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REGULAR SOBOLEV TYPE ORTHOGONAL POLYNOMIALS: THE BESSEL CASE

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ABSTRACT. In this paper, given a regular linear functional u on the linear space $\mathbf{P}$ of polynomials with real coefficients, we consider the bilinear symmetric form $\varphi(p,q) = \langle u, pq\rangle + \lambda p'(c)q'(c)$ where λ and c are real numbers and $p,\ q \in \mathbf{P}$. A necessary and sufficient condition to warrant the existence of a sequence of orthogonal polynomials with respect to φ is given, and different expressions in terms of the orthogonal polynomials associated to u are studied. Also, we consider the relations between these polynomials and the orthogonal polynomials associated to the linear functional $u_1 = (x-c)^2 u$. Finally, we illustrate these ideas with a nontrivial example, the functional associated to the Bessel polynomials.

1. Introduction. In the last years, a nonstandard class of orthogonal polynomials has attracted considerable attention. The so-called *Sobolev type orthogonal polynomials* (see references [**2, 3, 5, 9, 10**]) are associated to inner products like

$$(p,q)_w = \sum_{k=0}^{N} \int_{\mathbf{R}} p^{(k)}(x) q^{(k)}(x)\, d\mu_k(x)$$

where μ_0 is a finite positive Borel measure and μ_k, $k = 1, \dots, N$ are discrete measures.

In this work, if u is a regular linear functional on the linear space $\mathbf{P}$ of polynomials with real coefficients, we shall consider the bilinear symmetric form

$$\varphi(p,q) = \langle u, pq\rangle + \lambda p'(c)q'(c)$$

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where λ and c are real numbers and $p, q \in \mathbf{P}$. A necessary and sufficient condition to warrant the existence of a sequence $\{Q_n\}_n$ of orthogonal polynomials with respect to φ is given, and different expressions in terms of the orthogonal polynomials associated to u are obtained.

From the definition of φ, we deduce that the shift operator is not self-adjoint for the bilinear form φ and, therefore, the usual properties for the standard orthogonal polynomials are no longer valid. However, the operator associated to the multiplication by the polynomial $(x-c)^2$ is self-adjoint with respect to φ. Consequently, we can obtain a five term recurrence relation for the orthogonal polynomials associated to φ.

The functional $u_1 = (x-c)^2 u$ provides a very interesting interpretation of the Sobolev-type orthogonal polynomials. They are quasi-orthogonal polynomials of order two with respect to u_1, and therefore they can be expressed as a linear combination of three consecutive orthogonal polynomials associated with u_1.

These problems have been considered by several authors in the positive definite case (see Marcellán and Ronveaux [**9**], Bavinck and Meijer [**3**], Alfaro et al. [**2**]).

Finally, we consider the particular case of Bessel polynomials (see [**8**]). These polynomials constitute an interesting example of a nonpositive definite regular functional. We study the Sobolev-type orthogonal polynomials associated with the Bessel functional with $c = 0$. This point has been selected in order to preserve the classical character for the functional u_1. For these polynomials we get the asymptotic behavior for the coefficients of the recurrence relation. Moreover, differential properties for the polynomials are obtained. In particular, we obtain a Rodrigues-type formula and a second-order linear differential equation with polynomial coefficients, with their degrees not depending on n.

These results can be compared with those of Hendriksen (see [**7**]). He essentially studies

$$\langle u, pq\rangle + \lambda p(0)q(0)$$

where u is a regular (nonpositive definite) functional for the simple Bessel polynomials. He derives a second order differential equation for these Bessel type polynomials.

2. Regular Sobolev-type orthogonal polynomials. Let u be a regular linear functional on the linear space $\mathbf{P}$ of polynomials with real coefficients; that is, a linear functional u where the corresponding Hankel determinants $H_k(u)$ are different from zero. We will denote by $\{P_n\}_n$ the monic orthogonal polynomial sequence (MOPS) with respect to u. Then

$$\langle u, P_n P_m \rangle = k_n \delta_{nm}$$

where $k_n \neq 0$ for all $n \in \mathbf{N}$.

Let $\varphi : \mathbf{P} \times \mathbf{P} \to \mathbf{R}$ be the bilinear form defined by

$$\varphi(p, q) = \langle u, pq \rangle + \lambda p'(c) q'(c) \tag{2.1}$$

where $c \in \mathbf{R}$ and $\lambda \in \mathbf{R} - \{0\}$. If φ is not degenerate, we can construct a sequence of monic polynomials $\{Q_n\}_n$ such that

$$\begin{aligned} &\text{(i)} \quad \text{degree of } Q_n(x) = n, \forall n \in \mathbf{N}_0 = \{0, 1, \dots\}, \\ &\text{(ii)} \quad \varphi(Q_n, Q_m) = \tilde{k}_n \delta_{nm}, \tilde{k}_n \neq 0, \forall n \in \mathbf{N}_0. \end{aligned} \tag{2.2}$$

This system of polynomials will be called the *Sobolev-type monic orthogonal polynomial sequence* (MOPS) with respect to φ.

Denote by

$$K_n(x, y) = \sum_{k=0}^{n} \frac{P_k(x) P_k(y)}{\langle u, P_k^2 \rangle}$$

the reproducing kernel of order n associated with the family of orthogonal polynomials $\{P_n\}_n$, and denote by $K_n^{(r,s)}(x, y)$ the corresponding partial derivatives

$$K_n^{(r,s)}(x, y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x, y).$$

Conditions for the existence of the MOPS with respect to φ are given in the next proposition.

Proposition 2.1. *A necessary and sufficient condition for the existence of an* MOPS $\{Q_n\}_n$ *with respect to* φ *is*

$$1 + \lambda K_{n-1}^{(1,1)}(c, c) \neq 0, \quad \forall n \geq 1. \tag{2.3}$$

In this case, the polynomial $Q_n(x)$ can be expressed as

$$Q_n(x) = P_n(x) - \lambda \frac{P_n'(c)}{1+\lambda K_{n-1}^{(1,1)}(c,c)} K_{n-1}^{(0,1)}(x,c). \tag{2.4}$$

Proof. If a family of polynomials $\{Q_n\}_n$ satisfying (2.2) exists, we get

$$Q_n(x) = P_n(x) + \sum_{j=0}^{n-1} a_j P_j(x).$$

From the orthogonality we deduce that

$$\begin{aligned} a_j = \frac{\langle u, Q_n P_j\rangle}{\langle u, P_j^2\rangle} &= \frac{\varphi(Q_n, P_j) - \lambda Q_n'(c) P_j'(c)}{\langle u, P_j^2\rangle} \\ &= -\lambda \frac{Q_n'(c) P_j'(c)}{\langle u, P_j^2\rangle}, \end{aligned}$$

for $0 \le j \le n-1$, and, in this way

$$Q_n(x) = P_n(x) - \lambda Q_n'(c) K_{n-1}^{(0,1)}(x,c). \tag{2.5}$$

If we differentiate (2.5), and then evaluate at $x=c$, we obtain

$$Q_n'(c)(1+\lambda K_{n-1}^{(1,1)}(c,c)) = P_n'(c). \tag{2.6}$$

If $1+\lambda K_{n-1}^{(1,1)}(c,c) = 0$ for some value of n, equality (2.6) implies that $P_n'(c) = 0$, and therefore $1+\lambda K_n^{(1,1)}(c,c) = 0$. Thus, we get $P_m'(c) = 0$, for all $m \ge n$, and by derivation in the three term recurrence relation, we obtain $P_m(c) = 0$ for all $m > n$, which leads to a contradiction, since two consecutive standard orthogonal polynomials have no common zeros. In this way, $1+\lambda K_n^{(1,1)}(c,c) \ne 0$ for all $n \in \mathbf{N}$, and by substitution in (2.5), we obtain expression (2.4).

Conversely, if $1+\lambda K_n^{(1,1)}(c,c) \ne 0$ for all $n \in \mathbf{N}$, the polynomials defined by (2.4) have exact degree n and satisfy the orthogonality conditions (2.2). $\square$

We remark that, if u is a regular functional, φ is nondegenerate for every value of λ except for an infinite and discrete set of values. If u is positive definite, then $K_n^{(1,1)}(c,c) > 0$, and it suffices to take $\lambda \in \mathbf{R}^+$ to obtain a nondegenerate form. We will suppose that φ is nondegenerate for the rest of this paper. If we denote by

$$\lambda_n = 1 + \lambda K_n^{(1,1)}(c,c),$$
$$k_n = \langle u, P_n^2(x)\rangle,$$
$$\tilde{k}_n = \varphi(Q_n, Q_n),$$

then we have

Corollary 2.2.

(i) $Q_n'(c) = P_n'(c)/(1 + \lambda K_{n-1}^{(1,1)}(c,c))$

(ii) $\tilde{k}_n = (\lambda_n/\lambda_{n-1})k_n$.

3. Representation formulas for the polynomials Q_n. From the Christoffel-Darboux relation, we obtain

$$(3.1)\quad (x-c)^2 K_{n-1}^{(0,1)}(x,c) = \frac{1}{k_{n-1}}[P_n(x)T_1(P_{n-1},c)(x) - P_{n-1}(x)T_1(P_n,c)(x)]$$

where $T_i(P_j,c)(x)$ denotes the Taylor polynomial of degree i associated to $P_j(x)$ in c. By substitution in (2.4), we get a formula relating $Q_n(x)$, $P_n(x)$ and $P_{n-1}(x)$.

Proposition 3.1. *The polynomials $\{Q_n\}_n$ satisfy*

$$(3.2)\qquad (x-c)^2 Q_n(x) = q_2(x,n)P_n(x) + q_1(x,n)P_{n-1}(x),$$

with

$$\begin{aligned} q_2(x,n) &= (x-c)^2 - \lambda \frac{Q_n'(c)}{k_{n-1}} T_1(P_{n-1},c)(x) \\ &= (x-c)^2 - \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_{n-1}'(c)}{k_{n-1}}(x-c) \\ &\quad - \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_{n-1}(c)}{k_{n-1}}, \\ q_1(x,n) &= \lambda \frac{Q_n'(c)}{k_{n-1}} T_1(P_n,c)(x) \\ &= \frac{\lambda}{\lambda_{n-1}} \frac{[P_n'(c)]^2}{k_{n-1}}(x-c) + \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_n(c)}{k_{n-1}}. \end{aligned}$$

The above proposition shows that $(x-c)^2 Q_n(x)$ is a quasi-orthogonal polynomial of order four with respect to u; *that is*,

Proposition 3.2.

$$\begin{aligned} (x-c)^2 Q_n(x) &= P_{n+2}(x) + \alpha_{n+1}^{(n)} P_{n+1}(x) + \alpha_n^{(n)} P_n(x) \\ &\quad + \alpha_{n-1}^{(n)} P_{n-1}(x) + \alpha_{n-2}^{(n)} P_{n-2}(x), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \alpha_{n+1}^{(n)} &= (\beta_n + \beta_{n+1} - 2c) - \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_{n-1}'(c)}{k_{n-1}}, \\ \alpha_n^{(n)} &= \gamma_n + \gamma_{n+1} + (\beta_n - c)^2 \\ &\quad - (\beta_{n-1} + \beta_n - 2c) \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_{n-1}'(c)}{k_{n-1}} \\ &\quad - \gamma_{n-1} \frac{\lambda}{\lambda_{n-1}} \frac{P_n'(c)P_{n-2}'(c)}{k_{n-1}}, \\ \alpha_{n-1}^{(n)} &= (\beta_{n-1} + \beta_n - 2c)\gamma_n \frac{\lambda_n}{\lambda_{n-1}} + \frac{\lambda}{\lambda_{n-1}} \frac{P_{n+1}'(c)P_n'(c)}{k_{n-1}}, \\ \alpha_{n-2}^{(n)} &= \gamma_n \gamma_{n-1} \frac{\lambda_n}{\lambda_{n-1}} \neq 0. \end{aligned}$$

4. Five term recurrence relation. Christoffel-Darboux type formula.

Proposition 4.1. *The polynomials* $\{Q_n\}_n$ *satisfy the following five term recurrence relation*

$$\begin{aligned}(x-c)^2 Q_n(x) = Q_{n+2}(x) &+ c_{n+1}^{(n)} Q_{n+1}(x) + c_n^{(n)} Q_n(x)\\ &+ c_{n-1}^{(n)} Q_{n-1}(x) + c_{n-2}^{(n)} Q_{n-2}(x),\end{aligned}$$

where

$$c_{n+1}^{(n)} = (\beta_n + \beta_{n+1} - 2c) + \frac{\lambda}{\lambda_{n+1}} \frac{P'_{n+2}(c) P'_{n+1}(c)}{k_{n+1}} - \frac{\lambda}{\lambda_{n-1}} \frac{P'_n(c) P'_{n-1}(c)}{k_{n-1}},$$

$$\begin{aligned}c_n^{(n)} = &\frac{\lambda_{n-1}}{\lambda_n}[\gamma_n + \gamma_{n+1} + (\beta_n - c)^2]\\ &- \frac{\lambda}{\lambda_n} \frac{P'_n(c) P'_{n-2}(c)}{k_n} \gamma_n \gamma_{n-1} \left[1 + \frac{\lambda_n}{\lambda_{n-1}}\right]\\ &- \frac{\lambda}{\lambda_n} \frac{P'_n(c) P'_{n-1}(c)}{k_{n-1}} \left\{ [(\beta_{n-1} + \beta_n - 2c)] \left[1 + \frac{\lambda_n}{\lambda_{n-1}}\right] \right.\\ &\left. \qquad + \frac{\lambda}{\lambda_{n-1}} \frac{P'_{n+1}(c) P'_n(c)}{k_n} \right\},\end{aligned}$$

$$c_{n-1}^{(n)} = \frac{\lambda_n \lambda_{n-2}}{\lambda_{n-1}^2} \frac{k_n}{k_{n-1}} c_n^{(n-1)},$$

$$c_{n-2}^{(n)} = \frac{\lambda_n}{\lambda_{n-1}} \frac{\lambda_{n-3}}{\lambda_{n-2}} \frac{k_n}{k_{n-2}}.$$

Proof. It is sufficient to expand $(x-c)^2 Q_n(x)$ in terms of the polynomials $\{Q_i\}_i$

$$(x-c)^2 Q_n(x) = Q_{n+2}(x) + \sum_{k=0}^{n+1} c_k^{(n)} Q_k(x)$$

with

$$\begin{aligned}c_k^{(n)} &= \frac{\varphi((x-c)^2 Q_n(x), Q_k(x))}{\varphi(Q_k, Q_k)}\\ &= \frac{\varphi(Q_n(x), (x-c)^2 Q_k(x))}{\varphi(Q_k, Q_k)} = 0,\end{aligned}$$

$0 \le k \le n-3$. The expressions for the coefficients $c_i^{(n)}$ are deduced from Proposition 3.2. □

Finally, in the usual way, from the recurrence relation, we deduce a Christoffel-Darboux-type formula. First, we need the following lemma

Lemma 4.2. *If $n \in \mathbf{N}$, $i = 0, 1, \dots, n-2$ and $i-2 \le j \le i+2$, then*

$$\frac{c_{n-i}^{(n-j)}}{\tilde{k}_{n-j}} = \frac{c_{n-j}^{(n-i)}}{\tilde{k}_{n-i}}.$$

Proposition 4.3. *The following Christoffel-Darboux-type formula holds*

$$\begin{aligned}[(x-c)^2-(y-c)^2]\sum_{j=0}^{n}\frac{Q_j(x)Q_j(y)}{\tilde{k}_j} &= \frac{1}{\tilde{k}_n}[Q_{n+2}(x)Q_n(y)-Q_n(x)Q_{n+2}(y)]\\ &\quad+\frac{1}{\tilde{k}_{n-1}}[Q_{n+1}(x)Q_{n-1}(y)-Q_{n-1}(x)Q_{n+1}(y)]\\ &\quad+\frac{c_{n+1}^{(n)}}{\tilde{k}_{n-1}}[Q_{n+1}(x)Q_n(y)-Q_n(x)Q_{n+1}(y)].\end{aligned}$$

5. The kernels. We define by

$$L_n(x,y)=\sum_{i=0}^{n}\frac{Q_i(x)Q_i(y)}{\tilde{k}_i}$$

the n-kernel associated to the MOPS $\{Q_n\}_n$, and denote by

$$L_n^{(r,s)}(x,y)=\frac{\partial^{r+s}}{\partial x^r\partial y^s}L_n(x,y)$$

the corresponding partial derivatives. They satisfy the usual reproducing properties

Proposition 5.1. *If* $p(x) \in \mathbf{P}_n$ *and* $r = 0, 1, \dots,$

$$\varphi(L_n^{(0,r)}(x,y), p(x)) = p^{(r)}(y). \tag{5.1}$$

Proposition 5.2.

$$L_n(x,y) = K_n(x,y) - \lambda \frac{K_n^{(0,1)}(x,c) K_n^{(0,1)}(y,c)}{1 + \lambda K_n^{(1,1)}(c,c)}. \tag{5.2}$$

Proof. We can expand L_n as a polynomial in the variable x in terms of $P_j(x)$, with coefficients dependent on the parameter y

$$L_n(x,y) = \sum_{j=0}^{n} A_j^{(n)}(y) P_j(x),$$

where

$$\begin{aligned} A_j^{(n)}(y) &= \frac{\langle u, L_n(x,y) P_j(x) \rangle}{\langle u, P_j^2 \rangle} \\ &= \frac{P_j(y)}{\langle u, P_j^2 \rangle} - \lambda \frac{L_n^{(0,1)}(y,c) P_j'(c)}{\langle u, P_j^2 \rangle} \end{aligned}$$

by using the reproducing property. Then

$$L_n(x,y) = K_n(x,y) - \lambda L_n^{(0,1)}(y,c) K_n^{(0,1)}(x,c).$$

If we differentiate this expression with respect to x and then evaluate at $x = c$, we get

$$L_n^{(0,1)}(y,c)[1 + \lambda K_n^{(1,1)}(c,c)] = K_n^{(0,1)}(y,c)$$

and finally

$$L_n(x,y) = K_n(x,y) - \lambda \frac{K_n^{(0,1)}(x,c) K_n^{(0,1)}(y,c)}{1 + \lambda K_n^{(1,1)}(c,c)}. \qquad \square$$

6. Relation with the modification associated to $(x-c)^2$. The concept of quasi-orthogonality gives the Sobolev-type orthogonal polynomials another interesting aspect. In fact, let $p(x) \in \mathbf{P}_{n-3}$, and consider

$$\varphi(Q_n(x), (x-c)^2 p(x)) = \langle u, Q_n(x) p(x)(x-c)^2 \rangle = 0.$$

This equality shows $Q_n(x)$ to be a quasi-orthogonal polynomial of order 2 with respect to the functional u_1, defined by $u_1 = (x-c)^2 u$. This quasi-orthogonality condition implies that $Q_n(x)$ can be expressed as a linear combination of the monic orthogonal polynomials with respect to the functional u_1, if they exist, that is, if u_1 is a regular functional. In [**9**], the following necessary and sufficient condition to warrant the regularity of the functional u_1 is shown

Proposition 6.1. *u_1 is a regular functional if and only if $K_n(c,c) \neq 0$ for all $n \in \mathbf{N}$.*

From now on, we suppose that μ_1 is a regular linear functional. The relations between the polynomials $\{P_n\}_n$ and the MOPS associated to u_1 are expressed in the next lemma.

Lemma 6.2. *Let $\{P_n^{1,c}(x)\}$ be the* MOPS *associated to u_1. Then*

$$(x-c)P_{n-1}^{1,c}(x) = P_n(x) - \frac{P_n(c)}{K_{n-1}(c,c)} K_{n-1}(x,c), \tag{6.1}$$

$$P_{n-1}^{1,c}(c) = P_n'(c) - \frac{P_n(c)}{K_{n-1}(c,c)} K_{n-1}^{(0,1)}(c,c), \tag{6.2}$$

$$(x-c)(y-c)K_{n-1}^{1,c}(x,y) = K_n(x,y) - \frac{K_n(x,c)K_n(c,y)}{K_n(c,c)}, \tag{6.3}$$

$$(x-c)K_{n-1}^{1,c} = K_n^{(0,1)}(x,c) - \frac{K_n^{(0,1)}(c,c)}{K_n(c,c)} K_n(x,c), \tag{6.4}$$

If we denote by $k_{n-1}^c = \langle u_1, (P_{n-1}^{1,c})^2 \rangle$, *then*

$$k_{n-1}^c = k_n \frac{K_n(c,c)}{K_{n-1}(c,c)} \tag{6.5}$$

Proof. For (6.1)–(6.4) see Alfaro and others [**2**]. For (6.5), we use the relation (6.1) and the reproducing properties for the kernels

$$\begin{aligned} k_{n-1}^c &= \langle u_1, (P_{n-1}^{1,c})^2 \rangle = \langle u, (x-c)^2 (P_{n-1}^{1,c}(x))^2 \rangle \\ &= \left\langle u, \left[P_n(x) - \frac{P_n(c)}{K_{n-1}(c,c)} K_{n-1}(x,c) \right] \right\rangle \\ &= \langle u, P_n \rangle + \frac{P_n(c)^2}{K_{n-1}(c,c)^2} \langle u, K_{n-1}(x,c)^2 \rangle \\ &= k_n \left[1 + \frac{P_n(c)^2}{k_n K_{n-1}(c,c)} \right] = k_n \frac{K_n(c,c)}{K_{n-1}(c,c)}. \qquad \square \end{aligned}$$

Proposition 6.3. *Suppose that* u_1 *is a regular linear functional, and denote by* $\{P_n^{1,c}\}_n$ *the* MOPS *with respect to* u_1. *Then*

$$Q_n(x) = P_n^{1,c}(x) + a_{n-1}^{(n)} P_{n-1}^{1,c}(x) + a_{n-2}^{(n)} P_{n-2}^{1,c}(x), \tag{6.6}$$

where

$$\begin{aligned} a_{n-2}^{(n)} &= \frac{\tilde{k}_n}{k_{n-2}^c} = \frac{\lambda_n}{\lambda_{n-1}} \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \frac{k_n}{k_{n-1}} \neq 0, \\ a_{n-1}^{(n)} &= (\beta_n - c) + \frac{P_n(c) P_{n+1}(c)}{K_n(c,c) k_n} - \lambda \frac{P_n'(c) P_{n-1}'(c)}{\lambda_{n-1} k_{n-1}}. \end{aligned}$$

Proof. Expand $Q_n(x)$ in terms of the polynomials $\{P_n^{1,c}\}$:

$$Q_n(x) = P_n^{1,c}(x) + \sum_{j=0}^{n-1} a_j^{(n)} P_j^{1,c}(x),$$

where

$$a_j^{(n)} = \frac{\langle u_1, Q_n P_j^{1,c}\rangle}{\langle u_1, (P_j^{1,c})^2\rangle} = \frac{\langle u, (x-c)^2 Q_n P_j^{1,c}\rangle}{\langle u_1, (P_j^{1,c})^2\rangle}, \quad j = 0, 1, \dots, n-1,$$

and therefore $a_j^{(n)} = 0$ for $j < n-2$, since $Q_n(x)$ is orthogonal to every polynomial of degree less than n, and

$$\begin{aligned} a_{n-2}^{(n)} &= \frac{\langle u_1, Q_n P_{n-2}^{1,c}\rangle}{\langle u_1, (P_{n-2}^{1,c})^2\rangle} = \frac{\langle u, (x-c)^2 Q_n P_{n-2}^{1,c}\rangle}{\langle u_1, (P_{n-2}^{1,c})^2\rangle} \\ &= \frac{\varphi(Q_n, (x-c)^2 P_{n-2}^{1,c})}{\langle u_1, (P_{n-2}^{1,c})^2\rangle} \\ &= \frac{\tilde{k}_n}{k_{n-2}^c} = \frac{\lambda_n}{\lambda_{n-1}} \frac{K_{n-2}(c,c)}{K_{n-1}(c,c)} \frac{k_n}{k_{n-1}} \neq 0. \end{aligned}$$

To obtain $a_{n-1}^{(n)}$, we consider (2.4) and Lemma 6.2,

$$\begin{aligned} \langle u_1, Q_n P_{n-1}^{1,c}\rangle &= \langle u, (x-c)^2 Q_n P_{n-1}^{1,c}\rangle \\ &= \Big\langle u, \Big[P_n(x) - \frac{\lambda P_n'(c)}{\lambda_{n-1}} K_{n-1}^{(0,1)}(x,c)\Big](x-c) \\ &\qquad \Big[P_n(x) - \frac{P_n(c)}{K_{n-1}(c,c)} K_{n-1}(x,c)\Big]\Big\rangle \\ &= \langle u, (x-c)P_n(x)^2\rangle - \frac{P_n(c)P_{n-1}(c)k_n}{K_{n-1}(c,c)k_{n-1}} \\ &\quad - \lambda \frac{P_n'(c)P_{n-1}'(c)k_n}{\lambda_{n-1}k_{n-1}}\Big[1 + \frac{P_n(c)^2}{K_{n-1}(c,c)k_n}\Big]. \end{aligned}$$

Finally, using the three term recurrence relation, we get

$$\begin{aligned} a_{n-1}^{(n)} &= \frac{\langle u_1, Q_n P_{n-1}^{1,c}\rangle}{\langle u_1, (P_{n-1}^{1,c})^2\rangle} = \frac{K_{n-1}(c,c)}{K_n(c,c)}(\beta_n - c) \\ &\quad - \frac{P_n(c)P_{n-1}(c)}{K_n(c,c)k_{n-1}} - \lambda \frac{P_n'(c)P_{n-1}'(c)}{\lambda_{n-1}k_{n-1}} \\ &= \Big[1 - \frac{P_n(c)^2}{K_n(c,c)k_n}\Big](\beta_n - c) \\ &\quad - \frac{P_n(c)P_{n-1}(c)}{K_n(c,c)k_{n-1}} - \lambda \frac{P_n'(c)P_{n-1}'(c)}{\lambda_{n-1}k_{n-1}}. \qquad \square \end{aligned}$$

Corollary 6.4.

$$Q_n(x) = [x - (\beta^c_{n-1} - a^{(n)}_{n-1})]P^{1,c}_{n-1}(x) - (\gamma^c_{n-1} - a^{(n)}_{n-2})P^{1,c}_{n-2}(x), \tag{6.7}$$

where β^c_{n-1} *and* γ^c_{n-1} *are the coefficients of the three term recurrence relation for the polynomials* $\{P^{1,c}_j(x)\}$.

Proof. It is sufficient to substitute in (6.6) the three-term recurrence relation for the polynomials $\{P^{1,c}_j(x)\}$. □

Remark. This corollary shows that the polynomials $Q_n(x)$ can be obtained by a perturbation of the three-term recurrence relation for the orthogonal polynomials associated to u_1.

Conversely, the polynomials $P^{1,c}_n(x)$ can be expressed by means of three consecutive polynomials $Q_i(x)$.

Proposition 6.5.

$$(x-c)^2 P^{1,c}_n(x) = Q_{n+2}(x) + b^{(n)}_{n+1} Q_{n+1}(x) + b^{(n)}_n Q_n(x), \tag{6.8}$$

where

$$b^{(n)}_n = \frac{\lambda_{n-1}}{\lambda_{n+1}} \frac{K_{n+1}(c,c)}{K_{n-1}(c,c)} a^{(n+1)}_{n-1} \neq 0,$$
$$b^{(n)}_{n+1} = \frac{\lambda_n}{\lambda_{n+1}} \frac{K_{n+1}(c,c)}{K_n(c,c)} a^{(n+1)}_n.$$

Proof. Expand the polynomial $(x-c)^2 P^{1,c}_n(x)$ in terms of $\{Q_i(x)\}$:

$$(x-c)^2 P^{1,c}_n(x) = Q_{n+2}(x) + \sum_{i=0}^{n+1} b^{(n)}_i Q_i(x).$$

From the orthogonality, we deduce

$$b^{(n)}_i = \frac{\varphi(Q_i, (x-c)^2 P^{1,c}_n)}{\varphi(Q_i, Q_i)}, \quad i = 0, \dots, n+1,$$

and $b_i^{(n)} = 0$, $i = 0, \dots, n-1$. The coefficients $b_n^{(n)}$, $b_{n+1}^{(n)}$ are obtained from the relation (6.5)

$$\begin{aligned}
2b_n^{(n)} &= \frac{\varphi(Q_n, (x-c)^2 P_n^{1,c})}{\varphi(Q_n, Q_n)} = \frac{\langle u_1, Q_n P_n^{1,c}\rangle}{\varphi(Q_n, Q_n)} \\
&= \frac{k_n^c}{\tilde{k}_n} = \frac{\lambda_{n-1}}{\lambda_n} \frac{K_{n+1}(c,c)}{K_n(c,c)} \frac{k_{n+1}}{k_n} \\
&= \frac{\lambda_{n-1}}{\lambda_{n+1}} \frac{K_{n+1}(c,c)}{K_{n-1}(c,c)} a_{n-1}^{(n+1)} \neq 0, \\
b_{n+1}^{(n)} &= \frac{\varphi(Q_{n+1}, (x-c)^2 P_n^{1,c})}{\varphi(Q_{n+1}, Q_{n+1})} \\
&= \frac{\langle u_1, Q_{n+1} P_n^{1,c}\rangle}{\varphi(Q_{n+1}, Q_{n+1})} = a_n^{(n+1)} \frac{k_n^c}{\tilde{k}_{n+1}}. \qquad \square
\end{aligned}$$

From the above propositions, we can obtain again the five-term recurrence relation for the polynomials $\{Q_n(x)\}$.

Proposition 6.6.

$$\begin{aligned}
(x-c)^2 Q_n(x) = Q_{n+2}(x) &+ c_{n+1}^{(n)} Q_{n+1}(x) + c_n^{(n)} Q_n(x) \\
&+ c_{n-1}^{(n)} Q_{n-1}(x) + c_{n-2}^{(n)} Q_{n-2}(x)
\end{aligned}$$

where, if $n \geq 2$,

$$\begin{aligned}
c_{n+1}^{(n)} &= b_{n+1}^{(n)} + a_{n-1}^{(n)}, \\
c_n^{(n)} &= b_n^{(n)} + a_{n-1}^{(n)} b_n^{(n-1)} + a_{n-2}^{(n)}, \\
c_{n-1}^{(n)} &= a_{n-1}^{(n)} b_{n-1}^{(n-1)} + a_{n-2}^{(n)} b_{n-1}^{(n-2)}, \\
c_{n-2}^{(n)} &= a_{n-2}^{(n)} b_{n-2}^{(n-2)}.
\end{aligned} \tag{6.9}$$

Proof. From Propositions 6.3 and 6.5, we deduce

$$\begin{aligned}(x-c)^2 Q_n(x) &= (x-c)^2 P_n^{1,c}(x) + a_{n-1}^{(n)}(x-c)^2 P_{n-1}^{1,c}(x)\\ &\quad + (x-c)^2 a_{n-2}^{(n)} P_{n-2}^{1,c}(x)\\ &= Q_{n+2}(x) + (b_{n+1}^{(n)} + a_{n-1}^{(n)}) Q_{n+1}(x)\\ &\quad + (b_n^{(n)} + a_{n-1}^{(n)} b_n^{(n-1)} + a_{n-2}^{(n)}) Q_n(x)\\ &\quad + (a_{n-1}^{(n)} b_{n-1}^{(n-1)} + a_{n-2}^{(n)} b_{n-1}^{(n-2)}) Q_{n-1}(x)\\ &\quad + a_{n-2}^{(n)} b_{n-2}^{(n-2)} Q_{n-2}(x). \qquad \square\end{aligned}$$

Proposition 6.7. *The coefficients of the relations* (6.6) *and* (6.8) *satisfy*

$$\begin{aligned} a_{n+1}^{(n+2)} + b_{n+1}^{(n)} &= \beta_{n+1}^c + \beta_n^c - 2c,\\ a_n^{(n+2)} + b_{n+1}^{(n)} a_n^{(n+1)} + b_n^{(n)} &= \gamma_{n+1}^c + (\beta_n^c - c)^2 + \gamma_n^c,\\ b_{n+1}^{(n)} a_{n-1}^{(n+1)} + b_n^{(n)} a_{n-1}^{(n)} &= \gamma_n^c(\beta_n^c + \beta_{n-1}^c - 2c),\\ b_n^{(n)} a_{n-2}^{(n)} &= \gamma_n^c \gamma_{n-1}^c. \end{aligned} \tag{6.10}$$

Proof. By using relations (6.6) and (6.8), we obtain

$$\begin{aligned}(x-c)^2 P_n^{1,c}(x) &= Q_{n+2}(x) + b_{n+1}^{(n)} Q_{n+1}(x) + b_n^{(n)} Q_n(x)\\ &= P_{n+2}^{1,c}(x) + (a_{n+1}^{(n+2)} + b_{n+1}^{(n)}) P_{n+1}^{1,c}(x)\\ &\quad + (a_n^{(n+2)} + b_{n+1}^{(n)} a_n^{(n+1)} + b_n^{(n)}) P_n^{1,c}(x)\\ &\quad + (b_{n+1}^{(n)} a_{n-1}^{(n+1)} + b_n^{(n)} a_{n-1}^{(n)}) P_{n-1}^{1,c}(x)\\ &\quad + b_n^{(n)} a_{n-2}^{(n)} P_{n-2}^{1,c}(x).\end{aligned}$$

In this way, we only need to compare with the five-term recurrence relations satisfied by the polynomials $P_n^{1,c}(x)$. $\square$

Remark. The above results provide a recursive algorithm to compute the polynomials $\{Q_n(x)\}_n$ from the coefficients of the three-term recurrence relation of the polynomials $\{P_n^{1,c}(x)\}_n$. From the relation (6.10) we can deduce the coefficients $b_n^{(n)}$, $b_{n+1}^{(n)}$, $a_n^{(n+2)}$ and $a_{n+1}^{(n+2)}$ from $a_{n-2}^{(n)}$, $a_{n-1}^{(n)}$, $a_{n-1}^{(n+1)}$ and $a_n^{(n+1)}$, for $n \geq 2$,

$$\begin{aligned} b_n^{(n)} &= (1/a_{n-2}^{(n)})\gamma_n^c\gamma_{n-1}^c, \\ b_{n+1}^{(n)} &= (1/a_{n-1}^{(n+1)})[\gamma_n^c(\beta_{n+1}^c + \beta_n^c - 2c) - b_n^{(n)}a_{n-1}^{(n)}], \\ a_n^{(n+2)} &= \gamma_{n+1}^c + (\beta_n^c - c)^2 + \gamma_n^c - b_{n+1}^{(n)}a_n^{(n+1)} - b_n^{(n)}, \\ a_{n+1}^{(n+2)} &= \beta_{n+1}^c + \beta_n^c - 2c - b_{n+1}^{(n)}. \end{aligned} \tag{6.11}$$

The initial conditions are given by

$$a_0^{(1)} = (\beta_1 - c) + \frac{P_1(c)P_2(c)}{K_1(c,c)k_1},$$
$$b_0^{(0)} = \frac{k_0^c}{k_0}, \qquad b_1^{(0)} = a_0^{(1)}\frac{k_0^c}{\tilde{k}_1}.$$

7. The Bessel case. In this section we consider the particular case of Bessel polynomials (see [**6, 8**]). These polynomials constitute an interesting example of an MOPS with respect to a regular functional which is not positive definite.

The *generalized Bessel polynomials* $y_n(x;a,b)$ were introduced by Krall and Frink [**8**] as the polynomial solutions of the Bessel polynomial differential equation

$$x^2y'' + (ax+b)y' - n(n+a-1)y = 0 \tag{7.1}$$

where $b \neq 0$ and $a \neq 0, -1, -2, \dots$, satisfying $y_n(0;a,b) = 1$. It is easy to see that $y_n(bx;a,b)$ is independent of b. Consequently, we consider the polynomials

$$Y_n^{(\alpha)}(x) = y_n(x;\alpha+2,2)$$

where $\alpha \neq -2, -3, \dots$. From now on, they will be called the *Bessel polynomials*. In the same paper, Krall and Frink [**8**] give the orthogonality relation

$$\frac{1}{2\pi i}\int_T Y_n^{(\alpha)}(z)Y_m^{(\alpha)}(z)\rho^{(\alpha)}(z)\,dz = \frac{2^{\alpha+1}(-1)^{n+1}n!}{(2n+\alpha+1)\Gamma(n+\alpha+1)}\delta_{nm}, \tag{7.2}$$

where

$$\rho^{(\alpha)}(z) = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} \frac{1}{(\alpha+1)_k} \left(-\frac{2}{z} \right)^k \tag{7.3}$$

and the integration is around the unit circle. Here $(a)_n$ denotes the Pochhammer's symbol, defined by $(a)_n = a(a+1)\cdots(a+n-1)$. Thus, $\{Y_n^{(\alpha)}(x)\}$ is a quasi-definite or regular OPS.

Denote by $B_n^{(\alpha)}(x)$ the monic Bessel orthogonal polynomial. In this case, the following properties are known (see Krall and Frink [**8**], Grosswald [**6**])

Explicit representation.

$$B_n^{(\alpha)}(x) = \frac{2^n}{(n+\alpha+1)_n} \sum_{k=0}^{n} \binom{n}{k} (n+\alpha+1)_k \left(\frac{x}{2} \right)^k. \tag{7.4}$$

Orthogonality condition.

$$\begin{aligned} \frac{1}{2\pi i} \int_T B_n^{(\alpha)}(z) B_m^{(\alpha)}(z) \rho^{(\alpha)}(z)\, dz \\ = \left(\frac{2^n}{(n+\alpha+1)_n} \right)^2 \frac{2^{\alpha+1}(-1)^{n+1} n!}{(2n+\alpha+1)\Gamma(n+\alpha+1)} \delta_{nm}; \end{aligned} \tag{7.5}$$

in particular,

$$\begin{aligned} k_n^{(\alpha)} &= \frac{1}{2\pi i} \int_T (B_n^{(\alpha)}(z))^2 \rho^{(\alpha)}(z)\, dz \\ &= \left(\frac{2^n}{(n+\alpha+1)_n} \right)^2 \frac{2^{\alpha+1}(-1)^{n+1} n!}{(2n+\alpha+1)\Gamma(n+\alpha+1)}. \end{aligned} \tag{7.6}$$

The three-term recurrence relation.

$$x B_n^{(\alpha)}(x) = B_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)} B_n^{(\alpha)}(x) + \gamma_n^{(\alpha)} B_{n-1}^{(\alpha)}(x), \quad n \geq 1,$$

where

$$\beta_n^{(\alpha)} = -\frac{2\alpha}{(2n+\alpha)(2n+\alpha+2)}, \tag{7.7}$$

$$\gamma_n^{(\alpha)} = -\frac{4n(n+\alpha)}{(2n+\alpha+1)(2n+\alpha)^2(2n+\alpha-1)}, \tag{7.8}$$

and

$$B_0^{(\alpha)}(x) = 1, \qquad B_1^{(\alpha)}(x) = x + \frac{2}{(\alpha+2)}.$$

Differential equation.

$$x^2 \frac{d^2}{dx^2} B_n^{(\alpha)}(x) + [(\alpha+2)x+2] \frac{d}{dx} B_n^{(\alpha)}(x) - n(n+\alpha+1) B_n^{(\alpha)}(x) = 0. \tag{7.9}$$

Differential relation.

$$\frac{d}{dx} B_n^{(\alpha)}(x) = n B_{n-1}^{(\alpha+2)}(x). \tag{7.10}$$

Rodrigues formula.

$$B_n^{(\alpha)}(x) = \frac{1}{(n+\alpha+1)_n} x^{-\alpha} e^{2/x} D^n (x^{2n+\alpha} e^{-2/x}), \tag{7.11}$$

where $D^n = d^n/dx^n$.

Structure relation.

$$\begin{aligned} x^2 \frac{d}{dx} B_n^{(\alpha)}(x) &= n\left(x - \frac{2}{2n+\alpha}\right) B_n^{(\alpha)}(x) \\ &\quad + \frac{4n(n+\alpha)}{(2n+\alpha)^2(2n+\alpha-1)} B_{n-1}^{(\alpha)}(x). \end{aligned} \tag{7.12}$$

From the above properties, we deduce the values of the parameters which appear in the expressions of the Sobolev-type polynomials.

$$\begin{aligned} B_n^{(\alpha)}(0) &= \frac{2^n}{(n+\alpha+1)_n}, \qquad (B_n^{(\alpha)})'(0) = n\frac{2^{n-1}}{(n+\alpha+2)_{n-1}}, \\ K_{n-1}(0,0) &= \frac{(-1)^n}{2^{\alpha+1}} \frac{\Gamma(n+\alpha+1)}{(n-1)!}, \\ K_{n-1}^{(1,1)}(0,0) &= \frac{(-1)^n}{2^{\alpha+3}} \frac{\Gamma(n+\alpha+2)}{(n-2)!} [n(n+\alpha) - (\alpha+2)]. \end{aligned}$$

Let $\alpha > -2$, and let $u^{(\alpha)}$ be the functional associated to the Bessel polynomials $\{B^{(\alpha)})_n\}_n$. We consider the bilinear form $\varphi^{(\alpha)}$ defined by

$$\varphi^{(\alpha)}(f,g) = \langle u^{(\alpha)}, fg\rangle + \lambda f'(0)g'(0).$$

The point $c = 0$ has been selected in order to preserve the classical character for the functional $(u^{(\alpha)})_1$.

By Proposition 2.1 a necessary and sufficient condition to warrant the nondegeneracy of $\varphi^{(\alpha)}$ is

$$1 + \lambda K_{n-1}^{(1,1)}(0,0) \neq 0, \quad \forall n \geq 1.$$

Because the sequence $\{|K_{n-1}^{(1,1)}(0,0)|\}$ diverges (for $\alpha > -2$), the set of values λ, such that $\varphi^{(\alpha)}$ is degenerate, is contained in a bounded interval, and, therefore, $\varphi^{(\alpha)}$ is nondegenerate for

$$|\lambda| > |K_1^{(1,1)}(0,0)|^{-1} = \frac{2^{\alpha+3}}{\Gamma(\alpha+4)(\alpha+2)}.$$

Denote by $\{Q_n^{(\alpha)}\}_n$ the MOPS associated to a nondegenerate bilinear form $\varphi^{(\alpha)}$. By using the results in Proposition 2.1, we can obtain the first representation formula for the polynomials $\{Q_n^{(\alpha)}\}_n$.

Proposition 7.1.

$$Q_n^{(\alpha)}(x) = B_n^{(\alpha)}(x) - \frac{\lambda}{\lambda_{n-1}}(B_n^{(\alpha)})'(0)K_{n-1}^{(0,1)}(x,0), \tag{7.13}$$

where $\lambda_n = 1 + \lambda K_n^{(1,1)}(0,0)$.

Corollary 7.2.

(i) $Q_n^{(\alpha)}(0) = (1/\lambda_{n-1})(2n/(n+\alpha+1)_n)[1-\lambda((-1)^n/2^{\alpha+3})(\Gamma(n+\alpha+3)/(n-2)!)]$,

(ii) $(Q_n^{(\alpha)})'(0) = (n/\lambda_{n-1})(2^{n-1}/(n+\alpha+2)_{n-1})$.

Proposition 3.1 becomes

Proposition 7.3.

$$x^2 Q_n^{(\alpha)}(x) = q_2(x,n) B_n^{(\alpha)}(x) + q_1(x,n) B_{n-1}^{(\alpha)}(x), \tag{7.14}$$

where

$$q_2(x,n) = x^2 - \frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)} \left[1 - \frac{\lambda_{n-2}}{\lambda_{n-1}}\right]\left[x + \frac{2}{(n-1)(n+a)}\right], \tag{7.15}$$

$$q_1(x,n) = \left[\frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)}\right]^2 \left[1 - \frac{\lambda_{n-2}}{\lambda_{n-1}}\right]\left[x + \frac{2}{n(n+\alpha+1)}\right]. \tag{7.16}$$

The polynomials $\{Q_n^{(\alpha)}\}_n$ satisfy a five-term recurrence relation. Next, we get the asymptotic behavior of the coefficients in this relation. First, we need some preliminary results.

Lemma 7.4.

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} = \lim_{n\to\infty} \frac{1+\lambda K_n^{(1,1)}(0,0)}{1+\lambda K_{n-1}^{(1,1)}(0,0)} = -1.$$

Proof.

$$\begin{aligned}\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} &= \lim_{n\to\infty} \frac{1+\lambda \frac{(-1)^{n+1}}{2^{\alpha+3}} \frac{\Gamma(n+\alpha+3)}{(n-1)!}[(n+1)(n+\alpha+1)-(\alpha+2)]}{1+\lambda \frac{(-1)^n}{2^{\alpha+3}} \frac{\Gamma(n+\alpha+2)}{(n-2)!)}[n(n+\alpha)-(\alpha+2)]} \\ &= \lim_{n\to\infty} \frac{\frac{1}{z_n} - \frac{n+\alpha+2}{n-1}[(n+1)(n+\alpha+1)-(\alpha+2)]}{\frac{1}{z_n} + [n(n+\alpha)-(\alpha+2)]} = -1,\end{aligned}$$

because

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \lambda \frac{(-1)^n}{2^{\alpha+3}} \frac{\Gamma(n+\alpha+2)}{(n-2)!} = \infty. \qquad \square$$

Lemma 7.5. *Let* $\{a_n\}_n$ *and* $\{b_n\}_n$ *be the sequences*

$$\begin{aligned} a_n &= \frac{\lambda}{\lambda_{n-1}} \frac{(B_n^{(\alpha)})'(0)(B_{n-1}^{(\alpha)})'(0)}{k_{n-1}^{(\alpha)}} \\ &= \frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)} \left[1 - \frac{\lambda_{n-2}}{\lambda_{n-2}}\right], \\ b_n &= \frac{\lambda}{\lambda_n} \frac{(B_n^{(\alpha)})'(0)(B_{n-2}^{(\alpha)})'(0)}{k_n^{(\alpha)}} \\ &= \frac{(n-2)(2n+\alpha-3)(2n+\alpha-2)(2n+\alpha-1)(2n+\alpha)}{4n(n+\alpha)(n+\alpha+1)} \left[1 - \frac{\lambda_{n-1}}{\lambda_n}\right]. \end{aligned}$$

Then $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = +\infty$.

Proof.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\lambda}{\lambda_{n-1}} \frac{(B_n^{(\alpha)})'(0)(B_{n-1}^{(\alpha)})'(0)}{k_{n-1}^{(\alpha)}} \\ &= \lim_{n\to\infty} \frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)} \left[1 - \frac{\lambda_{n-2}}{\lambda_{n-1}}\right] = 0, \\ \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{\lambda}{\lambda_n} \frac{(B_n^{(\alpha)})'(0)(B_{n-2}^{(\alpha)})'(0)}{k_n^{(\alpha)}} \\ &= \lim_{n\to\infty} \frac{(n-2)(2n+\alpha-3)(2n+\alpha-2)(2n+\alpha-1)(2n+\alpha)}{4n(n+\alpha)(n+\alpha+1)} \\ &\qquad \left[1 - \frac{\lambda_{n-1}}{\lambda_n}\right] = +\infty. \qquad \square \end{aligned}$$

Proposition 7.6 (Five-term recurrence relation). *The polynomials* $\{Q_n^{(\alpha)}\}_n$ *satisfy*

$$\begin{aligned} x^2 Q_n^{(\alpha)}(x) &= Q_{n+2}^{(\alpha)}(x) + c_{n+1}^{(n)} Q_{n+1}^{(\alpha)}(x) \\ &\quad + c_n^{(n)} Q_n^{(\alpha)}(x) + c_{n-1}^{(n)} Q_{n-1}^{(\alpha)}(x) + c_{n-2}^{(n)} Q_{n-2}^{(\alpha)}(x), \end{aligned}$$

with

$$\lim_{n\to\infty} c_{n+1}^{(n)} = \lim_{n\to\infty} c_n^{(n)} = \lim_{n\to\infty} c_{n-1}^{(n)} = \lim_{n\to\infty} c_{n-2}^{(n)} = 0.$$

Proof. Recalling the expressions of the coefficients in the five-term recurrence relation, applying

$$\lim_{n\to\infty}\beta_n^{(\alpha)}=0,\qquad \lim_{n\to\infty}\gamma_n^{(\alpha)}=\lim_{n\to\infty}\frac{k_n^{\alpha}}{k_{n-1}^{\alpha}}=0,$$

and using the above lemmas, we obtain

$$\lim_{n\to\infty}c_{n+1}^{(n)}=\lim_{n\to\infty}c_{n-1}^{(n)}=\lim_{n\to\infty}c_{n-2}^{(n)}=0.$$

Finally,

$$\begin{aligned}c_n^{(n)}&=\frac{\lambda_{n-1}}{\lambda_n}[\gamma_n^{(\alpha)}+\gamma_{n+1}^{(\alpha)}+(\beta_n^{(\alpha)})^2]\\&\quad-\frac{4(n-1)(n-2)(n+\alpha-1)}{(n+\alpha+1)(2n+\alpha-2)(2n+\alpha-1)(2n+\alpha)(2n+\alpha+1)}\\&\qquad\qquad\left[1-\frac{\lambda_{n-1}}{\lambda_n}\right]\left[1+\frac{\lambda_n}{\lambda_{n-1}}\right]\\&\quad-\frac{\lambda_{n-1}}{\lambda_n}a_n\left\{(\beta_{n-1}^{(\alpha)}+\beta_n^{(\alpha)})\left[1+\frac{\lambda_n}{\lambda_{n-1}}\right]+\frac{\lambda_n}{\lambda_{n-1}}a_{n+1}\right\},\end{aligned}$$

and we deduce that

$$\lim_{n\to\infty}c_n^{(n)}=0.\qquad\square$$

Now we can get the expressions for $Q_n^{(\alpha)}(x)$ in terms of the orthogonal polynomials associated with the linear functional $(u^{(\alpha)})_1$. In this case the MOPS corresponding to $(u^{(\alpha)})_1$ is $\{B_n^{(\alpha+2)}\}_n$. Then Propositions 6.3, 6.4 and 6.5 give

Proposition 7.7.

(7.17) $$Q_n^{(\alpha)}(x)=B_n^{(\alpha+2)}(x)+a_{n-1}^{(n)}B_{n-1}^{(\alpha+2)}(x)+a_{n-2}^{(n)}B_{n-2}^{(\alpha+2)}(x),$$

where

(7.18)
$$a_{n-2}^{(n)}=\frac{\lambda_n}{\lambda_{n-1}}\frac{4n(n-1)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)},$$

$$a_{n-1}^{(n)}=\frac{4n}{(2n+\alpha)(2n+\alpha+2)}$$

(7.19) $$-\frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)}\left[1-\frac{\lambda_{n-2}}{\lambda_{n-1}}\right];$$

(7.20) $$Q_n^{(\alpha)}(x) = (x-\zeta_n)B_{n-1}^{(\alpha+2)}(x) - \xi_n B_{n-2}^{(\alpha+2)}(x),$$

where

(7.21)
$$\zeta_n = -\frac{2}{2n+\alpha} + \frac{2n(n+\alpha+1)}{(n-1)(2n+\alpha-1)(2n+\alpha)}\left[1-\frac{\lambda_{n-2}}{\lambda_{n-1}}\right],$$

(7.22)
$$\xi_n = -\frac{4(n-1)(n+\alpha+1)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}\left[1+\frac{n}{n+\alpha+1}\frac{\lambda_n}{\lambda_{n-1}}\right];$$

(7.23) $$x^2 B_n^{(\alpha+2)}(x) = Q_{n+2}^{(\alpha)}(x) + b_{n+1}^{(n)} Q_{n+1}^{(\alpha)}(x) + b_n^{(n)} Q_n^{(\alpha)}(x),$$

where

(7.24)
$$b_n^{(n)} = \frac{\lambda_{n-1}}{\lambda_n}\frac{4(n+\alpha+1)(n+\alpha+2)}{(2n+\alpha+1)(2n+\alpha+2)^2(2n+\alpha+3)},$$

(7.25)
$$b_{n+1}^{(n)} = -\frac{\lambda_n}{\lambda_{n+1}}\frac{n+\alpha+2}{n+1}a_n^{(n+1)}.$$

The classical character of the polynomials $B_n^{(\alpha+2)}$ allows us to obtain differential properties for the polynomials $Q_n^{(\alpha)}$.

Proposition 7.8 (Rodrigues-type formula). *The polynomials* $Q_n^{(\alpha)}(x)$ *satisfy*

(7.26)
$$x^{\alpha+2}e^{-2/x}Q_n^{(\alpha)}(x) = \frac{1}{(n+\alpha+1)_n}D^{n-2}\{x^{2n+\alpha-2}e^{-2/x}\rho(x;n)\},$$

where $\rho(x;n)$ *is a polynomial of degree* 2*, given explicitly by*

$$\begin{aligned}\rho(x;n) = \frac{(n+\alpha+1)(n+\alpha+2)}{(2n+\alpha+1)(2n+\alpha+2)}[(2n+\alpha+1)(2n+\alpha+2)x^2 \\ + 4(2n+\alpha+2)x+4] \\ + a_{n-1}^{(n)}(n+\alpha+1)[(2n+\alpha)x+2] \\ + a_{n-2}^{(n)}(2n+\alpha-1)(2n+\alpha).\end{aligned}$$

Proof. Write the Rodrigues formula, given in (7.11), for the polynomials $\{B_n^{(\alpha+2)}\}_n$

$$x^{\alpha+2}e^{-2/x}B_n^{(\alpha+2)}(x) = \frac{1}{(n+\alpha+3)_n}D^n[x^{2n+\alpha+2}e^{-2/x}].$$

By substitution in (7.17), we deduce

$$\begin{aligned} x^{\alpha+2}e^{-2/x}Q_n^{(\alpha)}(x) = & \frac{1}{(n+\alpha+3)_n}D^n[x^{2n+\alpha+2}e^{-2/x}] \\ & + a_{n-1}^{(n)}\frac{1}{(n+\alpha+2)_{n-1}}D^{n-1}[x^{2n+\alpha}e^{-2/x}] \\ & + a_{n-2}^{(n)}\frac{1}{(n+\alpha+1)_{n-2}}D^{n-2}[x^{2n+\alpha-2}e^{-2/x}]. \end{aligned}$$

Then

$$\begin{aligned} x^{\alpha+2}e^{-2/x}Q_n^{(\alpha)}(x) = \frac{1}{(n+\alpha+1)_n}D^{n-2}\bigg\{ & \frac{(n+\alpha+1)(n+\alpha+2)}{(2n+\alpha+1)(2n+\alpha+2)} \\ & D^2[x^{2n+\alpha+2}e^{-2/x}] \\ & + a_{n-1}^{(n)}(n+\alpha+1)D[x^{2n+\alpha}e^{-2/x}] \\ & + a_{n-2}^{(n)}(2n+\alpha-1)(2n+\alpha)x^{2n+\alpha-2}e^{-2/x}\bigg\}, \end{aligned}$$

and

$$x^{\alpha+2}e^{-2/x}Q_n^{(\alpha)}(x) = \frac{1}{(n+\alpha+1)_n}D^{n-2}\{x^{2n+\alpha-2}e^{-2/x}\rho(x;n)\},$$

where $\rho(x;n)$ is a polynomial of degree 2, whose expression is

$$\begin{aligned} \rho(x;n) = & \frac{(n+\alpha+1)(n+\alpha+2)}{(2n+\alpha+1)(2n+\alpha+2)}[(2n+\alpha+1)(2n+\alpha+2)x^2 \\ & \qquad + 4(2n+\alpha+1)x + 4] \\ & + a_{n-1}^{(n)}(n+\alpha+1)[(2n+\alpha)x+2] \\ & + a_{n-2}^{(n)}(2n+\alpha-1)(2n+\alpha). \qquad \square \end{aligned}$$

Finally, we can deduce a second order linear differential equation for the polynomials $\{Q_n^{(\alpha)}(x)\}$, with polynomial coefficients, whose degrees do not depend on n.

Proposition 7.9. *The polynomials* $\{Q_n^{(\alpha)}(x)\}_n$ *satisfy a second order linear differential equation*

(7.27)
$$A_4(x,n)\frac{d^2}{dx^2}Q_n^{(\alpha)}(x) + B_3(x,n)\frac{d}{dx}Q_n^{(\alpha)}(x) + C_2(x,n)Q_n^{(\alpha)}(x) = 0,$$

where $A_4(x,n)$ *is a polynomial of degree* 4*,* $B_3(x,n)$ *is of degree* 3*, and* $C_2(x,n)$ *is of degree* 2*.*

Proof. Taking the structure relation for the polynomials $\{B_i^{(\alpha+2)}\}$,

$$B_{n-2}^{(\alpha+2)}(x) = F_{n-1}x^2\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x) - F_{n-1}(n-1)(x-G_{n-1})B_{n-1}^{(\alpha+2)}(x),$$

where

$$G_{n-1} = \frac{2}{2n+\alpha} \quad \text{and} \quad F_{n-1} = \frac{(2n+\alpha-1)(2n+\alpha)^2}{4(n-1)(n+\alpha+1)}.$$

By substitution in (7.20), we get

$$Q_n^{(\alpha)}(x) = [x-\zeta_n+\xi_nF_{n-1}(n-1)(x-G_{n-1})]B_{n-1}^{(\alpha+2)}(x) \\ - \xi_nF_{n-1}x^2\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x);$$

then

$$Q_n^{(\alpha)}(x) = M_1(x,n)B_{n-1}^{(\alpha+2)}(x) + N_2(x,n)\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x), \tag{7.28}$$

where $M_1(x,n)$ is a polynomial of degree 1 and $N_2(x,n)$ is a polynomial of degree 2. By differentiation and substitution in the differential equation of the Bessel polynomials $\{B_i^{(\alpha+2)}\}$, we deduce that

$$\frac{d}{dx}Q_n^{(\alpha)}(x) = [M_1'(x,n)+\xi_nF_{n-1}(n-1)(n+\alpha+2)]B_{n-1}^{(\alpha+2)} \\ + [M_1(x,n)+N_2'(x,n) \\ + \xi_nF_{n-1}[-2-(\alpha+4)x]]\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x),$$

i.e.,

$$\frac{d}{dx}Q_n^{(\alpha)}(x) = \tilde{M}_0(x,n)B_{n-1}^{(\alpha+2)}(x) + \tilde{N}_1(x,n)\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x), \tag{7.29}$$

where $\tilde{M}_0(x,n)$ is a polynomial of degree 0 and $\tilde{N}_1(x,n)$ is a polynomial of degree 1.

Taking the expressions (7.28) and (7.29), we have a system of equations, whose solution is given by

$$\Delta_2(x,n)B_{n-1}^{(\alpha+2)}(x) = \begin{vmatrix} Q_n^{(\alpha)}(x) & M_2(x,n) \\ (d/dx)Q_n^{(\alpha)}(x) & \tilde{N}_1(x,n) \end{vmatrix}, \tag{7.30}$$

$$\Delta_2(x,n)\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x) = \begin{vmatrix} M_1(x,n) & Q_n^{(\alpha)}(x) \\ \tilde{M}_0(x,n) & (d/dx)Q_n^{(\alpha)}(x) \end{vmatrix}, \tag{7.31}$$

where $\Delta_2(x,n) = M_1(x,n)\tilde{N}_1(x,n) - N_2(x,n)\tilde{M}_0(x,n)$ is a polynomial of degree two. Taking derivatives in (7.30), we get

$$\begin{aligned} \Delta_2'(x,n)B_{n-1}^{(\alpha+2)}(x) &+ \Delta_2(x,n)\frac{d}{dx}B_{n-1}^{(\alpha+2)}(x) \\ &= \tilde{N}_1'(x,n)Q_n^{(\alpha)}(x) \\ &\quad + [\tilde{N}_1(x,n) - N_2'(x,n)]\frac{d}{dx}Q_n^{(\alpha)}(x) - N_2(x,n)\frac{d^2}{dx^2}Q_n^{(\alpha)}(x) \end{aligned}$$

and eliminating with equation (7.31), we obtain

$$\begin{aligned} \Delta_2(x,n)N_2(x,n)\frac{d^2}{dx^2}Q_n^{(\alpha)}(x) + [\Delta_2(x,n)N_2'(x,n) \\ - \Delta_2(x,n)\tilde{N}_1(x,n) + \Delta_2(x,n)M_1(x,n) \\ - \Delta_2'(x,n)N_2(x,n)]\frac{d}{dx}Q_n^{(\alpha)}(x) \\ + [\Delta_2'(x,n)\tilde{N}_1(x,n) - \Delta_2(x,n)\tilde{M}_0(x,n) \\ - \Delta_2(x,n)\tilde{N}_1'(x,n)]Q_n^{(\alpha)}(x) = 0. \qquad \square \end{aligned}$$

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