ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

ON GORENSTEIN MONOMIAL IDEALS OF CODIMENSION THREE

YUJI KAMOI

Introduction. Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K and $I \subset S$ be an ideal generated by monomials. Buchsbaum-Eisenbud's structure theorem [**1**] tells us that if I is a Gorenstein ideal of codimension three, then it is generated by pfaffians of certain skew-symmetric matrix.

Our aim in this paper is to give explicit description of generators of codimension three Gorenstein monomial ideals, and we state our result as follows.

Theorem 0.1. *We put $m = \mu(I)$ a number of minimal generators of I and $s = (m+1)/2$. Let $\nu : \mathbf{Z} \to \{1, \dots, m\}$ be a map such that $i \equiv \nu(i) \pmod{m}$ for $i \in \mathbf{Z}$. Then the following conditions are equivalent.*

(1) *I is a Gorenstein monomial ideal of codimension three.*

(2) *m is odd and $I = Pf_{m-1}(M)$ where $M = (a_{ij})$ is an $m \times m$ skew-symmetric matrix of monomials satisfying the following conditions.*

(a) $a_{ij} = \begin{cases} \text{nonzero}, & \text{if } j = \nu(i+s-1) \text{ or } \nu(i+s) \\ 0, & \text{otherwise.} \end{cases}$

(b) *$\{a_{ij} \mid a_{ij} \neq 0, i < j\}$ is the set of pairwise coprime monomials.*

(3) *m is odd and there exist m pairwise coprime monomials $b_1, \dots, b_m$ such that*

$$I = \left(\left\{ \prod_{k=1}^{s-1} b_{\nu(i+k)} \mid 1 \leq i \leq m \right\} \right).$$

Example 0.2. Let M be a matrix with coefficients in $K[X_1, \dots, X_7]$

Received by the editors on October 12, 1993, and in revised form on June 1, 1994.

such that

$$M = \begin{pmatrix} 0 & 0 & 0 & X_4 & -X_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_5 & -X_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_6 & -X_3 \\ -X_4 & 0 & 0 & 0 & 0 & 0 & X_7 \\ X_1 & -X_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_2 & -X_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_3 & -X_7 & 0 & 0 & 0 \end{pmatrix}.$$

Then M satisfies conditions (a) and (b) of (0.1), (2) and $Pf_6(M)$ is generated by

$$\{X_1X_2X_3, X_2X_3X_4, X_3X_4X_5, X_4X_5X_6, X_5X_6X_7, X_6X_7X_1, X_7X_1X_2\}.$$

1. A syzygy module of a monomial ideal. In this section we consider the first syzygy module of a monomial ideal and determine minimal generators of this module for a Gorenstein monomial ideal of codimension three.

Let $S = K[X_1, \dots, X_n]$ be a polynomial ring over a field K. We denote by $\mathbf{Z}$ (respectively $\mathbf{N}$) the set of all integers (respectively, the set of all nonnegative integers). We define a $\mathbf{Z}^n$-grading on S by $\deg(X_i) = (0, \dots 0, \overset{i}{1}, 0, \dots, 0) \in \mathbf{Z}^n$. For $\alpha = (a_1, \dots, a_n) \in \mathbf{Z}^n$, we set $X^\alpha = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$.

Let I be a monomial ideal of S with minimal generators $X^{\alpha_1}, \dots, X^{\alpha_m}$ and F be a free S-module with free basis $e_1, \dots, e_m$. We define a map $\varphi : F \to S$ by $\varphi(e_i) = X^{\alpha_i}$ for $1 \le i \le m$. We put $\deg(e_i) = \alpha_i$ for $1 \le i \le m$ and regard F as a $\mathbf{Z}^n$-graded module. Then φ preserves $\mathbf{Z}^n$-gradings and $\operatorname{Ker}(\varphi)$ can be regarded as a $\mathbf{Z}^n$-graded module.

We set

$$\begin{aligned} \Sigma &= \{X^\alpha e_i \mid \alpha \in \mathbf{N}^n, 1 \le i \le m\} \\ \Sigma_\beta &= \{X^\alpha e_i \mid \alpha + \alpha_i = \beta\}, \qquad \beta \in \mathbf{N}^n \\ I_i &= (\{X^{\alpha_j}\}_{j \ne i}), \qquad 1 \le i \le m. \end{aligned}$$

Definition 1.1. We define an equivalence relation on Σ_β, $\beta \in \mathbf{N}^n$, as follows. For $u, v \in \Sigma_\beta$, we denote $u \sim v$ if either $u = v$ or there exist $X^{\beta_1}e_{i_1}, \dots, X^{\beta_p}e_{i_p} \in \Sigma_\beta$ such that $u = X^{\beta_1}e_{i_1}$, $v = X^{\beta_p}e_{i_p}$ and $\mathrm{Gcd}\,(X^{\beta_i}, X^{\beta_{i+1}}) \neq 1$ for $1 \leq i \leq p-1$.

Let $X^{\beta_1}e_{i_1}, \dots, X^{\beta_p}e_{i_p} \in \Sigma_\beta$ be representatives of $\Sigma_\beta/\sim$ with $1 \leq i_1 < \cdots < i_p \leq m$. We put

$$G_\beta = \begin{cases} \{X^{\beta_1}e_{i_1} - X^{\beta_2}e_{i_2}, \dots, X^{\beta_1}e_{i_1} - X^{\beta_p}e_{i_p}\}, & \text{if } p \geq 2 \\ \phi, & \text{if } p \leq 1 \end{cases}$$

and $G = \cup_\beta G_\beta$. Then it is easy to see that G is a minimal basis of $\mathrm{Ker}\,(\varphi)$. (The proof is essentially the same as Proposition 1 in Eliahou [**2**] or Proposition 1.5 in Herzog [**3**].)

We define a graph $\mathcal{G} = (V, E)$ on vertices $V = \{e_1, \dots, e_m\}$ and the set of edges $E = \{e_ie_j \mid X^\alpha e_i - X^\beta e_j \in G$ or $X^\beta e_j - X^\alpha e_i \in G$ for some $X^\alpha e_i, X^\beta e_j \in \Sigma\}$. We put $\deg_V(e_i) =^\# \{e_j \mid e_ie_j \in E\}$.

Remark 1.2. (1) $\mathcal{G}$ is connected.

(2) For any $X^\alpha e_i - X^\beta e_j \in \mathrm{Ker}\,(\varphi)$, we have

$$X^\alpha e_i - X^\beta e_j = X^\gamma \left(\frac{\mathrm{Lcm}\,(X^{\alpha_i}, X^{\alpha_j})}{X^{\alpha_i}} e_i - \frac{\mathrm{Lcm}\,(X^{\alpha_i}, X^{\alpha_j})}{X^{\alpha_j}} e_j \right)$$

for some monomial X^γ. Thus, if $\pm(X^\alpha e_i - X^\beta e_j)$ and $\pm(X^\gamma e_i - X^\delta e_k)$, are distinct elements of G, then $j \neq k$. Therefore, we have $|G| = |E| = \sum_{i=1}^m \deg_V(e_i)/2$.

We want to determine elements of G more precisely.

Let $\Gamma = \{X^\alpha e_i \mid X^\alpha$ is in the minimal basis of $[I_i : X^{\alpha_i}], 1 \leq i \leq m\} \subset \Sigma$ and $\Gamma' = \{(X^\alpha e_i, X^\beta e_j), (X^\beta e_j, X^\alpha e_i) \mid X^\alpha e_i - X^\beta e_j \in G\} \subset \Sigma \times \Sigma$. We define a map $\Phi : \Gamma \to \Gamma'$ as follows.

Let $X^\alpha e_i \in \Gamma$ and $\beta = \alpha + \alpha_i$. Since $X^\alpha \in [I_i : X^{\alpha_i}]$, we have $|\Sigma_\beta| \geq 2$. If there exists $u \in \Sigma_\beta$ such that $u \neq X^\alpha e_i$ and $u \sim X^\alpha e_i$, then by definition of the relation, we have $X^\alpha \in m[I_i : X^{\alpha_i}]$ where $\mathfrak{m} = (X_1, \dots, X_n)$. This contradicts that X^α is in the minimal basis of $[I_i : X^{\alpha_i}]$. Thus the equivalence class of $X^\alpha e_i$ in Σ_β is equal to $\{X^\alpha e_i\}$. Since $|\Sigma_\beta| \geq 2$, we have $G_\beta \neq \phi$. We put $G_\beta = \{X^{\beta_1}e_{i_1} - X^{\beta_2}e_{i_2}, \dots, X^{\beta_1}e_{i_1} - X^{\beta_p}e_{i_p}\}$ with $1 \leq i_1 < \cdots < i_p \leq m$.

Then there exists $1 \leq q \leq p$ such that $X^{\beta_q} e_{i_q} = X^{\alpha} e_i$. Therefore we define a map Φ by

$$\Phi(X^{\alpha} e_i) = \begin{cases} (X^{\beta_1} e_{i_1}, X^{\beta_q} e_{i_q}), & q > 1 \\ (X^{\beta_q} e_{i_2}, X^{\beta_1} e_{i_1}), & q = 1. \end{cases}$$

Proposition 1.3. *We have* $\deg_V(e_i) \geq \mu([I_i : X^{\alpha_i}])$ *for any* $1 \leq i \leq m$. *Hence,*

$$\mu(\mathrm{Ker}\,(\varphi)) = \sum_{i=1}^{m} \frac{\deg_V(e_i)}{2} \geq \sum_{i=1}^{m} \frac{\mu([I_i : X^{\alpha_i}])}{2} \geq \frac{m(\mathrm{ht}\,(I) - 1)}{2}.$$

Proof. Considering the correspondence between Γ' and E (cf. Remark 1.2, (2), the first assertion is trivial. We show the last inequality of the second assertion. Since $\dim(S/I) \leq \dim(S/I_i) \leq \dim(S/I) + 1$, we have $\mathrm{ht}\,(I) - 1 \leq \mathrm{ht}\,([I_i : X^{\alpha_i}])$. Hence $\sum_{i=1}^{m} \mu([I_i : X^{\alpha_i}]) \geq \sum_{i=1}^{m} \mathrm{ht}\,([I_i : X^{\alpha_i}]) \geq m(ht(I) - 1)$.

Lemma 1.4. *Suppose that* I *is a Gorenstein ideal of* $\mathrm{ht}\,(I) = 3$. *Then* $\mathcal{G}$ *is a cycle of length* m.

Proof. Since $\mathcal{G}$ is a connected graph, we only need to show that $\deg_V(e_i) = 2$ for any $1 \leq i \leq m$.

Since I is Gorenstein and $\mathrm{ht}\,(I) = 3$, the second betti number $b_2(S/I)$ of S/I is equal to m. Hence, by (1.3), we have

$$\begin{aligned} m = b_2(S/I) = |G| = \sum_{i=1}^{m} \frac{\deg_V(e_i)}{2} &\geq \sum_{i=1}^{m} \frac{\mu([I_i : X^{\alpha_i}])}{2} \\ &\geq \frac{m(\mathrm{ht}\,(I) - 1)}{2} = m. \end{aligned}$$

Thus $\sum_{i=1}^{m} \deg_V(e_i) = \sum_{i=1}^{m} \mu([I_i : X^{\alpha_i}])$ and, by (1.3), $\deg_V(e_i) = \mu([I_i : X^{\alpha_i}])$ for any $1 \leq i \leq m$. Also, since $\mu([I_i : X^{\alpha_i}]) \geq \mathrm{ht}\,(I) - 1 = 2$ and $\sum_{i=1}^{m} \mu([I_i : X^{\alpha_i}]) = 2m$, we have $\deg_V(e_i) = \mu([I_i : X^{\alpha_i}]) = 2$. Hence, $\mathcal{G}$ is a cycle of length m. $\square$

Propostion 1.5. *Suppose that* I *is a Gorenstein ideal of* $\mathrm{ht}\,(I)=3$. *Then a homogeneous minimal basis of* $\mathrm{Ker}\,(\varphi)$ (*with respect to* $\mathbf{Z}^n$-*grading*) *is uniquely determined for* I (*up to multiplication by units*). *Furthermore, after the renumbering of* $X^{\alpha_1},\dots,X^{\alpha_m}$, G *can be written in the form*

$$G=\{X^{\alpha_{ii}}e_i-X^{\alpha_{ii+1}}e_{i+1},X^{\alpha_{mm}}e_m-X^{\alpha_{m1}}e_1\mid 1\le i\le m-1\}.$$

Proof. By the proof of (1.4), Φ is bijective and, by definition of Φ, if $G_\beta\neq\phi$, then $|G_\beta|=1$. If we put $G_\beta=\{X^{\beta_1}e_i-X^{\beta_2}e_j\}$, then $X^{\beta_1}e_i$, $X^{\beta_2}e_j\in\Gamma$ and, by the construction of Φ, $\Sigma_\beta=\{X^{\beta_1}e_i,X^{\beta_2}e_j\}$. Note that the set of all degrees of a homogeneous minimal basis of $\mathrm{Ker}\,(\varphi)$ is independent on the choice of a minimal basis and is equal to $\{\beta\mid G_\beta\neq\phi\}$. Therefore, G is the unique homogeneous minimal basis of $\mathrm{Ker}\,(\varphi)$.

The second assertion follows from (1.4). □

2. A proof of the main theorem. First, we determine $Pf(M)$ of a skew-symmetric matrix M satisfying condition (a) of Theorem 0.1, (2).

Let $m>0$ be an odd integer and $s=(m+1)/2$. We denote by $[m]=\{1,\dots,m\}$ and by $\nu:\mathbf{Z}\to[m]$ a map such that $i\equiv\nu(i)(\mathrm{mod}(m))$.

Let $\tau\in\mathfrak{S}_m$ be a permutation on $[m]$. We denote by $(\tau)=(\tau_{ij})$ an $m\times m$-matrix such that

$$\tau_{ij}=\begin{cases}1, & i=\tau(j))\\ 0, & i\neq\tau(j)).\end{cases}$$

For an $m\times m$ matrix $M=(a_{ij})$, we set $\tau M=(\tau)\cdot M$ and $M\tau=M\cdot(\tau)^{-1}$. Namely, if we put $\tau M=(b_{ij})$, respectively, $M\tau=(c_{ij})$, then $b_{\tau(i)j}=a_{ij}$, respectively, $c_{i\tau(j)}=a_{ij}$, for $1\le i,\ j\le m$.

Definition 2.1. Let $M=(a_{ij})$ be an $m\times m$ matrix.

(1) We say that M is of *type* (I), if

(a) for $1\le i,\ j\le m$,

$$a_{ij}=\begin{cases}\text{nonzero} & j=i \text{ or } \nu(i+1)\\ 0 & \text{otherwise}\end{cases}$$

(b) for $1 \le i \le m$, $a_{ii} = -a_{\nu(i+s-1)\nu(i+s)}$.

(2) We say that M is of *type* (II), if M is skew-symmetric and, for $1 \le i, j \le m$,

$$a_{ij} = \begin{cases} \text{nonzero} & j = \nu(i+s-1) \text{ or } \nu(i+s) \\ 0 & \text{otherwise.} \end{cases}$$

We define a permutation $\sigma \in \mathfrak{S}_m$ by $\sigma(i) = \nu(i-s+1)$ $(= \nu(i+s))$ for $i \in [m]$. Then it is easy to see that M is of type (I) if and only if σM is of type (II) for an $m \times m$ matrix M.

For an $m \times m$ matrix M, we denote by M_k the submatrix of M consisting of the first k rows and columns and by M_{ij} the $(m-1) \times (m-1)$ matrix obtained from M by deleting the i-th row and j-th column.

By definition of type (I), we have the following.

Lemma 2.2. *Let $M = (a_{ij})$ be an $m \times m$ matrix of type* (I)*.*

(1) *M_{ms} can be written in the form*

$$M_{ms} = \begin{pmatrix} M_{s-1} & 0 \\ 0 & {}^t(-M_{s-1}) \end{pmatrix}.$$

(2) *Let $1 \le k \le m$ and $\tau \in \mathfrak{S}_m$ such that $\tau(i) = \nu(i-k)$ for $i \in [m]$. Then $\tau M \tau$ is again of type* (I)*.*

Proposition 2.3. *Let m be an odd integer and $M = (a_{ij})$ be an $m \times m$ matrix of type* (I)*.*

(1) *$Pf_{m-1}(\sigma M)$ is generated by $\{\prod_{k=1}^{s-1} a_{\nu(i+k)\nu(i+k)} \mid 1 \le i \le m\}$ and, for any $1 \le i \ne j \le m$, there exists $1 \le k \le m$ such that $k \ne i, j$ and $Pf_{m-1}(\sigma M) \subset (a_{ii}, a_{jj}, a_{kk})$.*

(2) *Furthermore, we assume that coefficients of M are monomials. Then the following are equivalent.*

(a) $\mathrm{ht}\,(Pf_{m-1}(\sigma M)) = 3$.

(b) *$\{a_{11}, \ldots, a_{mm}\}$ is the set of pairwise coprime monomials.*

Proof. (1) By definition of σ, we have $\det((\sigma M)_{\sigma(i)\sigma(i)}) = (-1)^p \det (M_{i\nu(i+s)})$ for some $p \geq 0$. Let $\tau \in \mathfrak{G}_m$ such that $\tau(j) = \nu(j-i)$ for $j \in [m]$ and $N = \tau M \tau$. Then, by (2.2), we have $\det(M_{i\nu(i+s)}) = (-1)^q \det(N_{ms}) = (-1)^{q+1}(\det(N_{s-1}))^2$ for some $q \geq 0$. Thus, $\det((\sigma M)_{\sigma(i)\sigma(i)}) = (-1)^r (\prod_{k=1}^{s-1} a_{\nu(i+k)\nu(i+k)})^2$, $r \geq 0$, and

$$Pf_{m-1}(\sigma M) = \left(\left\{ \prod_{k=1}^{s-1} a_{\nu(i+k)\nu(i+k)} \mid 1 \leq i \leq m \right\}\right).$$

The second statement follows from the first.

(2) (a) $\Rightarrow$ (b). Suppose that $\operatorname{Gcd}(a_{ii}, a_{jj}) = b \neq 1$. Then, by (1), $Pf_{m-1}(\sigma M) \subset (b, a_{kk})$ for some $k \neq i, j$. This contradicts that $\operatorname{ht}(Pf_{m-1}(\sigma M)) = 3$.

(b) $\Rightarrow$ (a). By (1), we have already seen that $\operatorname{ht}(Pf_{m-1}(\sigma M)) \leq 3$. Conversely, for any $1 \leq i \neq j \leq m$, (a_{ii}, a_{jj}) contains only $m-1$ elements of $\{\prod_{k=1}^{s-1} a_{\nu(i+k)\nu(i+k)} \mid 1 \leq i \leq m\}$, since $a_{11}, \dots, a_{mm}$ are pairwise coprime. Hence, $2 < \operatorname{ht}(Pf_{m-1}(\sigma M)) = 3$. □

Proof of Theorem 0.1. By (2.3), we have already proved implications (2) $\Leftrightarrow$ (3). Furthermore, by Theorem 2.1 of Buchsbaum-Eisenbud [**1**], the statement (2) implies that I is a Gorenstein ideal.

(1) $\Rightarrow$ (2). Let I be a Gorenstein ideal of $\operatorname{ht}(I) = 3$ with minimal basis $X^{\alpha_1}, \dots, X^{\alpha_m}$. Then, by a theorem of Watanabe [**6**], we may assume that m is odd. We put $s = (m+1)/2$ and denote a $\mathbf{Z}^n$-graded minimal free resolution of S/I by

$$\mathbf{F} := 0 \longrightarrow S(-\gamma) \xrightarrow{d_3} \oplus_{i=1}^m S(-\beta_{p_i}) \xrightarrow{d_2} \oplus_{i=1}^m S(-\alpha_i) \xrightarrow{{}^t d_1} S$$

where $d_1 = (X^{\alpha_1} \cdots X^{\alpha_m})$ and $d_3 = ((-1)^{r_1} X^{\alpha_{p_1}} \cdots (-1)^{r_m} X^{\alpha_{p_m}})$, $r_i = 0$ or 1. Furthermore, by (1.5), we can write the $m \times m$ matrix $d_2 = (a_{ij})$ as

$$a_{ij} = \begin{cases} X^{\alpha_{ii}} & 1 \leq i \leq m,\ j = i, \\ -X^{\alpha_{ii+1}} & 1 \leq i < m,\ j = i+1, \\ -X^{\alpha_{m1}} & (i,j) = (m,1) \\ 0 & \text{otherwise.} \end{cases}$$

Since $d_3 \cdot d_2 = 0$, $r_1 = \cdots = r_m$, and we may assume that $d_3 = (X^{\alpha_{p_1}} \cdots X^{\alpha_{p_m}})$. Then we shall show that the matrix d_2 is of type (I) of (2.1).

Taking $S(-\gamma)$-dual of $\mathbf{F}$, we have a minimal free resolution

$$\mathbf{G}. = 0 \longrightarrow S(-\gamma) \xrightarrow{d_1} \oplus_{i=1}^m S(-\beta_i) \xrightarrow{{}^t d_2} \oplus_{i=1}^m S(-\alpha_{p_i}) \xrightarrow{{}^t d_3} S$$

of S/I. Then, by Theorem 1.5 and Theorem 2.1 of Buchsbaum-Eisenbud [**1**], there exists a $\mathbf{Z}^n$-graded isomorphism $t. : \mathbf{G}. \to \mathbf{F}.$ such that the matrix $t_2 \cdot d_2$ is skew-symmetric. Since $t.$ preserves a $\mathbf{Z}^n$-grading, t_1, respectively t_2, is determined by a permutation of free-basis of F_1, respectively F_2, and a multiplication of a unit. Hence, we have $t_1 = (-1)^q(\tau^{-1})$ and $t_2 = (-1)^{q'}(\tau)$ where $\tau \in \mathfrak{S}_m$ such that $\tau(p_i) = i$ for $i \in [m]$. Since t_0, respectively t_3, is an identity, we have $q = 0$, respectively $q' = 0$.

On the other hand, by the form of d_2 and ${}^t d_2$, we have either $p_i = \nu(p_1 - (i-1))$ for all $i \in [m]$ or $p_i = \nu(p_1 + (i-1))$ for all $i \in [m]$, cf. (1.5). Namely, τ is determined as $\tau(i) = \nu(p_1 - i + 1)$ for any $i \in [m]$ or $\tau(i) = \nu(i - p_1 + 1)$ for any $i \in [m]$.

We note that the matrix $t_2 \cdot d_2 = \tau d_2$ is skew-symmetric. It is only possible in the case that $\tau(i) = \nu(i - p_1 + 1)$ for any $i \in [m]$ and $p_1 = s$. This implies that τd_2 is of type (II) (or d_2 is of type (I)).

This completes the proof of Theorem 0.1. □

Corollary 2.4. *Let $I \subset S$ be a Gorenstein monomial ideal of codimension three. Then the Rees algebra $R(I) = \oplus_{i \geq 0} I^i$ is isomorphic to the symmetric algebra* Sym (I) *and is Cohen-Macaulay. Furthermore, the associated graded ring $G(I) = \oplus_{i \geq 0} I^i/I^{i+1}$ is Gorenstein.*

Proof. By a theorem of Huneke [**5**] and Theorem 2.6 of Herzog-Simis-Vasconcelos [**4**], we only need to show that $\mu(I_P) \leq \mathrm{ht}\,(P)$ for any $P \in V(I)$.

Let $P \in V(I)$ of $\mathrm{ht}\,(P) = t \leq n$. Then there exist $1 \leq i_1 < \cdots < i_{n-t} \leq n$ such that $X_{i_1}, \ldots, X_{i_{n-t}} \notin P$. We put $A = K[X_{i_1}, \ldots, X_{i_{n-t}}]$ and $T = S \otimes_A K(X_{i_1}, \ldots, X_{i_{n-t}})$. Then T is a polynomial ring over $K(X_{i_1}, \ldots, X_{i_{n-t}})$ with t variables and IT is a Gorenstein monomial ideal of codimension three. Hence, by (0.1), $\mu(IT) \leq t$ and thus $\mu(I_P) \leq \mu(IT) \leq t = \mathrm{ht}\,(P)$. □

Acknowledgments. The author would like to express his appreciation to Professor Junzo Watanabe and Dr. Kazuhiko Kurano for stimulating discussion. Thanks are also due to Professor Kei-ichi Watanabe and Professor Shiro Goto for several useful suggestions.

REFERENCES

1. D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), 447–485.

2. S. Eliahou, *Courbes monomiales et algèbre de Rees symbolique*, Thèse N°2080, Université de Genève, 1983.

3. J. Herzog, *Generators and relations of Abelian semigroups and semigroup rings*, Manuscripta Math. **3** (1970), 175–193.

4. J. Herzog, A. Simis and W.V. Vasconcelos, *Approximation complexes of blowing-up rings*, J. Algebra **74** (1982), 466–493.

5. C. Huneke, *Linkage and the Koszul homology of ideals*, Amer. J. Math. **104** (1982), 1043–1062.

6. J. Watanabe, *A note on Gorenstein rings of embedding codimension* 3, Nagoya Math. J. **50** (1973), 227–232.

Department of Mathematics, Tokyo Metropolitan University, Tokyo, 192-03 Japan