to 1^-} p(t)w'(t) = 0 \biggr\}.$$

In [**11, 13**] it was shown that $L^{-1} : L^2_{pq}[0,1] \to D(L)$ and L^{-1} is completely continuous with $\langle L^{-1}u, v\rangle = \langle u, L^{-1}v\rangle$ for $u, v \in L^2_{pq}[0,1]$. Consequently, the spectral theorem for compact self-adjoint operators [**16**] implies that L has a countably infinite number of real eigenvalues λ_i with corresponding eigenfunctions $\psi_i \in D(L)$. The eigenfunctions ψ_i may be chosen so that they form an orthonormal set and we may also arrange the eigenvalues so that

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

In addition [**16**], the set of eigenfunctions ψ_i forms a basis for $L^2_{pq}[0,1]$ and if $h \in L^2_{pq}[0,1]$ then h has a Fourier series representation and h satisfies Parseval's equality, i.e.,

$$h = \sum_{i=0}^{\infty} \langle h, \psi_i\rangle \psi_i \quad \text{and} \quad \int_0^1 pq|h|^2\,dt = \sum_{i=0}^{\infty} |\langle h, \psi_i\rangle|^2. \tag{1.8}$$

Also, we have a Rayleigh-Ritz minimization theorem:

Theorem 1.2. *Suppose* (1.3), (1.4), *and* (1.7) *hold. Then*

$$\lambda_0 \int_0^1 p(t)q(t)[y(t)]^2\,dt \le \int_0^1 p(t)[y'(t)]^2\,dt + \frac{a}{b}[y(1)]^2$$

for all functions $y \in D(L)$.

Remark. In fact notice Theorem 1.2 holds for all $y \in AC[0,1]$, $lim_{t\to 0^+} p(t)y'(t) = ay(1) + \lim_{t\to 1^-} p(t)y'(t) = 0$ with $y' \in L_p^2[0,1]$ and $py' \in AC[0,1]$.

Proof. Notice for $u \in D(L)$, we have

$$\langle Lu, u\rangle = \int_0^1 p(t)[u'(t)]^2\,dt + \frac{a}{b}[u(1)]^2.$$

From (1.8) any $u \in D(L)$ has a Fourier series representation so

$$\begin{aligned}\langle Lu, u\rangle &= \sum_{i=0}^{\infty} \lambda_i |\langle u, \psi_i\rangle|^2 \\ &\ge \lambda_0 \sum_{i=0}^{\infty} |\langle u, \psi_i\rangle|^2 \\ &= \lambda_0 \int_0^1 p(t)q(t)[u(t)]^2\,dt.\end{aligned}$$

Consequently, $\langle Lu, u\rangle \ge \lambda_0 \int_0^1 pqu^2\,dt$ with equality if $u = \psi_0$. □

In recent years several authors [**2, 3, 6–9, 14, 15**] have examined the nonsingular (usually when $p \equiv q \equiv 1$) resonant second order boundary value problem. However, very little is known concerning the resonant singular case; this paper is devoted to the study of such problems.

2. Existence theory. Existence theory is developed for the second order boundary value problem

$$\begin{cases} \frac{1}{pq}(py')' + \lambda_0 y = f(t, y, py') \quad \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, \qquad a > 0,\ b \ge 0 \end{cases} \tag{2.1}$$

where λ_0 is the first eigenvalue of (1.2).

Throughout this section, let

$$H_{\alpha,\theta}(u_1) = \begin{cases} |u_1|^{\theta+1}, & |u_1| \le 1 \\ |u_1|^{\alpha+1}, & |u_1| > 1. \end{cases}$$

Theorem 2.1. *Let $pqf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be an L^1-Carathéodory function with*

$$p \in C[0,1] \cap C^1(0,1) \quad \textit{with } p > 0 \quad \textit{on } (0,1) \tag{2.2}$$

$$q \in L^1_p[0,1] \quad \textit{with } q > 0 \quad \textit{a.e. on } (0,1) \tag{2.3}$$

and

$$\int_0^1 \frac{1}{p(s)} \bigg(\int_0^s p(x)q(x)\,dx \bigg)^{1/2} ds < \infty \tag{2.4}$$

holding. Also, suppose $f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2)$ with $pqg, pqh : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ L^1-Caratheodory functions and there exist constants

$$\begin{aligned} &A > 0,\ 0 < \alpha < 1 \quad \textit{with } u_1 g(t,u_1,u_2) \ge A H_{\alpha,\theta}(u_1) \\ &\textit{for } t \in [0,1], \quad u_1 \in \mathbf{R},\ u_2 \in \mathbf{R}; \quad \textit{here } \alpha \ge \theta; \end{aligned} \tag{2.5}$$

there exist

$$\begin{aligned} &\phi_i \in L^1_{pq}[0,1],\ i = 1,2,3 \quad \textit{and constants } \beta \textit{ and } \sigma \textit{ with} \\ &|h(t,u_1,u_2)| \le \phi_1(t) + \phi_2(t)|u_1|^\beta + \phi_3(t)|u_2|^\sigma \\ &\textit{for a.e. } t \in [0,1]; \quad \textit{here } \beta < \alpha \quad \textit{and} \quad \phi_3 > 0 \quad \textit{a.e.} \\ &\textit{on } [0,1] \quad \textit{or } \phi_3 \equiv 0 \quad \textit{on } [0,1]; \end{aligned} \tag{2.6}$$

there exist

$$\begin{aligned} &\phi_i \in L^1_{pq}[0,1],\ i = 4,5,6 \quad \textit{and constants } \gamma \le \alpha, \tau > \sigma \textit{ with} \\ &|g(t,u_1,u_2)| \le \phi_4(t) + \phi_5(t)|u_1|^\gamma + \phi_6(t)|u_2|^\tau \\ &\textit{for a.e. } t \in [0,1]; \quad \textit{here } \phi_6 > 0 \quad \textit{a.e.} \\ &\textit{on } [0,1] \quad \textit{or } \phi_6 \equiv 0 \textit{ on } [0,1]; \end{aligned} \tag{2.7}$$

$$\sigma < \min\{\alpha/\gamma, \alpha\} \quad \text{and} \quad \tau < 1 \tag{2.8}$$

$$\begin{gathered}\phi_1^{(\alpha+1)/\alpha} \in L^1_{pq}[0,1], \quad \phi_2^{(\alpha+1)/(\alpha-\beta)} \in L^1_{pq}[0,1], \\ \phi_5^{(\alpha+1)/(\alpha+1-\gamma)} \in L^1_{pq}[0,1];\end{gathered} \tag{2.9}$$

$$\begin{gathered}\phi_3^{(\alpha+1)/\alpha} \in L^1_{pq}[0,1], \quad \phi_6^{(\alpha+1)/\alpha} \in L^1_{pq}[0,1], \\ \phi_3^{((\alpha+1)\tau)/(\alpha(\tau-\sigma))} \phi_3^{-((\alpha+1)\sigma)/(\alpha(\tau-\sigma))} \in L^1_{pq}[0,1];\end{gathered} \tag{2.10}$$

and

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)\phi_i(x)\,dx\,ds < \infty, \qquad i = 1, \dots, 6 \tag{2.11}$$

holding. Then (2.1) *has at least one solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $py' \in AC[0,1]$.

Remark. Typical examples where (2.5) is satisfied are, say, (i) $g(t, u_1, u_2) = u_1^{m/n}$, m odd and n odd or (ii) $g(t, u_1, u_2) = u_1^{1/2}$, $u_1 \geq 0$ with $g(t, u_1, u_2) = -|u_1|^{1/2}$, $u_1 < 0$.

Proof. Let y be a solution to

$$\begin{cases} (1/pq)(py')' = \lambda[f(t, y, py') - \lambda_0 y] & \text{a.e. on } [0,1] \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, & a > 0,\ b \geq 0 \end{cases} \tag{2.12$_\lambda$}$$

for $0 < \lambda < 1$. Multiply the differential equation in $(2.12)_\lambda$ by $-y$ and integrate from 0 to 1 to obtain

$$\frac{a}{b}[y(1)]^2 + \int_0^1 p(t)[y'(t)]^2\,dt = \lambda\lambda_0 \int_0^1 pqy^2\,dt - \lambda \int_0^1 pqyf(t, y, py')\,dt.$$

This, together with Theorem 1.2, implies

$$\lambda \int_0^1 pqyg(t, y, py')\,dt \leq \lambda \int_0^1 pqyh(t, y, py')\,dt$$

and so

$$\begin{aligned}\int_0^1 pqyg(t,y,py')\,dt \le \int_0^1 pq\phi_1|y|\,dt \\ + \int_0^1 pq\phi_2|y|^{\beta+1}\,dt \\ + \int_0^1 pq\phi_3|y|\,|py'|^{\sigma}\,dt.\end{aligned} \tag{2.13}$$

In addition, (2.5) yields

$$\begin{aligned}\int_0^1 pqyg(t,y,py')\,dt &\ge A\int_0^1 pqH_{\alpha,\theta}(y)\,dt \\ &= A\int_0^1 pq|y|^{\alpha+1}\,dt \\ &\quad + A\int_{\{t:|y(t)|\le 1\}} pq[|y|^{\theta+1} - |y|^{\alpha+1}]\,dt \\ &\ge A\int_0^1 pq|y|^{\alpha+1}\,dt - A\int_0^1 pq\,dt\end{aligned}$$

and this together with (2.13) yields

$$\begin{aligned}A\int_0^1 pq|y|^{\alpha+1}\,dt \le A\int_0^1 pq\,dt + \int_0^1 pq\phi_1|y|\,dt \\ + \int_0^1 pq\phi_2|y|^{\beta+1}\,dt + \int_0^1 pq\phi_3|y|\,|py'|^{\sigma}\,dt.\end{aligned} \tag{2.14}$$

Holder's inequality together with (2.9) implies

$$\begin{aligned}\int_0^1 pq\phi_1|y|\,dt &\le Q_1\left(\int_0^1 pq|y|^{\alpha+1}\,dt\right)^{1/(\alpha+1)}; \\ \int_0^1 pq\phi_2|y|^{\beta+1}\,dt &\le Q_2\left(\int_0^1 pq|y|^{\alpha+1}\,dt\right)^{(\beta+1)/(\alpha+1)}; \\ \int_0^1 pq\phi_3|y|\,|py'|^{\sigma}\,dt &\le \left(\int_0^1 pq|y|^{\alpha+1}\,dt\right)^{1/(\alpha+1)} \\ &\quad \times \left(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\right)^{\alpha/(\alpha+1)}\end{aligned}$$

for some constants Q_1 and Q_2. Thus,

(2.15)
$$\begin{aligned} A\int_0^1 pq|y|^{\alpha+1}\,dt \le A\int_0^1 pq\,dt &+ Q_1\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{1/(\alpha+1)} \\ &+ Q_2\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{(\beta+1)/(\alpha+1)} \\ &+ \bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{1/(\alpha+1)} \\ &\times \bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)}. \end{aligned}$$

We now consider two cases $\int_0^1 pq|y|^{\alpha+1}\,dt > 1$ and $\int_0^1 pq|y|^{\alpha+1}\,dt \le 1$ separately.

Case (i). $\int_0^1 pq|y|^{\alpha+1}\,dt > 1$.

Divide (2.15) by $(\int_0^1 pq|y|^{\alpha+1}\,dt)^{1/(\alpha+1)}$ and use $\int_0^1 pq|y|^{\alpha+1}\,dt > 1$ to obtain

$$\begin{aligned} A\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{\alpha/(\alpha+1)} \le Q_3 &+ Q_2\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{\beta/(\alpha+1)} \\ &+ \bigg(\int_0^1 pq\phi_3^{(\alpha+1)/3}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)} \end{aligned}$$

for some constant Q_3. Since $\beta < \alpha$ there exist constants Q_4 and Q_5 with

$$\int_0^1 pq|y|^{\alpha+1}\,dt \le Q_4 + Q_5\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt. \tag{2.16}$$

Case (ii). $\int_0^1 pq|y|^{\alpha+1}\,dt \le 1$.

In this case (2.16) is clearly true with $Q_4 = 1$ and $Q_5 = 0$.

Consequently in all cases (2.16) is true. Returning to $(2.12)_\lambda$ we have

$$p(t)y'(t) = \lambda\int_0^t p(s)q(s)[f(s,y(s),p(s)y'(s)) - \lambda_0 y(s)]\,ds.$$

Thus for $t \in (0,1)$, we have using (2.6) and (2.7) that

$$\begin{aligned}|p(t)y'(t)| \le \int_0^1 pq\phi_1\,ds &+ \int_0^1 pq\phi_1|y|^\beta\,ds\\ &+ \int_0^1 pq\phi_3|py'|^\sigma\,ds + \int_0^1 pq\phi_4\,ds\\ &+ \int_0^1 pq\phi_5|y|^\gamma\,ds + \int_0^1 pq\phi_6|py'|^\tau\,ds + \lambda_0\int_0^1 pq|y|\,ds.\end{aligned}$$

Holder's inequality, together with (2.9) and (2.10), implies

$$\begin{aligned}|p(t)y'(t)| \le Q_6 &+ Q_7\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{\beta/(\alpha+1)}\\ &+ Q_8\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)}\\ &+ Q_9\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{\gamma/(\alpha+1)}\\ &+ Q_{10}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)}\\ &+ Q_{11}\bigg(\int_0^1 pq|y|^{\alpha+1}\,dt\bigg)^{1/(\alpha+1)}\end{aligned}$$

for some constants $Q_6,\dots,Q_{11}$. This, together with (2.16), implies for $t \in (0,1)$ that

$$\begin{aligned}|p(t)y'(t)| \le Q_{12} &+ Q_{13}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\beta/(\alpha+1)}\\ &+ Q_8\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)}\\ &+ Q_{14}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\gamma/(\alpha+1)}\\ &+ Q_{10}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\alpha/(\alpha+1)}\\ &+ Q_{15}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{1/(\alpha+1)}\end{aligned}\tag{2.17}$$

for some constants $Q_{12}, \dots, Q_{15}$. There are two cases to consider, namely $\phi_6 > 0$ almost everywhere on $[0,1]$ or $\phi_6 \equiv 0$ on $[0,1]$.

Case (i). $\phi_6 > 0$ almost everywhere on $[0,1]$.

Now (2.17) implies

$$\begin{aligned}\int_0^1 & pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt \\ &\le Q_{16} + Q_{17}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{(\sigma(\alpha+1))/\alpha}\,dt\bigg)^{\tau\beta/\alpha} \\ &\quad + Q_{18}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\tau} \\ &\quad + Q_{19}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\tau\gamma/\alpha} \\ &\quad + Q_{20}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\tau} \\ &\quad + Q_{21}\bigg(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\bigg)^{\tau/\alpha}\end{aligned}$$

for some constants $Q_{16}, \dots, Q_{21}$. Hölder's inequality there is a constant Q_{22} with

$$\begin{aligned}(2.18)\qquad \int_0^1 & pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt \\ &\le Q_{22}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\sigma/\tau}\end{aligned}$$

and putting this into the above inequality yields

$$\begin{aligned}\int_0^1 & pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt \\ &\le Q_{23} + Q_{24}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\sigma\beta/\alpha} \\ &\quad + Q_{25}\bigg(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\bigg)^{\sigma}\end{aligned}$$

$$+ Q_{26}\left(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\right)^{\sigma\gamma/\alpha}$$
$$+ Q_{27}\left(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\right)^{\tau}$$
$$+ Q_{28}\left(\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/\alpha}\,dt\right)^{\sigma/\alpha}$$

for some constants $Q_{23},\dots,Q_{28}$. Since $\max\{\sigma\beta/\alpha,\sigma,\sigma\gamma/\alpha,\tau,\sigma/\alpha\} < 1$, there exists a constant Q_{29} with

$$\int_0^1 pq\phi_6^{(\alpha+1)/\alpha}|py'|^{\tau(\alpha+1)/a}\,dt \le Q_{29}$$

and this together with (2.18), (2.17) and (2.16) imply that there are constants Q_{30} and Q_{31} with

$$|py'|_0 = \sup_{(0,1)}|p(t)y'(t)| \le Q_{30} \tag{2.19}$$

and

$$\int_0^1 pq|y|^{\alpha+1}\,dt \le Q_{31}. \tag{2.20}$$

Case (ii). $\phi_6 \equiv 0$ on $[0,1]$.

We may assume without loss of generality that $\sigma > 0$ and $\phi_3 > 0$ almost everywhere on $[0,1]$. Then (2.17) implies

$$\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt$$
$$\le Q_{32} + Q_{33}\left(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\right)^{\sigma\beta/\alpha}$$
$$+ Q_{34}\left(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\right)^{\sigma}$$
$$+ Q_{35}\left(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\right)^{\sigma\gamma/\alpha}$$
$$+ Q_{36}\left(\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt\right)^{\sigma/\alpha}$$

for some constants $Q_{32}, \dots, Q_{36}$. Thus there exists a constant Q_{37} with

$$\int_0^1 pq\phi_3^{(\alpha+1)/\alpha}|py'|^{\sigma(\alpha+1)/\alpha}\,dt \le Q_{37}$$

and once again (2.19) and (2.20) follow.

Thus in all cases (2.19) and (2.20) are true. Also $(2.12)_\lambda$ yields

$$\begin{aligned} y(t) = &-\frac{b}{a}\int_0^1 \lambda pq[f(x,y,py')-\lambda_0 y]\,dx \\ &-\int_t^1 \frac{1}{p(s)}\int_0^s \lambda pq[f(x,y,py')-\lambda_0 y]\,dx\,ds \end{aligned} \tag{2.21}$$

and this together with (2.6), (2.7) and (2.19) yields

$$\begin{aligned} \int_0^1 pq|y|^2\,dt \le &\left(2\left(\frac{b}{a}\right)^2\int_0^1 pq\,dx + 2\int_0^1 p(t)q(t)\left(\int_t^1 \frac{ds}{p(s)}\right)^2 dt\right) \\ &\times\bigg[\int_0^1 pq[\phi_1+\phi_2|y|^\beta+\phi_3 Q_{30}^\sigma \\ &\qquad + \phi_4+\phi_5|y|^\gamma+\phi_6 Q_{30}^\tau+\lambda_0|y|]\,dx\bigg]^2 \end{aligned}$$

so there exist constants $Q_{38}, \dots, Q_{44}$ with

$$\begin{aligned} \int_0^1 pq|y|^2\,dt &\le Q_{38} + Q_{39}\left(\int_0^1 pq\phi_2|y|^\beta\,dx\right)^2 \\ &\quad + Q_{40}\left(\int_0^1 pq\phi_5|y|^\gamma\,dx\right)^2 + Q_{41}\left(\int_0^1 pq|y|\,dx\right)^2 \\ &\le Q_{38} + Q_{42}\left(\int_0^1 pq|y|^{\alpha+1}\,dx\right)^{2\beta/(\alpha+1)} \\ &\quad + Q_{43}\left(\int_0^1 pq|y|^{\alpha+1}\,dx\right)^{2\gamma/(\alpha+1)} \\ &\quad + Q_{44}\left(\int_0^1 pq|y|^{\alpha+1}\,dx\right)^{2/(\alpha+1)}. \end{aligned}$$

This together with (2.20) implies that there exists a constant Q_{45} with

$$\int_0^1 pq|y|^2\,dt \le Q_{45}. \tag{2.22}$$

Returning to (2.21) again we obtain for $t \in [0,1]$ that

$$\begin{aligned}|y(t)| \le \frac{|b|}{|a|}\int_0^1 & pq[\phi_1 + \phi_2|y|^\beta + \phi_3 Q_{30}^\sigma + \phi_4 \\ &\qquad + \phi_5|y|^\gamma + \phi_6 Q_{30}^\tau + \lambda_0|y|]\,dx \\ &+ \int_0^1 \frac{1}{p(s)}\int_0^s pq[\phi_1 + \phi_2|y|^\beta + \phi_3 Q_{30}^\sigma \\ &\qquad + \phi_4 + \phi_5|y|^\gamma + \phi_6 Q_{30}^\tau + \lambda_0|y|]\,dx\,ds.\end{aligned}$$

Let $|y|_0 = \sup_{[0,1]}|y(t)|$ so the above inequality yields

$$\begin{aligned}|y|_0 &\le Q_{46} + Q_{47}|y|_0^\beta + Q_{48}|y|_0^\gamma + Q_{49}\int_0^1 pq|y|\,dx \\ &\quad + Q_{50}\int_0^1 \frac{1}{p(s)}\int_0^s pq|y|\,dx\,ds \\ &\le Q_{46} + Q_{47}|y|_0^\beta + Q_{48}|y|_0^\gamma + Q_{49}\left(\int_0^1 pq\,dx\right)^{1/2}\left(\int_0^1 pq|y|^2\,dx\right)^{1/2} \\ &\quad + Q_{50}\left(\int_0^1 pq|y|^2\,dx\right)^{1/2}\int_0^1 \frac{1}{p(s)}\left(\int_0^s pq\,dx\right)^{1/2} ds\end{aligned}$$

for some constants $Q_{46}, \dots, Q_{50}$. This together with (2.22) implies that there is a constant Q_{51} with

$$|y|_0 \le Q_{51} + Q_{47}|y|_0^\beta + Q_{48}|y|_0^\gamma.$$

Since $0 \le \beta,\ \gamma < 1$ there is a constant Q_{52} with

$$|y|_0 \le Q_{52}. \tag{2.23}$$

Now (2.19), (2.23) together with Theorem 1.1 establishes the existence of a solution to (2.1). □

The next theorem establishes the existence of a nonnegative solution to

$$(2.24)\qquad \begin{cases} (1/pq)(py')' + \lambda_0 y = \psi(t)f(t,y,py'), & 0<t<1 \\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, & a>0,\ b\ge 0 \end{cases}$$

where λ_0 is the first eigenvalue of (1.2). Let

$$(2.25)\qquad q \in L^1_p[0,1] \quad \text{with } q>0 \quad \text{on } (0,1)$$

and

$$(2.26)\qquad \psi \in L^1_{pq}[0,1] \quad \text{with } \psi>0 \quad \text{on } (0,1).$$

Let

$$H^*_{\alpha,\theta}(u_1) = \begin{cases} u_1^{\theta+1}, & 0\le u_1\le 1 \\ u_1^{\alpha+1}, & 1<u_1<\infty. \end{cases}$$

Theorem 2.2. *Let* $f:[0,1]\times \mathbf{R}^2 \to \mathbf{R}$ *be continuous with* (2.2), (2.4), (2.25), (2.26) *and*

$$(2.27)\qquad f(t,0,0)\le 0$$

holding. Suppose $\psi(t)f(t,u_1,u_2) = g(t,u_1,u_2) + h(t,u_1,u_2)$ *with* pqg, $pqh : [0,1]\times\mathbf{R}^2 \to \mathbf{R}$ L^1*-Caratheodory functions and there exist constants*

$$(2.28)\qquad \begin{aligned} &A>0, 0<\alpha<1 \quad \textit{with } u_1 g(t,u_1,u_2) \ge AH^*_{\alpha,\theta}(u_1) \\ &\textit{for } t\in(0,1), u_1\ge 0 \quad \textit{and } u_2\in\mathbf{R}; \quad \textit{here } \alpha\ge\theta \end{aligned}$$

there exist

$$(2.29)\qquad \begin{aligned} &\phi_i \in L^1_{pq}[0,1], i=1,2,3, \quad \textit{and constants } \beta \textit{ and } \sigma \textit{ with} \\ &|h(t,u_1,u_2)| \le \phi_1(t) + \phi_2(t)u_1^\beta + \phi_3(t)|u_2|^\sigma \\ &\textit{for } t\in(0,1),\ u_1\ge 0 \quad \textit{and } u_2\in\mathbf{R}; \quad \textit{here } \beta<\alpha \textit{ and} \\ &\phi_3>0 \textit{ a.e.} \quad \textit{on } [0,1] \quad \textit{or } \phi_3\equiv 0 \end{aligned}$$

and there exist

$$(2.30)\qquad \begin{aligned}&\phi_i \in L^1_{pq}[0,1], i = 4,5,6 \quad \textit{and constants } \gamma \le \alpha, \tau > \sigma \textit{ with}\\ &|g(t,u_1,u_2)| \le \phi_4(t) + \phi_5(t)u_1^\gamma + \phi_6(t)|u_2|^\tau\\ &\textit{for } t \in (0,1),\ u_1 \ge 0 \quad \textit{and } u_2 \in \mathbf{R}; \quad \textit{here } \phi_6 > 0\\ &\textit{a.e. on } [0,1] \textit{ or } \phi_6 \equiv 0\end{aligned}$$

hold. Finally, suppose (2.8), (2.9), (2.10) *and* (2.11) *are satisfied. Then* (2.24) *has at least one nonnegative solution* $y \in C[0,1] \cap C^1(0,1)$ *with* $py' \in AC[0,1]$.

Proof. Consider the family of problems

$$(2.31)_\lambda \qquad \begin{cases} (1/(pq))(py')' = \lambda f^*(t,y,py'), & 0 < t < 1\\ \lim_{t\to 0^+} p(t)y'(t) = 0 \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = 0, & a > 0,\ b \ge 0\end{cases}$$

where $0 < \lambda < 1$ and

$$f^*(t,u_1,u_2) = \begin{cases} \psi(t)f(t,u_1,u_2) - \lambda_0 u_1, & u_1 \ge 0\\ \psi(t)f(t,0,u_2) + u_1, & u_1 < 0.\end{cases}$$

Remark. Notice $pqf^* : [0,1]\times\mathbf{R}^2 \to \mathbf{R}$ is an L^1-Cartheodory function.

Let y be a solution to $(2.31)_\lambda$ for some $0 < \lambda < 1$. We claim that $y \ge 0$ on $[0,1]$. If not, then y would have a negative absolute minimum somewhere on $[0,1]$, say at t_0. If $t_0 \in (0,1)$, then $y'(t_0) = 0$ and this together with the differential equation and (2.27) yields

$$y''(t_0) = \frac{1}{p(t_0)}(p(t_0)y'(t_0))' = \lambda q(t_0)\psi(t_0)f(t_0,0,0) + \lambda q(t_0)y(t_0) < 0,$$

a contradiction. Next suppose the negative absolute minimum were to occur at $t_0 = 0$. Now $f(0,0,0) \le 0$ and this together with the differential equation implies that there exists $\delta > 0$ with $(p(t)y'(t))' < 0$ for $t \in (0,\delta)$. Thus, the boundary condition implies $p(t)y'(t) < 0$ for $t \in (0,\delta)$, a contradiction. It remains to consider the case $t_0 = 1$. Of course, we need only consider $b \ne 0$. Then

$$y(1)\lim_{t\to 1^-} p(t)y'(t) = -\frac{a}{b}y^2(1) < 0,$$

which implies $y^2(t)$ is a decreasing function near 1, a contradiction. Thus, $y \geq 0$ on $[0,1]$ for any solution y to $(2.31)_\lambda$. Consequently, y satisfies

$$\frac{1}{pq}(py')' = \lambda(\psi(t)f(t,y,py') - \lambda_0 y), \qquad 0 < t < 1.$$

Essentially the same reasoning as in Theorem 2.2 (in this case we look at $\int_0^1 pqy^{\alpha+1}\,dt$) guarantees the existence of a solution y to $(2.31)_1$. Of course, y is automatically a solution of (2.24) since $y \geq 0$ on $[0,1]$. □

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