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SMOOTH PARTITIONS OF UNITY IN BANACH SPACES

JULIEN FRONTISI

ABSTRACT. We show that if a Banach space X has an LUR norm, and if every Lipschitz convex function on X can be approximated by C^k-smooth functions, then X admits C^k-smooth partitions of unity, and thus every continuous function on X is a uniform limit of C^k-smooth functions.

1. Introduction and notation. Partitions of unity and smooth approximation on Banach spaces have been studied since the 1950's. For early references on the subject, the reader may refer to the bibliography in [**1**] or [**2**, Chapter 8].

In [**1**], Bonic and Frampton obtained results for classical separable spaces. The nonseparable cases were settled by Toruńczyk [**12**] who used homeomorphic coordinatewise smooth embeddings into spaces $c_0(\Gamma)$. A refinement of this method, in [**4**], was used to extend Bonic and Frampton's results to weakly compactly generated spaces. Building on the idea, McLaughlin [**7**] proved similar results to those we obtain here. In fact, he proved that if a w-LUR norm on a Banach space X can be uniformly approximated on bounded sets by equivalent C^{k+1}-norms, then X admits C^k-smooth partitions of unity.

The more geometrical approach we are following here originates in a paper of Milman [**8**]. It has already provided first-order smoothness results as in Theorem 2.1 in [**14**] (see also [**2**, Theorem 8.3.12]), Theorem 2.2 in [**10**] and in [**11**]. Milman's ideas have also been used in [**9**] to obtain smooth partitions with Lipschitz derivative.

Let us finally mention that our Proposition 2.5 extends Proposition 8.3.10 of [**2**] and provides a result of transfer for smooth partitions of unity.

We recall that a Banach space X admits C^k-smooth partitions of unity if for any open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of X there exists a family of C^k-smooth functions $\{\Psi_\alpha\}_{\alpha\in\Lambda}$ with the following properties:

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i) The family $\{\text{supp}\, \Psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite and $\text{supp}\, \Psi_\alpha \subseteq U_\alpha$ for all α in Λ.

ii) $\sum_{\alpha\in\Lambda} \Psi_\alpha = 1$ on X.

We say that a norm $|| \quad ||$ on a Banach space X is locally uniformly rotund (LUR) if for all x_0 in X, $\lim ||x_n - x_0|| = 0$ whenever $\{x_n\}_{n\geq 1} \subset X$ satisfies $\lim ||x_n|| = ||x_0||$ and $\lim ||x_n + x_0|| = 2||x_0||$. A norm is said to have the Kadeč-Klee property if the norm and weak topologies on the unit ball coincide at each point of the unit sphere of X. A norm is average locally uniformly rotund (ALUR) if and only if every point of S_X is a point of dentability. This condition is equivalent to the fact that the norm has the Kadeč-Klee property and is strictly convex.

The ALUR property is weaker than LUR but we know by Troyanski's theorem [**13**] that the existence of an ALUR norm implies the existence of an LUR norm (see also [**2**, Section 4.2]). So this doesn't actually weaken the hypothesis of LUR in our theorem.

In the following, X denotes a Banach space, S_X, B_X and $\overline{B}_X$ are its unit sphere, open and closed unit balls; $k \in \mathbf{N}$ or $k = \infty$, and $\mathcal{U}^k$ is the collection of sets:

$$\mathcal{U}^k = \{f^{-1}((0, +\infty)), f \in C^k(X)\}$$

where $C^k(X)$ is the collection of all real valued C^k-smooth functions defined on X. We refer to [**2**, Chapter 8] for further details.

2. The main result. We are going to show the following:

Theorem 2.1. *Suppose that X has an* LUR *norm, and that every Lipschitz convex function on X is the uniform limit on bounded sets of C^k-smooth functions. Then X admits C^k-smooth partitions of unity.*

The result relies on the lemma:

Lemma 2.2. *If the norm on X is* ALUR, *then the norm topology on S_X has a σ-locally finite basis $(O_\alpha)_{\alpha\in\Lambda}$ such that for all $\alpha \in \Lambda$, $O_\alpha = C_\alpha \cap S_X$ where C_α is a bounded convex open set in X.*

Proof. We follow the proof in [**6**, p. 234] for the existence of a σ-

locally finite basis of the topology in a metric space. If $|| \quad ||$ is ALUR, every x in S_X admits a basis of neighborhoods in S_X consisting of the sets:

$$\mathcal{S}_{\varepsilon,f} = \{y \in X, f(y) > 1 - \varepsilon\} \quad \text{where } \varepsilon > 0 \text{ and } f \in B_{X^*}.$$

If $\{\mathcal{S}_\alpha\}_{\alpha<\alpha_0}$ is any covering of S_X with such slices, let $\mathcal{S}_\alpha^p = \{x \in X, d(x, \mathcal{S}_\alpha^c) > 2^{-p}\}$, we have $\mathcal{S}_\alpha = \cup_{p\geq 0}\mathcal{S}_\alpha^p$, each set $\mathcal{S}_\alpha^p$ is open, and

$$d(\mathcal{S}_\alpha^p, (S_\alpha^{p+1})^c) \geq 2^{-p-1}.$$

Then the set $\mathcal{T}_\alpha^p = \mathcal{S}_\alpha^p \backslash \overline{(\cup_{\beta<\alpha}\mathcal{S}_\beta^{p+1})}$ is a convex open set (since all $\mathcal{S}_\alpha^p$ are half spaces, hence so are $(\mathcal{S}_\alpha^p)^c$, thus $\mathcal{T}_\alpha^p = \mathcal{S}_\alpha^p \cap [\cap_{\beta<\alpha}(\mathcal{S}_\beta^{p+1})^c]^o$ is convex).

For $\alpha_1 \neq \alpha_2$, $d(\mathcal{T}_{\alpha_1}^p, \mathcal{T}_{\alpha_2}^p) \geq 2^{-p-1}$ so the family $\{\mathcal{T}_\alpha^p\}_{\alpha<\alpha_0}$ is discrete and $\{\mathcal{T}_\alpha^p\}_{\alpha,p}$ is σ-discrete. It is not difficult to show that this family covers the sphere again. So we constructed a σ-discrete refinement for $\{\mathcal{S}_\alpha\}_{\alpha<\alpha_0}$.

Now, considering for each n the family $H_n = \{\mathcal{S}_{\varepsilon,f}; f \in B_{X^*}, \varepsilon \leq 1/n\}$, we construct as above a σ-discrete refinement F_n of H_n, and the base $\cup_{n>0}F_n$ gives the result.

If we want to obtain bounded sets, we just need to consider their intersection with $2B_X$. □

Let us proceed with the proof of Theorem 2.1. By a result of Toruńczyk [**12**] (see [**2**, Lemma 8.3.6]) it is known that the following statements are equivalent:

i) The space X admits C^k-smooth partitions of unity.

ii) If $A \subset W \subset X$, A closed, W open, then there exists $U \in \mathcal{U}^k$ such that $A \subset U \subset W$.

iii) The family $\mathcal{U}^k$ contains a σ-locally finite basis for the topology of X.

Hence, all we need to show is the existence of such a basis for X. Now let $|| \quad ||_X$ be an LUR norm on X. By assumption, $|| \quad ||_X$ can be approximated by C^k-smooth functions. As in [**14**], consider $Y = X \oplus_2 \mathbf{R}$ with the norm $||(x,t)||_Y^2 = ||x||_X^2 + t^2$, $|| \quad ||_Y$ is an LUR norm. Moreover, if $|| \quad ||_X$ is a limit of C^k-smooth functions uniformly on bounded sets, $|| \quad ||_Y$ clearly has the same property.

For $n \geq 1$, let $f_n : Y \to \mathbf{R}$ be a C^k-smooth function such that $|f_n(y) - ||y||_Y| < 1/(16n)$ for all y in nB_Y. For $y \in X \times \{1\}$, if $y \in nB_Y$, then $f_n(y) \geq 1/2$. Let $p_n : (X \times \{1\}) \cap nB_Y \to Y$, be defined by $p_n(y) = y/f_n(y)$. The map p_n is C^k-smooth. Denote by $p : X \times \{1\} \to S_Y$ the projection on the sphere $p(y) = y/||y||_Y$. The map p is a homeomorphism from $X \times \{1\}$ to its image. For $y \in (X \times \{1\}) \cap nB_Y$,

$$\begin{aligned} ||p_n(y) - p(y)||_Y &= ||y||_Y \left| \frac{1}{f_n(y)} - \frac{1}{||y||_Y} \right| \\ &= \frac{|||y||_Y - f_n(y)|}{f_n(y)} \leq \frac{1}{8n}. \end{aligned}$$

From Lemma 2.2, we have a σ-locally finite basis $(O_\alpha)_{\alpha \in \Lambda}$ of the topology of S_Y with $O_\alpha = C_\alpha \cap S_Y$, where C_α is a bounded convex open set. So $(p^{-1}(O_\alpha))_{\alpha \in \Lambda}$ is a σ-locally finite basis of the topology of $X \times \{1\}$.

Now let us fix α, and let $O = O_\alpha$ and $C = C_\alpha$. We want to write C as a union of elements of $\mathcal{U}^k(Y)$, $C = \cup_n g_n^{-1}((0, +\infty))$, and then, for x in $p^{-1}(O)$, approximate $g_n(p(x))$ by $g_n(p_n(x))$.

Lemma 2.3. *If C is a bounded convex open set of Y, then $C = g^{-1}((0, +\infty))$ where $g : Y \to \mathbf{R}$ is a one-lipschitzian concave function.*

Proof. We may assume that $C \neq \varnothing$. For all $y \in Y$, let $F(y) = d(y, C^c) = \inf\{||y - z|| : z \notin C\}$. For $x \in C$, let $A(x)$ be the set of affine continuous functions a defined on Y such that $(a - F)$ is nonnegative on C and $a(x) = F(x)$. Let $A = \cup_{x \in C} A(x)$.

The function F is concave continuous on C, so for all x in C, $A(x) \neq \varnothing$. Define $g(y) = \inf\{a(y) : a \in A\}$. The mapping F is one-lipschitzian on C, hence so is every a in A. Since C is bounded and $g(x) = F(x)$ for all x in C, g is well defined and one-lipschitzian on the whole space Y. The function is clearly concave, and, for all x in C, $g(x) > 0$.

If $y \notin C$, let $x \in C$. We can write $[x, y] \cap C = [x, z)$ with $z \in Y$. If $x_n \in [x, z)$ with $x_n \to z$, we have $F(x_n) \to 0$ so if $(a_n) \subseteq A$ is such that $a_n(x_n) \to 0$ we have $\liminf_{n\to\infty} a_n(x) \geq F(x) > 0 = \liminf_{n\to\infty} a_n(z)$, but $z \in [x, y]$ and a_n is affine, so $g(y) \leq \liminf a_n(y) \leq 0$ and $C = g^{-1}((0, +\infty))$, which concludes the proof of Lemma 2.3. □

So for the set C defined above, we take g as in Lemma 2.3.

Claim. *The function g can be approximated by C^k-smooth functions uniformly on bounded sets of Y.*

Proof. The map $g : X \times \mathbf{R} \to \mathbf{R}$ is concave and one-lipschitzian, hence so is $g(\cdot, t) : X \to \mathbf{R}$ for all real numbers. By hypothesis, $g(\cdot, t)$ can be approximated uniformly on bounded sets by C^k-smooth functions. Let $n \in \mathbf{N}$, $p \in \mathbf{Z}$ with $|p| \leq n^2$. We can find a C^k-smooth function $f_n^p : X \to \mathbf{R}$ such that $|f_n^p(x) - g(x, p/n)| < 1/n$ for all $x \in nB_X$.

Let $(\phi_p)_{p\in\mathbf{Z}}$ be a C^∞-smooth partition of unity on $\mathbf{R}$ subordinated to the covering $(](p-1)/n, (p+1)/n[)_{p\in\mathbf{Z}}$, the function $g_n(x, t) = \sum_{p=-n^2}^{n^2} \phi_p(t) f_n^p(x)$ is C^k-smooth, and since g is one-lipschitzian, we get $|g_n(x, t) - g(x, t)| \leq 2/n$ for $x \in nB_X$, $t \in [-n, n]$.

We now come back to the proof of Theorem 2.1. We may and do assume that $p^{-1}(O_\alpha) = p^{-1}(O)$ is included in some ball $n_0 B_Y$. Let g be as in Lemma 2.3. By the claim, there exist C^k-smooth functions g_n such that

$$|g_n(y) - g(y)| \leq 1/(8n) \quad \text{for all } y \in (n_0 + 1)B_Y.$$

Let $x \in X \times \{1\}$ be in the ball $(n_0 + 1)B_Y$. If $p(x) \in O$, there exists $n \in \mathbf{N}$ such that $d(p(x), O^c) \geq 1/n$ so that $g(p(x)) \geq 1/n$. Assuming that $n \geq n_0$, we have $||p(x) - p_n(x)||_Y \leq 1/(8n)$ and g is one-lipschitzian so $g(p_n(x)) \geq 7/(8n)$ and $g_n(p_n(x)) \geq 3/(4n)$.

On the other hand, if $p(x) \notin O$, then if $n \geq n_0$,

$$g_n(p_n(x)) \leq g(p_n(x)) + \frac{1}{8n} \leq g(p(x)) + \frac{1}{4n} \leq \frac{1}{4n}.$$

Let r_n be a C^∞-smooth function defined on the real numbers, such that

$$r_n(t) = 0 \quad \text{if } t \leq \frac{1}{2n}, \qquad r_n(t) > 0 \quad \text{if } t > \frac{1}{2n},$$

then define

$$\psi_n(x) = r_n g_n p_n(x) \quad \text{for } x \in (n_0+1)B_Y$$
$$\psi_n(x) = 0 \quad \text{for } x \notin n_0 B_Y.$$

The map ψ_n is C^k-smooth, and we can write

$$p^{-1}(O) = \cup_{n=n_0}^{\infty} O_n \quad \text{where} \quad O_n = \psi_n^{-1}((0,+\infty)) \quad \text{is in } \mathcal{U}^k.$$

Now considering for every α the set $p^{-1}(O_\alpha) = \cup_{n=n_0^\alpha}^{\infty} O_n^\alpha$, we obtain that $\{O_\alpha^n\}_{n,\alpha}$ is a σ-locally finite basis of the topology of X which is contained in $\mathcal{U}^k$. Now the aforementioned argument in [**12**] concludes the proof of Theorem 2.1. □

Note that this result extends Vanderwerff's Theorem 2.1 in [**14**]. Indeed, if the dual norm on X^* is LUR, then it is not difficult to show, using infimal convolutions, that every continuous convex function f bounded on bounded sets can be approximated uniformly on bounded sets by C^1-smooth convex functions:

$$f_n(x) = \inf_{y \in X} \{f(x-y) + n||y||^2\}.$$

The next proposition is to ensure that the assumption of approximating convex functions does not yield trivially the approximation of all continuous functions.

Proposition 2.4. *For any infinite dimensional Banach space X, there exists a continuous function defined on the unit ball of X that is not in the closed span of convex continuous functions for the topology of uniform convergence on B_X.*

Although the result is probably well-known, we have been unable to find a reference for Proposition 2.4.

Proof. If X is infinite-dimensional, then $\overline{B}_X$ is not compact, so there exists a bounded continuous function F defined on $\overline{B}_X$ which is not 3-uniformly continuous, that is, we can find $x_n, y_n \in \overline{B}_X$ with

$$||x_n - y_n|| \to 0 \quad \text{but} \quad |F(x_n) - F(y_n)| > 3.$$

We can also find a sequence $(u_n) \subseteq (1/2)S_X$ and $\delta > 0$ such that for all $n \neq m$, $||u_n - u_m|| > 2\delta$. Hence the balls $\overline{B}_p = \overline{B}(u_p, \delta/2^p)$ are disjoint balls all centered on the sphere $(1/2)S_X$.

Let us define $\overline{B}_p' = \overline{B}(u_p, (1/2)(\delta/2^p))$ and $f(u_p + (1/2)(\delta/2^p)x) = F(x)$, the map f is defined on $\cup\overline{B}_p'$, continuous and bounded. Let us still denote by f a bounded continuous extension of f to B_X.

Suppose that f is in the closed span of continuous convex functions for the topology of uniform convergence. There exist g, h convex functions such that for all x in B_X, $|f(x) - (g-h)(x)| \leq 1/2$.

Fix p and write $\bar{x}_n^p = u_p + (\delta/2^{p+1})x_n$, $\bar{y}_n^p = u_p + (\delta/2^{p+1})y_n$

$$\begin{aligned}|g(\bar{x}_n^p) - g(\bar{y}_n^p)| + |h(\bar{x}_n^p) - h(\bar{y}_n^p)|& \\ &\geq |(g-h)(\bar{x}_n^p) - (g-h)(\bar{y}_n^p)| \\ &\geq |f(\bar{x}_n^p) - f(\bar{y}_n^p)| - |f(\bar{x}_n^p) - (g-h)(\bar{x}_n^p)| \\ &\quad - |f(\bar{y}_n^p) - (g-h)(\bar{y}_n^p)| \\ &\geq 3 - \frac{1}{2} - \frac{1}{2} = 2.\end{aligned}$$

So for all n, $|g(\bar{x}_n^p) - g(\bar{y}_n^p)| \geq 1$ or $|h(\bar{x}_n^p) - h(\bar{y}_n^p)| \geq 1$. So at least one of the functions g and h, say g, has to verify this for an infinite sequence $(n_k)_{k\in N}$. As g is convex and $\bar{x}_n^p, \bar{y}_n^p \in \overline{B}_p'$, $(\bar{x}_n^p - \bar{y}_n^p) \to 0$, it is not difficult to see that g is unbounded on $\overline{B}_p$.

Therefore, at least one of g and h is unbounded on $\overline{B}_p$ for an infinite sequence of p, s. Without loss of generality, we can assume that g is unbounded on $\overline{B}_p$ for all $p \in \mathbf{N}$. The function g being continuous convex, g is bounded below on $\overline{B}_X$, so if $p \in \mathbf{N}$, there exists $z_p \in B_X$ such that $g(u_p + (\delta/2^p)z_p) \geq g(2u_p)/2 + p$. By convexity, $2g(u_p + (\delta/2^p)z_p) \leq g(2(\delta/2^p)z_p) + g(2u_p)$ hence $g((\delta/2^{p-1})z_p) \geq 2p$. But $||(\delta/2^{p-1})z_p||_X \leq \delta/2^{p-1} \to 0$ as $p \to \infty$ so g cannot be continuous at zero. Hence, f is not approximable by differences of continuous convex functions. □

Differences of continuous convex functions are called delta-convex functions and have been studied by Veselý and Zajíček in [**15**]. In particular, they prove that $C^{1,1}$-functions (i.e., differentiable with Lipschitz derivative) defined on Hilbert spaces are delta-convex. Note that, by the work of Toruńczyk, the approximation of Lipschitz functions suffices to obtain smooth partitions of unity.

The next result extends Proposition 8.3.10 in [**2**] and relates transfer technique (see [**2**, Section 2.2]) to smooth partitions of unity.

Proposition 2.5. *Suppose* $T : X \to Y$ *is a bounded operator between two Banach spaces* X *and* Y *such that* T^{**} *is one-to-one. Assume*

1) Y *admits* C^k*-smooth partitions of unity.*

2) X *has a Kadeč-Klee norm which is limit of* C^k*-smooth functions uniformly on bounded sets.*

Then X *admits* C^k*-smooth partitions of unity.*

Note. The class of Y's satisfying the assumptions of Proposition 2.5 is much more general than the class of spaces $c_0(\Gamma)$. For instance, Haydon proved in [**5**] that it contains the spaces $C_0(L)$ for certain locally compact sets, such as trees. We may also take $Y = C(K)$, where K is a scattered compact such that $K^{(\omega_0)} = \varnothing$ [**3**].

Proof. We follow the proof of Proposition 8.3.10 in [**2**]. By a result of Toruńczyk ([**12**], see [**2**, Theorem 8.3.2]), the space Y admits C^k-smooth partitions if and only if there exists a set Γ, a homeomorphic embedding $\varphi : Y \hookrightarrow c_0(\Gamma)$ such that for all γ in Γ, φ_γ is a C^k-smooth function, where $\varphi = (\varphi_\gamma)_{\gamma\in\Gamma}$.

We want to show that there exists a similar embedding for X. Let $\varphi : Y \to c_0(\Gamma)$ be such a function. The mapping $\varphi \times \mathrm{Id} : Y \oplus c_0(\mathbf{N}) \to c_0(\Gamma) \oplus c_0(\mathbf{N})$ is a homeomorphic embedding with C^k-smooth components.

Let $(f_n)_{n\geq 1}$ be a seqeunce of bounded C^k-smooth functions defined on X such that, for all $n \in \mathbf{N}$, for all $x \in nB_X$, $|f_n(x) - ||x||\,| \leq 1/n$. We can also assume that $f_n(x) = 2n$ for $||x|| \geq 2n$.

Define $S : X \to Y \oplus c_0(\mathbf{N})$ by $Sx = (Tx, (2^{-n}f_n(x))_{n\in\mathbf{N}})$. The map T is one-to-one, hence so is S. If the sequence x_k converges to x, then $\{x_k\}$ is bounded. Moreover, $Tx_k \to Tx$ and for all n, $f_n(x_k) \to f_n(x)$. Since f_n is uniformly bounded on bounded sets, $(2^{-n}f_n(x_k))_{n\geq 1} \to (2^{-n}f_n(x))_{n\geq 1}$ in the $c_0(\mathbf{N})$ norm. Hence, S is continuous.

To prove that S^{-1} is continuous, let x_k, x be in X such that $Sx_k \to Sx$. We have, for all n in $\mathbf{N}$, $f_n(x_k) \to f_n(x)$. Suppose $n_0 > ||x||$. For $n \geq n_0$, $f_n(x) \leq n_0 + 1/n$, so for k large enough, $f_n(x_k) \leq n_0 + 1/n + 1 < 2n$. Hence $\{x_k\} \subseteq 2nB_X$. As $f_n \to || \;||$ uniformly on bounded sets, we have $||x_k|| \to ||x||$. On the other hand, $Tx_k \to Tx$ implies that $x_k \overset{w}{\to} x$. Indeed, if $f \in Y^*$, $T^*f(x_k) = f(Tx_k) \to f(Tx) = T^*f(x)$ and T^*Y^* is dense in X^* for T^{**} is one-to-one. So we have $||x_k|| \to ||x||$ and $x_k \overset{w}{\to} x$. So if $|| \;||$ has the Kadeč-Klee property, $x_k \to x$ in the norm topology. Hence, S^{-1} is continuous.

Now $(\varphi \times \mathrm{Id})S$ is a homeomorphic embedding from X to $c_0(\Gamma) \oplus c_0(\mathbf{N})$, and it is easy to check that its components are C^k-smooth. And this concludes the proof since $c_0(\Gamma) \oplus_\infty c_0(\mathbf{N}) \equiv c_0(\Gamma \cup \mathbf{N})$. □

Let us mention that the method in the proof of Theorem 2.1 can still be used here, but we have to assume the existence of Lipschitz C^k-smooth partitions on Y. This supplementary assumption is not needed in the present approach. Note that it is not known whether it is actually stronger than the existence of C^k-smooth partitions of unity (see [**2**, p. 89]).

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Université P. et M. Curie, Equipe d'Analyse, Tour 46, 4 Place Jussieu, 75005 Paris, France.