ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

SPECTRUM WHERE THE BOUNDARY OF THE NUMERICAL RANGE IS NOT ROUND

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ABSTRACT. For a bounded linear operator A on a complex Hilbert space, we prove that the boundary points of the numerical range $W(A)$ with infinite curvature of the convex boundary curve are included in the spectrum of both A and A^*. If, additionally, $W(A)$ is closed, then the 'non-round' boundary points are eigenvalues of A and A^*.

The numerical range $W(A)$ of the operator A is defined as the set of complex numbers (Au, u) where u runs through the vectors of norm 1. The basic fact concerning numerical range is the Toeplitz-Hausdorff theorem which states that the numerical range of a bounded linear operator on a Hilbert space $\mathcal{H}$ is convex [**2**]. The closure $\overline{W(A)}$ of the numerical range contains the spectrum of A, is convex too and is compact because of boundedness of the operator A. The boundary of $W(A)$ is a Jordan curve and will be called $C(A)$. For some related material on the numerical range of operators, see [**3, 4**].

Convex compact sets have enough extreme points and we would like to ask whether extreme points of $\overline{W(A)}$ belong to the spectrum. The example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

on a 2-dimensional Hilbert space shows that this is not necessarily so; the matrix is nilpotent, has spectrum $\{0\}$ and numerical range equal to the closed disk with center 0 and radius 1/2. On the other hand, Donoghue considered in [**1**] the *corners*, which are the points of $C(A) \cap W(A)$ where $C(A)$ fails to have a unique tangent, and proved that they are eigenvalues of A. For normal operators, where we can use the spectral theorem, it is easy to prove that $\overline{W(A)}$ equals the convex hull of the spectrum.

Our proposal to generalize Donoghue's result is to consider points where $C(A)$ is *not round*, i.e., where the curvature is infinite. To be

Received by the editor on October 19, 1993.

explicit, we translate and multiply the operator with complex numbers so that $0 \in C(A)$ and the real axis is a (possibly nonunique) tangent to $C(A)$ at 0. The first assumption is

$$\inf_{||u||=1} \operatorname{Im}(Au,u) = 0. \tag{1}$$

The second assumption is that $C(A)$ is not round at 0, in the sense of infinite curvature:

$$\operatorname*{Lim\,inf}_{\delta\to 0\, ||u||=1, |\operatorname{Re}(Au,u)|<\delta} \frac{\operatorname{Im}(Au,u)}{(\operatorname{Re}(Au,u))^2} = \infty. \tag{2}$$

Because of the Toeplitz-Hausdorff theorem, this implies that 0 is the only real point in $\overline{W(A)}$. To see how non-round boundary points can appear, consider the curve $y = c|x|^p$ in the Cartesian plane, x, y real and $c > 0$. Then $p = 1$ corresponds to a corner at $x = 0$, and for $p \geq 2$ the curve is twice continuously differentiable, with finite curvature everywhere. The cases $1 < p < 2$ are the interesting ones for us: unique tangent at 0, but infinite curvature there. Our aim is to prove the

Theorem. *If* (1) *and* (2) *are fulfilled,* 0 *belongs to the spectrum of both A and A^*.*

The strategy to prove the theorem is to take an appropriate sequence (u_n) of unit vectors from Hilbert space and to show that $||Au_n|| \to 0$. A sequence of unit vectors with

$$(Au_n, u_n) =: i\varepsilon_n \to 0 \quad \text{as } n \to \infty, \tag{3}$$

with purely imaginary $i\varepsilon_n$ will be appropriate. If $C(A)$ has a corner at 0, i.e., the left and right tangent include an angle smaller than π, it may be impossible to choose such a sequence immediately (the imaginary axis may be disjoint from $W(A)$). But then we can rotate $W(A)$ about 0 by multiplication of A with a unimodular number, such that both assumptions (1) and (2) remain fulfilled. In case $W(A) = \{0\}$ the theorem is trivially true.

Because of (3), we can decompose Au_n as

$$Au_n =: i\varepsilon_n u_n + x_n v_n, \qquad (u_n, v_n) = 0, \qquad ||v_n|| = 1.$$

Since $\varepsilon_n \to 0$, we have to show that $x_n \to 0$. Motivated by the 2-dimensional case, we try the linear combination $u_n + z_n v_n$ as an input to assumption (1). Taking care of normalization and introducing $y_n := (Av_n, u_n)$, we get

$$\mathrm{Im}\, \frac{i\varepsilon_n + z_n y_n + \bar{z}_n x_n + |z_n|^2 (Av_n, v_n)}{1 + |z_n|^2} \geq 0.$$

A good choice is now $z_n = \sqrt{\varepsilon_n} e^{i\phi}$ with the phase ϕ still free. We get for all real ϕ,

$$\begin{aligned} |\mathrm{Im}\,(e^{i\phi} y_n + e^{-i\phi} x_n)| &\leq \sqrt{\varepsilon_n}(1 + \mathrm{Im}\,(Av_n, v_n)) \\ &\leq (1 + ||A||)\sqrt{\varepsilon_n}. \end{aligned} \tag{4}$$

Thus, application of (1) provided us only with knowledge that the imaginary parts of sums involving x_n must be small. To get more information about x_n, we now apply (2) to $u_n + z_n v_n$ with $z_n = \sqrt{\varepsilon_n} e^{i\phi}$ again. As the real part of $(A(u_n + z_n v_n), v_n + z_n v_n)$ goes to 0, we have

$$\frac{\varepsilon_n + \sqrt{\varepsilon_n}\mathrm{Im}\,(e^{i\phi} y_n + e^{-i\phi} x_n) + \varepsilon_n \mathrm{Im}\,(Av_n, v_n)}{[\sqrt{\varepsilon_n}\mathrm{Re}\,(e^{i\phi} y_n + e^{-i\phi} x_n) + \varepsilon_n \mathrm{Re}\,(Av_n, v_n)]^2} \to \infty,$$

which implies using (4),

$$\frac{2(1 + ||A||)\varepsilon_n}{\varepsilon_n [\mathrm{Re}\,(e^{i\phi} y_n + e^{-i\phi} x_n) + \sqrt{\varepsilon_n}\mathrm{Re}\,(Av_n, v_n)]^2} \geq: M_n \to \infty$$

with a sequence M_n diverging to ∞ such that the inequality holds uniformly in ϕ. We estimate the real part from above, uniformly in ϕ.

$$|\mathrm{Re}\,(e^{i\phi} y_n + e^{-i\phi} x_n)| \leq \sqrt{\frac{2(1 + ||A||)}{M_n}} - \sqrt{\varepsilon_n}\mathrm{Re}\,(Av_n, v_n).$$

$\mathrm{Re}\,(Av_n, v_n)$ and $\mathrm{Im}\,(Av_n, v_n)$ are bounded by $||A||$, so the righthand side of the last inequality tends to 0. We choose now for every n, the angle ϕ_n so that the complex numbers x_n, y_n have the same phase or, loosely speaking, point in the same direction on the complex plane. Then

$$|x_n| + |y_n| = \sqrt{[\mathrm{Re}\,(e^{i\phi_n} y_n + e^{-i\phi_n} x_n)]^2 + [\mathrm{Im}\,(e^{i\phi_n} y_n + e^{-i\phi_n} x_n)]^2}$$

converges to 0. This implies

$$||Au_n||^2 = \varepsilon_n^2 + x_n^2 \to 0, \quad \text{hence } 0 \in \text{ spec } A.$$

The numerical range $W(A^*)$ of the adjoint operator is the complex conjugate set of $W(A)$ and has, consequently, the same geometric boundary properties. If (1) and (2) hold for the operator A, then they hold for the adjoint A^* too, with the only change that the imaginary parts are replaced by their opposites. So 0 belongs to $\mathrm{spec}\, A^*$, too, which completes the proof of the theorem. □

What about variation of the exponent in condition (2)? As this infimum condition is only sensitive to changes in the behavior of $C(A)$ for *small* $\mathrm{Re}\,(Au, u)$, we see that the theorem holds a posteriori with exponent 2 replaced by any positive exponent less than or equal to 2 in condition (2). On the contrary, condition (2) with an exponent greater than 2 is insufficient to conclude that $0 \in \mathrm{spec}\, A$; the matrix above provides a counterexample.

In case that 0 is *known* to belong to $W(A)$, it should be possible to obtain a stronger conclusion and indeed this is so. Let $(Au, u) = 0$ and $Au = xv$, $||v|| = 1$, $(Av, u) = y$. Trying again the linear combination $u + zv$ for small complex z, we see that x, y must be complex conjugate to each other, otherwise complex numbers with negative imaginary part would enter $W(A)$, contrary to (1). If $x = \bar{y} \neq 0$, the curve

$$(Au(t), u(t)) = \frac{(|x| + |y|)t + (Av, v)t^2}{1 + t^2}, \qquad u(t) := \frac{u + te^{i\phi} v}{\sqrt{1 + t^2}}$$

parametrized by the real t and with appropriately chosen ϕ, belongs to $W(A)$ and has finite curvature at 0 which contradicts (2). Hence $x = 0$.

Corollary. *If* (1) *and* (2) *are fulfilled and* $0 \in W(A)$, *then* 0 *belongs to the point spectrum of both* A *and* A^*.

This is a slight generalization to Donoghue's theorem, but our argument is similar to his. We conjecture that if 0 is *not* a corner, i.e., $C(A)$ has a unique tangent at 0, then the 'non-round' condition (2) implies that 0 belongs even to the essential spectrum of A. This has not yet been proved, as far as I know.

Acknowledgment. I would like to thank Stephan Scheidl for introducing me to TEXTURES on the Macintosh and Hubert Kalf for a useful corresondence.

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