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STRONG SEMICONTINUITY FOR UNBOUNDED OPERATORS

HYOUNGSOON KIM

ABSTRACT. Let A be a C^*-algebra and A^{**} its enveloping von Neumann algebra. Pedersen and Akemann developed four concepts of lower semi-continuity for elements of A^{**}. Later, Brown suggested using only three classes: strongly lsc, middle lsc and weakly lsc. In this paper we generalize the concept of strong semi-continuity to the case of unbounded operators affiliated with A^{**}. First, we identify our generalized strongly lsc elements with weak* lsc affine $(-\infty, \infty]$-valued functions on $Q(A)$ vanishing at 0, and then we generalize various results of the theory of strong semicontinuity. Also, we discuss some interpolation problems and examples.

1. Introduction. In [**3**], Akemann and G. Pedersen defined four concepts of semi-continuity for elements of A^{**}, the enveloping von Neumann algebra of a C^*-algebra A. Later, L. Brown [**5**] suggested using only three classes $\overline{A^m_{sa}}$, $\tilde{A}^m_{sa}$, and $(\tilde{A}^m_{sa})^-$, and named them *strongly lsc, middle lsc* and *weakly lsc*, respectively. There are also three corresponding concepts of continuity, where "continuous" means "both lower and upper semi-continuous": the strong, respectively middle, weakly, continuous elements are elements in A_{sa}, respectively, $M(A)_{sa}$, $QM(A)_{sa}$. (All these terms are explained in Section 2.) Then L. Brown asked three questions, each of which is three-fold.

(Q1) Is every lsc element the limit of a monotone increasing net of continuous elements?

(Q2) Is every positive lsc element the limit of a monotone increasing net of positive continuous elements?

(Q3) If $h \geq k$, where h is lsc and k is usc, does there exist a continuous x such that $h \geq x \geq k$?

He provided reasonably satisfactory answers and made an extensive study on semi-continuity [**5**].

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As a generalization of the above, this paper considers semi-continuity for unbounded self-adjoint operators affiliated with A^{**} with the above three questions. D. Taylor mentioned this question in [**23**]. In order to deal with unbounded operators, we will use certain Möbius maps and the theory of quadratic forms which was developed by Kato, Robinson, Davies and Simon [**7, 10, 19, 21, 22**].

2. Preliminaries. Throughout this paper, A will denote a (nonunital) C^*-algebra and $Q(A)$ the quasi-state space of A. Equipped with the weak* topology inherited from A^*, $Q(A)$ is a compact convex set. It is well known that the enveloping von Neumann algebra of A can be identified with the second dual of A, so it will be denoted by A^{**}. Let H_u denote the universal Hilbert space of A. For $M \subset A^{**}$, let $\overline{M}$ denote the norm closure of M in $B(H_u)$,

$$M_{sa} = \{x \in M \mid x^* = x\}, \quad \text{and} \quad M_+ = \{x \in M \mid x \geq 0\}.$$

For $M \subset A_{sa}^{**}$, M^m, respectively M_m, denotes the set of limits in A^{**} of monotone increasing nets, respectively monotone decreasing nets, of elements of M. Let $\mathcal{Z}$ denote the center of A^{**}, $\tilde{A}$ the C^*-algebra generated by A and the unit 1 of A^{**}, $M(A)$ the multiplier algebra of A, and $QM(A)$ the quasi-multipliers of A. For operators x and h, $\sigma(x)$ denotes the spectrum of x and $E_S(h)$ the spectral projection of h corresponding to the Borel set $S \subset \mathbf{R}$ (h is self-adjoint).

Now we will briefly introduce the notion of strong semi-continuity for bounded operators. We refer the readers to [**3, 5, 14**] for further theory and applications. It is well known that the evaluation map $\hat{}$ on A_{sa}^{**} given by $\hat{x}(\varphi) = \varphi(x)$ for x in A_{sa}^{**} and φ in $Q(A)$ is an order preserving isometry of A_{sa}^{**} onto the set of bounded affine real-valued functions on $Q(A)$ vanishing at zero. The image of A_{sa} under $\hat{}$ is the set of continuous affine functions on $Q(A)$ vanishing at zero when $Q(A)$ is equipped with weak* topology inherited from A^*.

Definition-Theorem 2.1 (Akemann and Pedersen [**3**, Theorem 2.1]). *For x in A_{sa}^{**}, x is called* strongly lsc *if it satisfies one of the following equivalent conditions.*

(a) $x \in \overline{A_{sa}^m}$.

(b) *$\hat{x}$ is lower semi-continuous on $Q(A)$.*

(c) *There is a bounded, monotone increasing net* $(x_i + \alpha_i 1)$ *in* $\tilde{A}_{sa}$ *with limit* x *such that* $x_i \in A_{sa}$, $\alpha_i \in \mathbf{R}$ *and* $\alpha_i \nearrow 0$.

(d) $x + \varepsilon 1 \in A_{sa}^m$ *for every* $\varepsilon > 0$.

Recall that an operator h on some subspace of H_u is said to be affiliated with A^{**}, written $h\eta A^{**}$, if $uhu^{-1} = h$ for every unitary operator u in the commutant of A^{**}. If h is self-adjoint, it is equivalent to the condition that the spectral projections of h belong to A^{**}. One of the main difficulties with unbounded operators is that they are not everywhere defined. Especially, when we want to find a notion of convergence of unbounded operators there is big trouble since the domains of the operators may not have any common vector. For densely defined self-adjoint operators, it was overcome by making use of the resolvents of the operators. We will generalize this notion to semi-bounded, not necessarily densely defined, self-adjoint operators. Let H be a Hilbert space and $\mathcal{L}_b(H)$ denote the set of bounded below self-adjoint operators on some subspace of H. For h in $\mathcal{L}_b(H)$, $D(h)$ denotes the domain of h and p_h the projection on $\overline{D(h)}$. Then the bounded operator $(\lambda p_h - h)^{-1} \oplus 0(I - p_h)$, $\lambda \notin \sigma(h)$, will be called the *pseudo-resolvent* of h and denoted by $R'_\lambda(h)$.

Definition 2.2. Let h_i and h be in $\mathcal{L}_b(H)$ such that $\alpha \le h_i$, h for some α in $\mathbf{R}$. Then h_i is said to *converge to* h *in the revised strong resolvent sense* (*revised s.r.s.*) if $R'_\lambda(h_i) \to R'_\lambda(h)$ strongly for all λ in $\mathbf{C}\backslash[\alpha, \infty)$.

Remarks. (a) This concept was used (but not named) in [**7**] and [**22**] in the context of monotone increasing nets.

(b) If $\lambda_0 < \alpha$ and if $R'_{\lambda_0}(h_i) \to R'_{\lambda_0}(h)$ strongly, then $h_i \to h$ in the revised s.r.s. This follows the proof of [**18**, Theorem 8.19].

Next we will review some theory of quadratic forms. It is well known that there is one-to-one correspondence between $\mathcal{L}_b(H)$ and the set of closed quadratic forms on H bounded from below. If q_1 and q_2 are closed quadratic forms bounded from below, then so is the sum $q = q_1 + q_2$. Let h, h_1 and h_2 be the self-adjoint operators associated with q, q_1 and q_2, respectively. Then h may be regarded as the sum of h_1 and h_2 in a generalized sense, and we write $h = h_1 \dotplus h_2$. This condition is strictly weaker than the requirement that the ordinary sum $h_1 + h_2$ be well defined. Equipped with the operation $\dotplus$ we can regard the above one-to-one correspondence as an isomorphism.

Definition 2.3. Let q be a quadratic form bounded from below. Define $\tilde{q}$ on H by $\tilde{q}(u) = q(u)$ if $u \in D(q)$ and $\tilde{q}(u) = +\infty$ if $u \notin D(q)$. For q bounded from above, we define $\tilde{q}$ by $\tilde{q}(u) = -(-q)^{\sim}(u)$.

Theorem 2.4 (Davies [**7**] and Kato (see [**21**]), independently). *Let q be a quadratic form bounded from below. Then q is closed if and only if $\tilde{q}$ is lower semi-continuous.*

It is very natural to adopt the order structure for operators bounded below from the ordering in quadratic forms bounded from below. We write $h_1 \leq h_2$ if $\tilde{q}_1 \leq \tilde{q}_2$ where q_i's are the quadratic forms induced from h_i's, respectively.

Notations. (a) Let

$$f_\delta(x) = \frac{x}{1+\delta x} \quad \text{on} \quad \begin{cases} (-1/\delta, \infty) & \text{if } \delta > 0 \\ (-\infty, -1/\delta) & \text{if } \delta < 0. \end{cases}$$

Note that f_δ is operator monotone on its domain such that $f_\delta \cdot f_{-\delta} = f_{-\delta} \cdot f_\delta = \mathrm{id}$ and $f_\delta \cdot f_\varepsilon = f_{\delta+\varepsilon}$ where defined. For a self-adjoint operator h which is bounded below by $\alpha < 0$, $\tilde{f}_\delta(h)$, $0 < \delta < -1/\alpha$, denotes the bounded self-adjoint operator $f_\delta(h) \oplus (1/\delta)(I - p_h)$. For k bounded above by $\beta > 0$, we write $\tilde{f}_{-\delta}(k) = f_{-\delta}(k) \oplus (-1/\delta)(I - p_h)$, $0 < \delta < 1/\beta$.

(b) Let h_i, $i \in I$, and h be in $\mathcal{L}_b(H)$. We will write $h_i \nearrow h$ if (h_i) is monotone increasing such that $h_i \to h$ in the revised s.r.s., or equivalently, $f_\delta(h_i) \nearrow f_\delta(h)$ for $\delta > 0$ with $-1/\delta < h_i$, h.

Proposition 2.5 (Davies [**7**] for (a) and (b)). *Let h_1, h_2 be in $\mathcal{L}_b(H)$ such that $h_1, h_2 \geq \gamma$ (assume that $\gamma \leq 0$) and q_1, q_2 the quadratic forms induced from h_1, h_2, respectively. Then the following are equivalent.*

(a) $h_1 \leq h_2$.

(b) $R'_\lambda(h_1) \leq R'_\lambda(h_2)$ *for* (*one or all*) $\lambda < \gamma$.

(c) $\tilde{f}_\delta(h_1) \leq \tilde{f}_\delta(h_2)$ *for* δ *positive and* $-1/\delta < \gamma$.

Proof. (a) $\Leftrightarrow$ (b). See [**7**, Lemma 2.2].

(b) $\Leftrightarrow$ (c). If $\delta = 1/\lambda$, then $\tilde{f}_\delta(h) = -\lambda 1 + \lambda^2 R'_\lambda(h)$ for $h \geq \gamma$. This implies the result. □

We will end this section with a very useful approximation theorem for semi-bounded quadratic forms (semi-bounded self-adjoint operators). It is a significant strengthening of the theorem in [**10**, p. 459] and was independently discovered by Robinson [**19**], Davies [**7**], Kato (see [**21**]), and Simon [**22**]. (See also Pedersen and Takesai [**16**].) We refer the reader to [**18**, p. 385] for the history.

Proposition 2.6. *If* $q_1 \leq q_2 \leq \cdots$ *is an increasing sequence of closed quadratic forms bounded from below, then the form* q_∞, *where* $D(q_\infty) = \{u \in \cap_n D(q_n) \mid \sup_n q_n(u) < \infty\}$ *and* $q_\infty(u,v) = \lim_{n\to\infty} q_n(u,v)$, *is closed and* $h_n \nearrow h_\infty$ *where* h_i *is the operator associated with* q_i.

Remark. We may replace the sequence q_n by a monotone increasing net q_i, $i \in I$. This provides the existence of the limit of monotone increasing net (h_i) of semi-bounded operators.

3. Unbounded strong semi-continuity. We will generalize the concept of strong semi-continuity from bounded operators to semi-bounded self-adjoint operators. Since we are more interested in densely defined operators, we may restrict our attention to that case. However, because of the way we use the theory of quadratic forms, it is convenient to proceed without the restriction.

First, we extend the isomorphism between A^{**}_{sa} and the Banach space of all real valued bounded affine functions on $Q(A)$ which vanish at zero to the semi-bounded case. Let H_u denote the universal Hilbert space of A,

$$\mathcal{F}_0(A) = \{f : Q(A) \to (-\infty, \infty] \mid f \text{ is bounded below, affine, norm lower semi-continuous, and vanishes at } 0\}, \text{and}$$
$$\mathcal{R}_b(A) = \{h\eta A^{**} \mid h \text{ is a bounded below self-adjoint operator on a subspace of } H_u\}.$$

We are not assuming h to be densely defined. Note that p_h, the projection on $\overline{D(h)}$, belongs to A^{**}.

Definition 3.1. For $h \in \mathcal{R}_b(A)$, define $\hat{h} : Q(A) \to (-\infty, \infty]$ by

$$\hat{h}(\varphi_v) = \begin{cases} ||(h+\lambda 1)^{1/2}v||^2 - \lambda ||v||^2 (= (hv, v)) & \text{if } v \in D([h+\lambda 1]^{1/2}), \\ & h \geq -\lambda \\ \infty & \text{otherwise} \end{cases}$$

where φ_v is the linear functional $a \mapsto (av, v)$ on A. Then $\hat{h}$ is well-defined since $\tilde{f}_\delta(h+\lambda 1) \in A_{sa}^{**}$ and $\tilde{f}_\delta(h+\lambda 1) \nearrow (h+\lambda 1)$ as $\delta \searrow 0$.

Remark. Note that $h \geq k$ if and only if $\hat{h} \geq \hat{k}$ on $Q(A)$. Hence, for (h_i) and h in $\mathcal{R}_b(A)$, $h_i \nearrow h$ if and only if $\hat{h}_i \nearrow \hat{h}$ pointwise on $Q(A)$.

Theorem 3.2. *The map* $\hat{}$ *is an order preserving isomorphism between* $\mathcal{R}_b(A)$ *and* $\mathcal{F}_0(A)$.

Proof. For h in $\mathcal{R}_b(A)$, note that $\tilde{f}_\delta(h)$ is in A_{sa}^{**} for all $\delta > 0$ with $-1/\delta < h$. Since $\tilde{f}_\delta(h) \nearrow h$ as $\delta \searrow 0$, $\hat{h}$ is the limit of an increasing net $((\tilde{f}_\delta(h))^\wedge)$ of bounded, norm continuous affine functions on $Q(A)$ which vanish at zero. Thus $\hat{h}$ is in $\mathcal{F}_0(A)$. Obviously, h and $\hat{h}$ have the same lower bound, and the map $\hat{}$ is injective.

On the other hand, for a given function g in $\mathcal{F}_0(A)$, we define a form q by

$$\tilde{q}(v) = ||v||^2 g(\varphi_{(v/||v||)}), \qquad q(0) = 0, \quad \text{and} \quad q = \tilde{q}|_{D(q)},$$

where $D(q) = \{v \in H_u \mid q(v) < \infty\}$. Then it is easy to see that

$$\tilde{q}(\alpha v) = |\alpha|^2 \tilde{q}(v),$$

and

$$\tilde{q}(v+w)+\tilde{q}(v-w)=2\tilde{q}(v)+2\tilde{q}(w), \quad \forall\, v,w\in H_u.$$

Hence $D(q)$ is a vector space and q is a quadratic form (see Kurepa [**12**]). By the definition of q, g and q have the same lower bound and $\tilde{q}$ is lower semi-continuous on H_u. Theorem 2.4 implies that q is a closed quadratic form bounded from below, and hence there exists an operator h in $\mathcal{L}_b(H_u)$ such that $q(v)=(hv,v)$. Now we need to show that $h\eta A^{**}$. Since $\varphi_v=\varphi_{uv}$ for each unitary operator u in A', $\tilde{q}(uv)=g(\varphi_{uv})=g(\varphi_v)=\tilde{q}(v)$ for all unit vectors v. This implies that $u^{-1}hu=h$, and hence $h\eta A^{**}$. Therefore, $\hat{}$ is a surjective map. Moreover, it preserves addition when $\mathcal{R}_b(A)$ is equipped with the formal sum $\dot{+}$ induced by form sum. □

Remark-Notation. If (h_i) is a monotone increasing net in $\mathcal{R}_b(A)$, then the above theorem and Proposition 2.6 show that there exists h in $\mathcal{R}_b(A)$ such that $h_i\nearrow h$. For $U\subset A^{**}_{sa}$, we denote by U^M the set of limits of monotone increasing nets of elements in U. We will use "+" for the formal sum, $\dot{+}$ for operators in $\mathcal{R}_b(A)$.

Definition 3.3. For h in $\mathcal{R}_b(A)$, h is called *unbounded strongly lsc*, denoted by $h\in \mathrm{SLSC}(A)$, if there exists a monotone increasing net (h_i) in $\tilde{A}_{sa}, h_i=a_i+\lambda_i 1$, such that $h_i\nearrow h$ and $\lambda_i\nearrow 0$. h is called *unbounded strongly usc*, denoted by $h\in \mathrm{SUSC}(A)$, if $-h$ is in $\mathrm{SLSC}(A)$. As a special class of $\mathrm{SLSC}(A)$, respectively $\mathrm{SUSC}(A)$, we denote by $\mathrm{SLSC}^d(A)$, respectively $\mathrm{SUSC}^d(A)$, the set of all densely defined h in $\mathrm{SLSC}(A)$, respectively $\mathrm{SUSC}(A)$.

Theorem 3.4. *The map $\hat{}$ is an order preserving isomorphism between* $\mathrm{SLSC}(A)$ *and* $\mathcal{F}_1(A):=\{f\in\mathcal{F}_0(A)\mid f$ *is weak* lower semicontinuous*$\}$.

Proof. Let $h\in\mathrm{SLSC}(A)$. Then there exists a net (h_i), $h_i=a_i+\lambda_i 1$ in $\tilde{A}_{sa}$ such that $h_i\nearrow h$, $\lambda_i\nearrow 0$. Hence $\hat{h}$ is the limit of the increasing net $(\hat{a}_i+\lambda_i)$ of weak* continuous affine functions on $Q(A)$. This implies that $\hat{h}$ is weak* lower semicontinuous.

If g is in $\mathcal{F}_1(A)$, then there exists a net (g_i) of weak* continuous affine real valued functions on $Q(A)$ such that $g_i\nearrow g$ pointwise by Alfsen [**4**, Corollary I.1.4]. Since $g_i - g_i(0)$ is continuous on $Q(A)$ and vanishes at zero, there exists a_i in A_{sa} such that $\hat{a}_i = g_i - g_i(0)$ for all i. Let $h_i = a_i + g_i(0)1$. Then

$$\hat{h}_i = \hat{a}_i + g_i(0)\hat{1} = [g_i - g_i(0) + g_i(0)\hat{1}] \nearrow g.$$

Thus, $h_i \nearrow h$ where $\hat{h} = g$. □

Theorem 3.5. *The map* $\hat{}$ *is an order preserving isomorphism between* $\mathrm{SLSC}^d(A)$ *and* $\mathcal{F}_1^d(A) := \{f \in \mathcal{F}_1(A) \mid f^{-1}(\mathbf{R})$ *is norm dense in* $Q(A)\}$.

Proof. Since the map $v \mapsto \varphi_v$ is norm continuous on the closed unit ball of H_u, $\hat{}$ maps $\mathrm{SLSC}^d(A)$ into $\mathcal{F}_1^d(A)$. It remains to show that h is densely defined when $\hat{h} = f$ for a given f in $\mathcal{F}_1^d(A)$. Shifting by a constant, we may assume that $h \geq 0$. Then it suffices to show that $D(h^{1/2})$ is dense in H_u. Let p be the projection on $\overline{D(h^{1/2})}$, and let $p\varphi p$ denote the linear functional $a \mapsto \varphi(pap)$ on A. Then

$$\begin{aligned} pQ(A)p &= \{p\varphi p \mid \varphi \text{ is in } Q(A)\} \\ &\supset \{\varphi_{pv} \mid v \in D(h^{1/2}), ||v|| \leq 1\} \\ &= \{\varphi_v \mid f(\varphi_v) < \infty\} \\ &= f^{-1}(\mathbf{R}). \end{aligned}$$

Hence $pQ(A)p$ is dense in $Q(A)$. Since $pQ(A)p$ is norm closed (cf. Effros [**9**] and Prosser [**17**]), $pQ(A)p = Q(A)$ and hence $p = 1$. □

Theorem 3.6. *Let* $h \in \mathcal{R}_b(A)$. *The following conditions are equivalent.*

(a) $h \in \mathrm{SLSC}(A)$.

(b) $\hat{h} \in \mathcal{F}_1(A)$.

(c) $h + \varepsilon 1 \in A_{sa}^M$ *for all* $\varepsilon > 0$.

(d) $\tilde{f}_\delta(h) \in \overline{A_{sa}^m}$ *for one* (*all*) $\delta > 0$ *with* $-1/\delta < h$.

(e) *There exists a sequence* (*net*) (h_i) *in* $\overline{A_{sa}^m}$ *such that* $h_i \nearrow h$.

Moreover, if h *satisfies* (a)–(e) *and* $h \geq 0$, *then* $h + \varepsilon 1 \in A_+^M$ *for all* $\varepsilon > 0$.

Proof. (a) $\Leftrightarrow$ (b) is proved in Theorem 3.4.

(a) $\Rightarrow$ (c). This follows the proof (iii) $\Rightarrow$ (iv) of [**3**, Theorem 2.1].

(c) $\Rightarrow$ (b). $\hat{h}^{-1}(\alpha,\infty] = \cup_{\varepsilon>0}[(h+\varepsilon 1)^\wedge]^{-1}(\alpha+\varepsilon,\infty]$ is open in $Q(A)$, for all α in $\mathbf{R}$, since each $(h+\varepsilon 1)^\wedge$ is lower semi-continuous on $Q(A)$.

(a) $\Rightarrow$ (d). Let $h_i = a_i + \lambda_i 1$ be in $\tilde{A}_{sa}$ such that $h_i \nearrow h$ and $\lambda_i \nearrow 0$. Fix an i_0 and let $\alpha \leq 1/(||h_{i_0}|| + 1)$. Then $h_i > -1/\alpha$ for all $i \geq i_0$, and hence $f_\alpha(h_i) \in \tilde{A}_{sa}$ and $f_\alpha(h_i) \nearrow \tilde{f}_\alpha(h)$ as $i \nearrow \infty$. Since $f_\alpha(h_i) \in f_\alpha(\lambda_i) + A_{sa}$ and $f_\alpha(\lambda_i) \nearrow 0$, we get $\tilde{f}_\alpha(h) \in \overline{A^m_{sa}}$ by [**3**, Theorem 2.1]. Note that $\tilde{f}_\delta(h) = f_{\delta-\alpha}(\tilde{f}_\alpha(h)) \in \overline{A^m_{sa}}$ whenever $-1/\delta < h$ and $\delta > 0$ by [**5**, Propositions 2.30, 2.31].

(d) $\Rightarrow$ (e). Let $\tilde{f}_\delta(h) \in \overline{A^m_{sa}}$ for some $\delta > 0$. Then for any positive $\varepsilon < \delta$, $f_{-\varepsilon}(\tilde{f}_\delta(h)) = \tilde{f}_{\delta-\varepsilon}(h) \in \overline{A^m_{sa}}$ by [**5**, Proposition 2.30], and $\tilde{f}_{\delta-\varepsilon}(h) \nearrow h$ as $\varepsilon \nearrow \delta$. Choose a positive sequence (ε_n) such that $\varepsilon_n \nearrow \delta$. Then $\tilde{f}_{\delta-\varepsilon_n}(h) \nearrow h$.

(e) $\Rightarrow$ (b). $\hat{h}$ is the limit of a monotone increasing net $(\hat{h}_i)$ of lower semi-continuous functions on $Q(A)$. Thus, $\hat{h}$ is lower semi-continuous.

For the last statement we follow the proof of [**3**, Proposition 2.2]. □

Remarks. (a) $\overline{A^m_{sa}} = \{h \in \mathrm{SLSC}^d(A) \mid h \text{ is bounded}\}$.

(b) $A_{sa} = \mathrm{SLSC}^d(A) \cap \mathrm{SUSC}^d(A)$. So the concept of strong continuity is still the same as in the bounded case for densely defined operators.

Corollary 3.7. *If h is in* SLSC(A)*, then there exists an element a in A_{sa} such that $a \leq h$.*

Proof. By [**5**, Corollary 3.22] there exists $a \in A_{sa}$ such that $a \leq \tilde{f}_\delta(h) \leq h$. □

Corollary 3.8. *If h is in SLSC$(A)_+$, then there exists a sequence (h_n) in $\overline{A^m_+}$ such that $h_n \nearrow h$.*

Proof. If $h \in \mathrm{SLSC}(A)_+$, then $\tilde{f}_\delta(h) \in \overline{A^m_{sa}}$ for any fixed $\delta > 0$. Then

$f_{-\varepsilon_n}(\tilde{f}_\delta(h))$ is in $\overline{A_+^m}$ for all positive $\varepsilon_n \le \delta$ by [**5**, Proposition 2.31] and $f_{-\varepsilon_n}(\tilde{f}_\delta(h)) = \tilde{f}_{\delta-\varepsilon_n}(h) \nearrow h$ as $\varepsilon_n \nearrow \delta$. ◻

Corollary 3.9. $\mathrm{SLSC}(A)^M = \mathrm{SLSC}(A)$.

Proof. Lower semicontinuity (for functions on $Q(A)$) is preserved under monotone increasing limits. ◻

Corollary 3.10. *Let h be in $\overline{A_{sa}^m}$ and $h \le 1/\varepsilon$, $\varepsilon > 0$. Then $f_{-\varepsilon}(h) \in \mathrm{SLSC}(A)$. (Note that $f_{-\varepsilon}(h)$ is a self-adjoint operator in $\mathcal{R}_b(A)$ such that the closure of its domain is the range of the spectral projection $E_{(-\infty,1/\varepsilon)}(h)$ of h.)*

Proof. By [**5**, Proposition 2.31], $f_{-\delta}(h) \in \overline{A_{sa}^m}$ for all positive $\delta < \varepsilon$. Since $f_{-\delta}(h) \nearrow f_{-\varepsilon}(h)$ as $\delta \nearrow \varepsilon$, $f_{-\varepsilon}(h)$ satisfies (e) of Theorem 3.6. ◻

If A_i is a C^*-algebra, for all $i \in I$, then by the c_0-direct sum of the A_i's we mean the C^*-algebra of functions f on I such that $f(i) \in A_i$ and $||f(i)|| \to 0$ as $i \to \infty$. If A is a c_0-direct sum of the A_i, then A^{**} is the l_∞-direct sum of the A_i^{**}'s.

Proposition 3.11. *Let A be the c_0-direct sum of C^*-algebras A_i, $i \in I$, and let $h = \oplus_{i\in I} h_i$ be in $\mathcal{R}_b(A)$. Then*

$$h \in \mathrm{SLSC}(A) \iff h_i \in \mathrm{SLSC}(A_i), \quad \forall\, i \in I \text{ and } \forall\, \varepsilon > 0,\ h_i \ge -\varepsilon \text{ for all but finitely many } i \text{ in } I.$$

Proof. ($\Rightarrow$). This follows from Theorem 3.6 and [**5**, Proposition 2.11].

($\Leftarrow$). For given $\varepsilon > 0$, we will show that $h + \varepsilon 1 \in A_{sa}^M$. By the given condition, $h_i \ge -\varepsilon/2$ for all i except for $i_1, \dots, i_n$. Theorem 4.6 implies that $h_i + \varepsilon 1 \in (A_i)_+^M$ for all i except for $i_1, \dots, i_n$. For each i, choose a net (k_j^i) in $(A_i)_{sa}$ such that $k_j^i \nearrow h_i + \varepsilon 1$ as $j \to \infty$, and $k_j^i \ge 0$ if $i \ne i_1, \dots, i_n$. Now consider the collection of finite sets $F = \{(i, k_j^i)\}$

such that all of $i_1, \ldots, i_n$ appear and each i appears only one time. Then it forms a directed set by the order

$$F_1 \leq F_2 \quad \Leftrightarrow \quad \text{if } (i, k^i_j) \in F_1, \quad \text{then } \exists\, (i, k^i_{j'}) \in F_2 \text{ s.t. } j' \geq j.$$

Let $x_F = \oplus_F k^i_j$. Then x_F is in A_{sa} and $x_F \nearrow h + \varepsilon 1$. □

Proposition 3.12. *Let I be an ideal of A with open central projection z. Then*

$$h \in \mathrm{SLSC}(A)_+ \Longrightarrow zh \in \mathrm{SLSC}(A)_+, \qquad zh \in \mathrm{SLSC}(I)_+.$$

Proof. Combine Theorem 3.6 and [**5**, Proposition 2.18]. □

Proposition 3.13. *Let A be a C^*-algebra with* Prim A *Hausdorff, and let I and J be ideals of A with open central projections z and w, respectively, such that $A = I + J$, and $h \in \mathcal{R}_b(A)$. Then*

$$zh \in \mathrm{SLSC}(I) \quad \text{and} \quad wh \in \mathrm{SLSC}(J) \quad \Longrightarrow \quad h \in \mathrm{SLSC}(A).$$

Proof. Combine Theorem 3.6 and [**5**, Proposition 2.25]. □

The following theorem gives affirmative answers to (Q1) and (Q2) for separable C^*-algebras like those in the bounded case.

Theorem 3.14. *Let A be a separable C^*-algebra. Then*

(a) $h \in \mathrm{SLSC}(A)_+$ *implies that there exists a sequence (h_n) in A_+ such that $h_n \nearrow h$.*

(b) $h \in \mathrm{SLSC}(A)$ *implies that there exists a sequence (h_n) in A_{sa} such that $h_n \nearrow h$.*

Proof. (a) Let $h \in \mathrm{SLSC}(A)_+$. Then $\tilde{f}_\delta(h) \in \overline{A^m_+}$ for any fixed $\delta > 0$. Since $\overline{A^m_+} = A^\sigma_+$, we can choose a sequence (a_n) in A_+ such that

$a_n \nearrow \tilde{f}_\delta(h)$. Let (ε_n) be a sequence such that $0 < \varepsilon_n \nearrow \delta$, $\varepsilon_n < \delta$. Since $0 \le a_n \le 1/\delta < 1/\varepsilon_n$, $f_{-\varepsilon_n}(a_n)$ is in A_+ by [**5**, Proposition 2.30]. Moreover, $(f_{-\varepsilon_n}(a_n))$ increases to $f_{-\delta}(\tilde{f}_\delta(h)) = h$ as $n \nearrow \infty$.

(b) Combine Corollary 3.7 and part (a). □

Theorem 3.15. *For an arbitrary C^*-algebra A, let $h \in SLSC(A)_+$. Then there exists a net (a_i) in A_+ such that $0 \le a_i \le h$, $a_i \to h$ in the revised s.r.s, and for all $\lambda > 0$, for all $c \in A_{sa}$ such that $c \le h$, $c \le a_i + \lambda 1$ for i sufficiently large.*

Proof. Fix $\delta > 0$ so that $\tilde{f}_\delta(h) \in \overline{A_+^m}$. By [**5**, Theorem 3.24], there exists a net (b_α) in A_{sa} such that $0 \le b_\alpha \le \tilde{f}_\delta(h)$ and $b_\alpha \to \tilde{f}_\delta(h)$ strongly, and for all $\eta > 0$, for all $c \in A$ such that $c \le \tilde{f}_\delta(h)$, $c \le b_\alpha + \eta$ for α sufficiently large. Let $I = \{(\tau, \alpha) \mid 0 < \tau < \delta\}$ and define a partial order on I by

$$(\tau, \alpha) \le (\tau', \alpha') \quad \Longleftrightarrow \quad \alpha < \alpha' \quad \text{and} \quad \tau \le \tau'.$$

Then I is a directed set. Let $a_i = f_{-\tau}(b_\alpha)$, $i = (\tau, \alpha) \in I$. Then a_i is in A_+ since $\tau < \delta$. We claim that $(a_i)_I$ satisfies all the conditions. Clearly $f_\delta(a_i) = f_{\delta-\tau}(b_\alpha) \to \tilde{f}_\delta(h)$ strongly as $i \to \infty$. Since

$$f_\delta(a_i) = \frac{1}{\delta}1 + \frac{1}{\delta^2}R_{-1/\delta}(a_i) \quad \text{and} \quad \tilde{f}_\delta(h) = \frac{1}{\delta}1 + \frac{1}{\delta^2}R'_{-1/\delta}(h),$$

$R_{-1/\delta}(a_i) \to R'_{-1/\delta}(h)$ strongly. Therefore, $a_i \to h$ in the revised s.r.s.

Let $\lambda > 0$, $c \in A_{sa}$ such that $c \le h$. Choose a $\delta_0 > 0$ small enough such that $f_{\delta_0}(c) \ge c - (\lambda/2)1$ and $\delta_0 < \delta$. A little computation shows that if $\delta - \delta_0 < \tau_0 < \delta$, then there exists a small $\eta > 0$ such that $1/\delta + \eta < 1/(\delta - \delta_0)$ and $f_{-(\delta-\delta_0)}(t + \eta) \le f_{-\tau_0}(t) + \lambda/2$ for all $t \in [0, 1/\tau_0)$. Since $f_\delta(c) \le \tilde{f}_\delta(h)$ (by Proposition 2.5), there exists $\alpha_0 = \alpha_0(\eta, f_\delta(c))$ such that $f_\delta(c) \le b_\alpha + \eta 1$ if $\alpha \ge \alpha_0$. Hence, we have

$$f_{-(\delta-\delta_0)}(b_\alpha + \eta 1) \ge f_{-(\delta-\delta_0)}(f_\delta(c)) = f_{\delta_0}(c) \ge c - \frac{\lambda}{2}1.$$

Let $i_0 = (\tau_0, \alpha_0)$; then, for $i = (\tau, \alpha) \ge i_0$,

$$\begin{aligned} a_i &= f_{-\tau}(b_\alpha) \ge f_{-\tau_0}(b_\alpha) \\ &\ge f_{-(\delta-\delta_0)}(b_\alpha + \eta 1) - \frac{\lambda}{2}1 \\ &\ge c - \frac{\lambda}{2}1 - \frac{\lambda}{2}1 = c - \lambda 1. \qquad \square \end{aligned}$$

Corollary 3.16. *Let $h \in \mathrm{SLSC}(A)$. Then there exists a net (a_i) in A_{sa} such that $a_i \leq h$, $a_i \to h$ in the revised s.r.s. and for all $\lambda > 0$, for all $c \in A_{sa}$ such that $c \leq h$, $c \leq a_i + \lambda 1$ for i sufficiently large.*

Proof. Combine Theorem 3.15 and Corollary 3.7. □

Corollary 3.17. *Let $h \in \mathrm{SLSC}(A)_)$. Then there exists a net $(b_i + \lambda 1)$ in $\tilde{A}_{sa}$ such that $b_i \in A_+$, $\lambda_i \nearrow 0$, and $b_i + \lambda 1 \nearrow h$.*

Proof. Assume that (a_j) in A_+ such that $0 \leq a_j \leq h$, $a_j \to h$ in the revised s.r.s., and for all $\lambda > 0$, $\forall\, c \in A_{sa}$ such that $c \leq h$, $c \leq a_j + \lambda 1$ for j sufficiently large. Let $I = \{(j, \varepsilon) \mid \varepsilon > 0\}$ and define a partial order on I by

$$(j, \varepsilon) \leq (j', \varepsilon') \quad \Longleftrightarrow \quad j \leq j', \quad \varepsilon \geq \varepsilon' \quad \text{and} \quad a_j - \varepsilon 1 \leq a_{j'} - \varepsilon' 1.$$

Then I is a directed set. For $i = (j, \varepsilon)$, let $b_i = a_j$ and $\lambda_i = -\varepsilon$. Then $b_i + \lambda_i 1 = a_j - \varepsilon 1 \nearrow h$. □

The following theorem is an analogue of [**3**, Proposition 3.5] and [**5**, Proposition 2.1] that justifies the definition of $\mathrm{SLSC}(A)$ in some sense.

Theorem 3.18. *Let $0 \leq h \in A_{sa}^{**}$. Then*

(a) $h^{-1} \in \mathrm{SLSC}(A) \Leftrightarrow h \in [(\tilde{A}_{sa})_m]^-$.

(b) $h \in (\tilde{A}_{sa})_m \Leftrightarrow$ *there exists a $\delta > 0$ such that $h^{-1} - \delta 1 \in \mathrm{SLSC}(A)$.*

Proof. (a). Choose, by Corollary 3.17, a net $h_i = a_i + \lambda_i 1$, $i \in I$, in $\tilde{A}_{sa}$ such that $a_i \in A_+$, $\lambda_i \nearrow 0$, and $h_i \nearrow h$, and let $p = E_{(0,\infty)}(h)$. Note that p is the projection on $\overline{D(h^{-1})}$. For $\varepsilon > 0$ fixed, $h_i + \varepsilon 1 \nearrow h^{-1} + \varepsilon p (= h^{-1} + \varepsilon 1)$. Since $\varepsilon + \lambda_i > 0$ for i sufficiently large, $h_i + \varepsilon 1 = a_i + (\varepsilon + \lambda_i) 1 \geq (\varepsilon + \lambda_i) > 0$ and $(h_i + \varepsilon 1)^{-1} \in \tilde{A}_{sa}$. Therefore, we have

$$(h_i + \varepsilon 1)^{-1} \searrow (h^{-1} + \varepsilon p)^{-1} \oplus 0(1 - p) \in (\tilde{A}_{sa})_m.$$

Since $(h^{-1} + \varepsilon p)^{-1} \oplus 0(1 - p) \to h$ in norm, $h \in [(\tilde{A}_{sa})_m]^-$.

For the converse, let $h \in [(\tilde{A}_{sa})_m]^-$ and $h_n = h + (1/n)1$. Then $h_n \searrow h$ as $n \nearrow \infty$. By [**3**, Proposition 3.5] (cf. [**5**, Proposition 2.1 (a)]), $h_n^{-1} \in \overline{A_+^m}$. Since $h_n^{-1} \nearrow h^{-1}$, $h^{-1} \in \mathrm{SLSC}(A)$ by Theorem 3.6.

(b) If $h \in (\tilde{A}_{sa})_m$, then there exists a net $h_i = a_i + \lambda_i 1$, $i \in I$, in $\tilde{A}_{sa}$ such that $h_i \searrow h$ and $\lambda_i \searrow \lambda$, $\lambda > 0$. Here we may assume that $h_i \geq \delta_i > 0$ for some δ_i for all $i \in I$. Then $h^{-1} \in \lambda_i^{-1} + A_{sa}$ and $h_i^{-1} \nearrow h^{-1}$. Choose $0 < \delta < \lambda_{i_0}^{-1}$. Then $h_i^{-1} - \delta 1 \nearrow h^{-1} - \delta 1$ and $h_i^{-1} - \delta 1 \in (\lambda_i^{-1} - \delta)1 + A_{sa}$. Since $\lambda_i^{-1} - \delta > 0$ for $i > i_0$, $h_i^{-1} - \delta 1 \in \overline{A_{sa}^m}$. By Theorem 3.6, this implies that $h^{-1} - \delta 1 \in \mathrm{SLSC}(A)$.

For the converse, we may assume that δ is small enough that $h^{-1} - \delta 1$ is still positive. Then $h^{-1} - (\delta/2)1 \in A_+^M$ by Theorem 3.6. If $b_i \nearrow h^{-1} - (\delta/2)1$, $b_i \in A_+$, then $b_i + (\delta/2)1 \nearrow h^{-1}$. Therefore $(b_i + (\delta/2)1)^{-1} \searrow h$. ◻

Theorem 3.19. *If (I_α) is an increasing net of ideals with open central projections z_α such that $A = (\cup I_\alpha)^-$, then*

$$h \in \mathrm{SLSC}(A)_+ \iff z_\alpha h \in \mathrm{SLSC}(I_\alpha)_+, \quad \textit{for all } \alpha.$$

Proof. If $h \in \mathrm{SLSC}(A)_+$, then $\tilde{f}_\delta(h) \in \overline{A_+^m}$ for $\delta > 0$. By [**5**, Proposition 2.24], $z_\alpha \tilde{f}_\delta(h) \in \overline{(I_\alpha)_+^m}$ for all α. Then $z_\alpha h = f_{-\delta}(z_\alpha \tilde{f}_\delta(h)) \in \mathrm{SLSC}(I_\alpha)_+$ by Corollary 3.10.

For the converse, fix $\delta > 0$. If $z_\alpha h \in \mathrm{SLSC}(I_\alpha)_+$, then there exists a net $(b_i + \lambda_i z_\alpha)$ in $(\tilde{I}_\alpha)_{sa}$, $b_i \in (I_\alpha)_+$, $\lambda_i \nearrow 0$ such that $b_i + \lambda_i z_\alpha \nearrow z_\alpha h$ by Corollary 3.17. Thus, $f_\delta(b_i + \lambda_i z_\alpha) \in f_\delta(\lambda_i) z_\alpha + (I_\alpha)_{sa}$ whenever $\lambda_i > -\delta^{-1}$ and $f_\delta(b_i + \lambda_i z_\alpha) \nearrow \tilde{f}_\delta(z_\alpha h)$ as $i \nearrow \infty$. Since $f_\delta(\lambda_i) \nearrow 0$, $\tilde{f}_\delta(z_\alpha h) \in \overline{(I_\alpha)_+^m}$ for all α. By [**5**, Proposition 2.24], this implies $\tilde{f}_\delta(h) \in \overline{A_+^m}$ and hence $h \in \mathrm{SLSC}(A)_+$. ◻

Before we end this section, we check the relation between strong semi-continuity and strong q-semi-continuity. Recall that a projection $p \in A^{**}$ is called *open* if it is strongly (or weakly) lsc, *closed* if $1 - p$ is open, and *compact* if p is closed and $p \leq a$ for some a in A. We refer to [**5**, p. 905] for the history of q-semi-continuity.

Definition 3.20. Assume $h\eta A^{**}$ and h is self-adjoint on a subspace of H_u.

(a) h is called q-LSC if $[E_{(t,\infty)}(h)+(1-p)]$ is open for all $t \in \mathbf{R}$.

(b) h is called *strongly* q-LSC if $-h$ is q-LSC and $E_{(-\infty,-\varepsilon]}(h)$ is compact for all $\varepsilon > 0$.

(c) h is called *strongly* q-USC if $-h$ is strongly q-LSC.

Remark. The following lemma shows that strongly q-LSC implies bounded below.

Lemma 3.21. *h is strongly q-LSC implies that h is bounded below.*

Proof. Let $q_n = E_{(-\infty,-n]}(h)$. Then (q_n) is a decreasing sequence of compact projections and $\wedge_n q_n = 0$. By Akemann [**1**, Theorem 2.10], this implies that $q_{n_0} = 0$ for n_0 sufficiently large. Therefore, h is bounded below. □

Theorem 3.22. *h is strongly q-LSC implies that $h \in \mathrm{SLSC}(A)$.*

Proof. By the above lemma, we may assume $h \geq -n$ for some n. Then for $0 < \delta < 1/n$, $\tilde{f}_\delta(h)$ is strongly q-lsc. Therefore, $\tilde{f}_\delta(h) \in \overline{A_{sa}^m}$ by [**5**, Proposition 2.50], and so h belongs to SLSC(A). □

Theorem 3.23. *Let $h \in \mathrm{SLSC}(A)$ and $h\eta\mathcal{Z}$. Then h is strongly q-LSC.*

Proof. Combine Theorem 3.6 and [**5**, Proposition 2.55]. □

4. Interpolation theorems and examples. For the commutative C^*-algebra $A = C_0(X)$, where X is a locally compact Hausdorff space, the strong interpolation problem can be reduced to the bounded case easily, so that we can give an affirmative answer to (Q3). Unlike the commutative case, we have a noncommutative counterexample for this problem. But, under some extra hypotheses, we still get positive results.

Example. Let $A = c_0 \otimes M_2$. Then $A^{**} \cong l_\infty \otimes M_2$. Let $h = (h_n)$ and $k = (k_n)$ where

$$h_n = \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} -\beta_n & 0 \\ 0 & \alpha_n \end{pmatrix} \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix},$$

$$k_n = \begin{pmatrix} -\alpha_n & 0 \\ 0 & \beta_n \end{pmatrix},$$

$$1 \le \alpha_n \nearrow \infty, \quad 1 \ge \beta_n \searrow 0, \qquad \theta_n = \cos^{-1}\left(\frac{2\sqrt{\alpha_n\beta_n}}{\alpha_n + \beta_n}\right).$$

Then $h_n \ge k_n$ for all n, and so $h \ge k$. By Proposition 3.11, we can see that $h \in \mathrm{SLSC}(A)$ and $k \in \mathrm{SUSC}(A)$. Now, if a_n exists in $(M_2)_{sa}$ such that $h_n \ge a_n \ge k_n$, then a little computation shows that $||a_n|| \ge \sqrt{\alpha_n\beta_n}$. Therefore, (a_n) cannot be in A_{sa} unless $\sqrt{\alpha_n\beta_n} \to 0$.

Theorem 4.1. (a) *Let $k \in [(A_{sa})_m]^-$ and $h \in \mathrm{SLSC}(A)$ such that $k \le h$. Then there exists $a \in A_{sa}$ such that $k \le a \le h$.*

(b) *Let $k \in SUSC(A)$ and $h \in \overline{A_{sa}^m}$ such that $k \le h$. Then there exists $a \in A_{sa}$ such that $k \le a \le h$.*

Proof. (a) By Theorem 3.6 above, there exists $n_0 \in \mathbf{N}$ such that $\tilde{f}_{1/n}(h) \in \overline{A_{sa}^m}$ for all $n \ge n_0$. And we also have $f_{1/n}(k) \in [(A_{sa})_m]^-$ for all n such that $-n \le k - 1$ by [**5**, Proposition 2.30]. Proposition 2.5 implies $f_{1/n}(k) \le \tilde{f}_{1/n}(h)$ for n such that $n \ge n_0$ and $-n \le k-1$. We may assume that $-n_0 \le k-1$. Let B be the c_0-direct sum of countably many copies of A, $B = \oplus_{n=n_0}^\infty A_n$, and let

$$\tilde{k} = (\tilde{k}_n) = \left(\frac{1}{n} f_{1/n}(k)\right)_{n=n_0}^\infty$$

and

$$\tilde{h} = (\tilde{h}_n) = \left(\frac{1}{n} \tilde{f}_{1/n}(h)\right)_{n=n_0}^\infty.$$

Then by Proposition 3.11 above, $\tilde{k} \in [(B_{sa})_m]^-$ and $\tilde{h} \in \overline{B_{sa}^m}$. Since $\tilde{k}_n \le \tilde{h}_n$ for all $n \ge n_0$, $\tilde{k} \le \tilde{h}$ so that we can apply the strong interpolation theorem [**5**, Corollary 3.16]. Therefore there exists a

$b \in B_{sa}$ such that $\tilde{k} \le b \le \tilde{h}$, that is, there exists $b_n \in A_{sa}$ for all $n \ge n_0$, such that $||b_n|| \to 0$ and $(1/n)f_{1/n}(k) \le b_n \le (1/n)\tilde{f}_{1/n}(h)$. We fix an integer $m \ge n_0$ such that $||b_m|| < 1/2$. Then $f_{1/m}(k) \le mb_m \le \tilde{f}_{1/m}(h)$ and $||mb_m|| < m/2$. Hence $f_{-1/m}(mb_m)$ is in A_{sa} and $k \le f_{-1/m}(mb_m) \le h$.

(b) Apply part (a) to $-h \le -k$. □

Theorem 4.2. (a) *Let $k \in$ SUSC(A) and $h \in$ SLSC(A) such that $k \le h$. Then for all $\varepsilon > 0$, there exists an $a \in A_{sa}$ such that $k - \varepsilon 1 \le a \le h$.*

(b) *Let $k \in SUSC(A)$ and $h \in SLSC(A)$ such that $k \le h$. Then for all $\varepsilon > 0$, there exists an $a \in A_{sa}$ such that $k \le a \le h + \varepsilon 1$.*

Proof. (a) Since $-k \in$ SLSC(A), $\tilde{f}_{1/n}(-k) \in \overline{A_{sa}^m}$ for sufficiently large n. Note that $\tilde{f}_{1/n}(-k) \nearrow (-k)$ as $n \to \infty$. Hence, we have

$$h + \tilde{f}_{1/n}(-k) \nearrow (h-k) \ge 0, \quad \text{and} \quad h + \tilde{f}_{1/n}(-k) \in \mathrm{SLSC}(A).$$

Dini theorem (for lsc functions on $Q(A)$) implies that $-\varepsilon \le h + \tilde{f}_{1/n}(-k)$ for sufficiently large n. Applying the previous theorem for $-\tilde{f}_{1/n}(-k) - \varepsilon 1$ and h, we can find a in A_{sa} such that $k - \varepsilon 1 \le -\tilde{f}_{1/n}(-k) - \varepsilon 1 \le a \le h$.

(b) Apply part (a) to $-h \le -k$. □

Definition 4.3 (cf. [**5**, Definition–Lemma 3.39]). Let h and k be self-adjoint operators on some subspaces of H_u such that $h, k \eta A^{**}$. Then we write $h \overset{q}{\ge} k$ if and only if $E_{(-\infty,s]}(h) \cdot E_{[t,\infty)}(k) = 0$ for all $s, t \in \mathbf{R}$ such that $s < t$.

Remark. Note that $h \overset{q}{\ge} k$ implies $h \ge k$.

Theorem 4.4. *Assume $h \overset{q}{\ge} k$. If either h is strongly q-LSC and k is in* SUSC(A) *or k is strongly q*-USC *and h is in* SLSC(A)*, then there exists an a in A_{sa} such that $k \le a \le h$.*

Proof. Assume that h is strongly q-LSC, $k \in \mathrm{SUSC}(A)$ and $h \overset{q}{\geq} k$. Consider the function g given by $g(t) = t$ if $t \in (-\infty, ||k_+||)$, and $g(t) = ||k_+||$ otherwise. Let $\tilde{g}(h) = g(h) \oplus ||k_+||(1 - p_h)$. Then $g(k) = k$, $\tilde{g}(h)$ is strongly q-lsc, and $\tilde{g}(h) \overset{q}{\geq} g(k)$. Therefore, $\tilde{g}(h) \geq g(k)$ and we can apply Theorem 4.1. Then there exists $a \in A_{sa}$ such that $k = g(k) \leq a \leq \tilde{g}(h) \leq h$. The other case follows by a similar argument. $\square$

Example A. $C_0(X)$. Let $A = C_0(X)$ where X is a locally compact Hausdorff space. Since each bounded strongly lsc element is determined completely by its atomic part (Pedersen [**15**, Theorem 4.3.15]) so is every element in $\mathrm{SLSC}(A)$ by Theorem 3.6. Therefore, it is enough to consider $z_{at}A^{**}$. Note that the atomic part, $z_{at}A^{**}$ can be identified with $L^\infty(X)$, the set of bounded functions on X. The diffuse part, $(1 - z_{at})A^{**}$, is isomorphic to a direct sum of $L^\infty(X, \mu)$'s for some continuous measures μ. It is known and easy to see that the bounded strongly lsc elements correspond to the bounded lower semi-continuous functions f on X such that f_- vanishes at infinity. Using this and Theorem 3.6 above, it is obvious that $\mathrm{SLSC}(A)$ corresponds to the set of $(-\infty, \infty]$-valued lower semicontinuous functions on X such that f_- vanishes at infinity.

Example B. $\mathcal{K}$. Let $A = \mathcal{K}$, the algebra of compact operators on a separable infinite dimensional Hilbert space. It is well known that $A^{**} = M(A) = B(H)$. Since every (not necessarily densely defined) positive operator h can be approximated from below by finite rank operators on $\overline{D(h)}$ and $1 - p_h \in \overline{A^m_{sa}}$, it follows directly that $\mathrm{SLSC}(A) = \{h \mid h_- \oplus 0(1 - p_h) \in \mathcal{K}\}$.

Example C. $c \otimes \mathcal{K}$. Let $A = c \otimes \mathcal{K}$; then A^{**} can be identified with bounded collections $\{h_n \mid 1 \leq n \leq \infty,\ h_n \in B(H)\}$. We give the following criterion which is deduced from Brown [**5**, Criterion 5.13].

Criterion for unbounded semicontinuity for $c \otimes \mathcal{K}$. *If $h \in \mathcal{R}_b(A)$, then $h \in SLSC(A)$ if and only if*

(1) *Each h_n is in $SLSC(\mathcal{K})$, $1 \leq n \leq \infty$.*

(2) *If $K \in \mathcal{K}$, $K \le h_\infty$, and if $\varepsilon > 0$, then there exists an N such that $K \le h_n + \varepsilon$ for all $n > N$.*

Proof. This is a special case of the criterion for $C_0(X) \otimes \mathcal{K}$ below. □

Example D. $C_0(X) \otimes \mathcal{K}$. Let A denote $C_0(X) \otimes \mathcal{K}$ where X is a second countable, locally compact Hausdorff space. Then $z_{at}A^{**}$ can be identified with the space of bounded functions from X to $B(H)$. So we will say that, for a self-adjoint operator h affiliated with $z_{at}A^{**}$, h is strongly LSC if $h = z_{at}\tilde{h}$ for some (unique) $\tilde{h}$ in SLSC(A). We give a criterion for $h \eta z_{at}A^{**}$ as follows (cf. [**5**, Criterion 5.19]).

Criterion for unbounded semicontinuity for $C_0(X)\otimes\mathcal{K}$. *$h\eta z_{at}A^{**}$ is strongly LSC if and only if*

(1) *Each $h(x)$ is in SLSC($\mathcal{K}$) for all $x \in X$.*

(2) *For all $\varepsilon > 0$, there exists a compact $F \subset X$ such that $h(x) \ge -\varepsilon$ for all $x \notin F$, and*

(3) *If $x_0 \in X$, $\mathcal{K} \ni K \le h(x_0)$, and $\varepsilon > 0$, then there is a neighborhood U of x_0 such that $K \le h(x) + \varepsilon 1$ for all $x \in U$.*

Proof. ($\Rightarrow$). This follows from the same proof as for [**5**, 5.13].

($\Leftarrow$). Assume that h satisfies (1), (2) and (3). By (2) and (3), h is bounded below. Let $-m \le h$, $m > 0$, and fix $\delta > 0$ such that $\delta < 1/m$. Now we will show that $\tilde{f}_\delta(h) \in z_{at}\overline{A^m_{sa}}$ using the criterion 5.19 of [**5**]. First, it is clear that $\tilde{f}_\delta(h)(x) \in \overline{\mathcal{K}^m_{sa}}$ by (1). By the properties of f_δ, the second condition of [**5**, 5.19] follows from (2). For the last condition, let $x_0 \in X$, $\mathcal{K} \ni K \le \tilde{f}_\delta(h)(x_0)$, and $\varepsilon > 0$. Let $K' = g(K)$ where $g = \mathrm{id} \wedge (1/\delta - \varepsilon/2)$ (assume $1/\delta > \varepsilon/2$). Then $K' \le K \le K' + (\varepsilon/2)1 \le 1/\delta$ and $f_{-\delta}(K')$ is a compact operator such that $f_{-\delta}(K') \le h(x_0)$ by Proposition 2.5. Choose small $\alpha > 0$ such that $f_\delta(-m) + \varepsilon/2 \ge f_\delta(-m+\alpha)$. Then $f_\delta(\cdot) + \varepsilon/2 \ge f_\delta(\cdot + \alpha)$ on $[-m, \infty)$. By (3), there is a neighborhood U of x_0 such that $f_{-\delta}(K') \le h(x) + \alpha 1$ for all $x \in U$. Therefore,

$$K \le K' + \frac{\varepsilon}{2}1 \le \tilde{f}_\delta(h(x) + \alpha 1) + \frac{\varepsilon}{2}1 \le \tilde{f}_\delta(h(x)) + \varepsilon 1$$

for all $x \in U$. Hence, we are done by Theorem 3.6. □

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Department of Mathematics, Yonsei University, Kangwondo, 222-701, South Korea
E-mail address: kimh@bubble.yonsei.ac.kr